EXERCISES FOR THE MINI-COURSE.

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1. Formula from [B] for $\mathrm{Dim}_{s,t}\mathcal{E}(\mathfrak{g})$:

$$\frac{1}{|W|}\sum_{w\in W}\frac{\det(1+s\cdot w)}{\det(1-t\cdot w)}=\prod_{i=1}^{l}\frac{1+st^{e_i}}{1-t^{e_i+1}}.$$

2. Formula for the q-order of an affine Weil group:

$$|W_{aff}(R)|_q=|W(R)|_q\cdot|W_{aff}^{++}|_q$$

where W_{aff}^{++} is the subset of W_{aff} whose elements transform C_{aff}^{+} in a affine chambre inside C_+.

In particular, for $R=\mathbf{A}_n$ the second factor is $\prod_{i=1}^{n}(1-q^i)^{-1}$.

3. Problem (from [B]). Show that $|W_{aff}^{++}|_q=\prod_{i=1}^{l}(1-q^{e_i})^{-1}$.
Corollary:

$$|W_{aff}|_q=\frac{1}{(1-q)^l}\prod_{i=1}^{l}\frac{[e_i+1]_q}{[e_i]_q}.$$

4. Formula for the (s,t)-dimensions of isotypic components in $\mathcal{E}(\mathfrak{g})$ (generalization of problem 1 to general irreps of $\mathfrak{g}$).

5. Structure of some Weyl groups.

a) The group $W(\mathbf{E}_6)$ is an extension of $\mathbb{Z}_2$ by the simple group of order 25920, isomorphic to $SO_+(Q(\mathbf{E}_6)/3P(\mathbf{E}_6),\ B)$ where the symmetric bilinear form B comes from the Killing form.

b) The group $W(\mathbf{E}_7)$ is an extension by $\mathbb{Z}_2$ of the simple group of order $2^9\cdot 3^4\cdot 5\cdot 7$ $Sp(Q(\mathbf{E}_7)/2P(\mathbf{E}_7),\ B)$ where the alternate bilinear form B comes from the Killing form.

Note, that $Sp(6,\ \mathbb{F}_2)\simeq O(7,\ \mathbb{F}_2)$ for an appropriate quadratic form.

c) The group $W^+(\mathbf{E}_8)$ is an extension by $\mathbb{Z}_2$ of the simple group isomorphic to $SO_+(Q(\mathbf{E}_8)/2P(\mathbf{E}_8),\ B)$ where the alternate bilinear form B comes from the half of the standard form.

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6. Formula for the dimensions of unirreps of $Sp(2n), SO(2n), SO(2n+1)$ in terms of the partition defined by the highest weight:

$$\dim \pi_\lambda = \prod_{(i,j)\in\lambda} ???.$$

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7. Prove that there is exactly 2^{n-1} Coxeter elements in $W(\mathbf{A}_n)$.

8. Prove that the action of the Borel subgroup B on the tangent space to the flag manifold $\mathcal{F} = G/B$ is equivalent to the coadjoint action of B in $\mathfrak{b}^*$.

9. Double Bruhat cells as symplectic leaves of the appropriate Poisson structure on $\mathcal{F}$.

10*. Comparative statistics of double Bruhat cells and coadjoint orbits.

11. Generating function for the quantity $n(w) := \dim\ker(w-1)$ for $W(\mathbf{A}_n)$:

$$\sum_{w\in W} q^{n(w)} = \prod_{k=1}^{n}(q+n-k) = \frac{q}{n} + \frac{q^2}{n}\sum_{k=1}^{n-1}\frac{1}{k} + \ \dots\ + \frac{q^n}{n!}.$$

Hint. Show that $n(w)$ is equal to the numer of cycles in the permutation w.

12. The substitutional exponent and substitutional logarithm.

Let $\xi = \sum_{k\geq 1}\alpha_k x^k \cdot \frac{d}{dx}$ be a formal vector field, $f(x) = \exp\xi = x(1+\sum_{k\geq 1} a_k x^k)$ the corresponding formal transformation.

Then $\{\alpha_k\}$ and $\{a_k\}$ are related by one-to-one transformations called **substitutional exponent** and **substitutional logarithm**. The problem is to find a manageable expession for these maps.

In particular $a_1(\alpha) = \alpha_1,\ a_2(\alpha) = \alpha_2 + \alpha_1^2,\ a_3 = \alpha_3 + \frac{5}{2}\alpha_1\alpha_2 + \alpha_1^3,\ \dots .$

Hint. If α_k are all zero except α_n, then $f(x) = \frac{x}{\sqrt[n]{1-n\alpha_n x^n}}$.

References

[B] Bourbaki N., *Groupes et algèbres de Lie*, Hermann, Paris, 1968, 19?? (english edition).

[H] Humphreys J., *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972, 1980 (second edition).

[K] Kostant B., *Lie group representations on polynomial ring*, Amer. J. Math. **85** (1963), 327-404.

[OV] Onishchik A. and Vinberg E., *A seminar on Lie groups and algebraic groups*, Nauka, Moscow, 1988, 1995 (second edition), pp. 344 (In Russian); English translation: Series in Soviet Mathematics. Springer-Verlag, 1990, xx+328 pp.

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