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COMBINATORICS

Lecture 4: Combinatorial species

According to N. Bourbaki, a structure in a set U is an element of a tower of sets over U. In the combinatorial context, A. Joyal substitutes this organic definition by a functional one.

Definition. A *species of structures* is a functor from the category $\mathbb{B}$ of finite sets and bijections to the category of finite sets and arbitrary maps $\mathbb{S}$:

$$F : \mathbb{B} \to \mathbb{S}.$$

In other words, a species F is a rule that assigns, to every finite set U, a finite set $F[U]$ and to every bijection $\sigma : U \to V$, a mapping $F[\sigma] : F[U] \to F[V]$ in such a way that

1°. $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$.

2°. $F[\mathrm{id}_U] = \mathrm{id}_{F[U]}$.

Every element of $F[U]$ is referred to as a *structure of species* F, or an F*-structure* in the set U. The mapping $F[\sigma]$ is called the *transport of* F*-structures along* σ. If $\sigma : U \to U$ and $s \in F[U]$, then the structures s and $s' = F[\sigma](s)$ are called *equivalent* or *isomorphic*, notation: $s \sim s'$. It immediately follows from the definition that all maps $F[\sigma]$ are bijections. thus a species is in fact a functor $F : \mathbb{B} \to \mathbb{B}$.

Examples.

1. Empty species $0[U] = \emptyset$ for any U.
2. Species of empty sets $1[U] = \{U\}$, if $U = \emptyset$, and $\emptyset$ otherwise.
3. Species of singletons $X[U] = \{U\}$, if $|U| = 1$, and $\emptyset$ otherwise.
4. Species of sets $E[U] = \{U\}$ for any U.
5. Species of elements $\varepsilon[U] = U$ for any U.
6. Species of linear orders $L[U] = \{\text{linear orders in the set } U\}$.
7. Species of permutations $S[U] = \{\text{permutations of the set } U\}$.

Definition. (1) Two species, F_1 and F_2, are said to be *equipotent*, if the sets $F_1[U]$ and $F_2[U]$ have equal cardinality for any finite set U.

(2) Two species, F_1 and F_2, are said to be *isomorphic*, if the corresponding functors are isomorphic in category theoretic sense, i.e. there is a family of bijections $F_1[U] \to F_2[U]$, natural with respect to U.

The distinction between these two notions can be understood by comparing the two species L and S which are equipotent but not isomorphic.

With every species F, one can associate three important formal power series: F, $\tilde{F}$ and Z_F.

Before giving the definitions, let us note that the number of elements in the set $F[U]$ is uniquely defined by the cardinality of the set U, therefore $|F[U]| = |F[n]|$, if $|U| = n$, where $[n]$ stands for the set $\{1, 2, ..., n\}$ and $F[n]$ is the simplified way to write $F[[n]]$.

Definition. The *generating series* of a combinatorial species F is

$$F(x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n,$$

where $f_n = |F[n]|$.

Definition. The *type generating series* of a combinatorial species F is

$$\tilde{F}(x) = \sum_{n=0}^{\infty} \tilde{f}_n x^n,$$

where $\tilde{f}_n = |F[n]/\sim|$ is the number of F-structures in a set of n elements considered up to isomorphism.

Definition. The *cycle index series* of a combinatorial species F is the formal power series in an infinite number of variables $x_1, x_2, ...$

$$Z_F(x_1, x_2, ...) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{\sigma \in S_n} \operatorname{fix} F[\sigma] \, x_1^{\sigma_1} x_2^{\sigma_2} \cdots \Big),$$

where, for a permutation $\sigma : [n] \to [n]$, the symbol σ_k is the number of cycles of length k in the decomposition of σ into disjoint cycles and $\operatorname{fix} F[\sigma]$ means the number of fixed points of the mapping $F[\sigma] : F[n] \to F[n]$.

Examples.

Notation	Name	$F(x)$	$\tilde{F}(x)$	$Z_F(x_1, x_2, ...)$
0	empty species	0	0	0
1	empty sets	1	1	1
X	singletons	x	x	x_1
E	sets	e^x	$\frac{1}{1-x}$	$\exp \sum_{k=1}^{\infty} \frac{x_k}{k}$
ε	elements	xe^x	$\frac{x}{1-x}$	$x_1 \exp \sum_{k=1}^{\infty} \frac{x_k}{k}$
L	linear orders	$\frac{1}{1-x}$	$\frac{1}{1-x}$	$\frac{1}{1-x_1}$
S	permutations	$\frac{1}{1-x}$	$\prod_{k=1}^{\infty} \frac{1}{1-x^k}$	$\prod_{k=1}^{\infty} \frac{1}{1-x_k}$

Each of the two series $F(x)$ and $\tilde{F}(x)$ can be deduced from the cycle index series.

Theorem. For any species F, we have:

$$\begin{array}{lrcl} (a) & F(x) & = & Z_F(x, 0, 0, \ldots), \\ (b) & \tilde{F}(x) & = & Z_F(x, x^2, x^3, \ldots). \end{array}$$