ROOT SYSTEMS AND LIE ALGEBRAS. MINI-COURSE.

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Abstract.

Contents

0. Introduction

In this mini-course I want to give an introduction to the theory of semi-simple Lie algebras and their representations. I hope that it enables young mathematicians to work on the problems related to the new class of associative algebras related to simple complex Lie algebras (or root systems) – so-called family algebras.

It turns out that several important questions about semi-simple algebras and their representations can be formulated, studied and sometimes solved in terms of these family algebras.

This program was started in [K] and continued in [KR].

1. Root systems and semi-simple Lie algebras

1.1. Root systems.

1.1.1.. A finite set R of non-zero vectors in $\mathbb{R}^n$ is called a **root system** if the following two conditions are satisfied:

R1. The set R is symmetric with respect to any **mirror** $M_\alpha := \alpha^\perp,\ \alpha \in R$.
In other words, if α, β are in R, then $s_\alpha\beta := \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is also in R.

R2. For any $\alpha, \beta \in R$ the number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

Besides these main conditions often some additional properties are required which define special kinds of root systems.

We formulate these requirements here together with the name of corresponding kind of root systems.

R3a. The set R spans the whole $\mathbb{R}^n$. (Non-degenerate root system).

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R3b. The set R can not be represented in the form $R_1 \sqcup R_2$ with $R_1 \perp R_2$. (Indecomposable root system).

R3c. If $\alpha \in R$, then $2\alpha \notin R$. (Reduced root system).

R3d. All vectors from R have the same length. (Simply-laced root system).

Comments to additional requirements.

a) If this condition is not satisfied, we can simply replace R^n by a smaller space $\mathbb{R}^m$ spanned by R. The number m is called the **rank** of the root system R.

b) If $R_1 \sqcup R_2$ and $R_1 \perp R_2$, then both subsets R_1 and R_2 are root systems and we can study them separately.

c) We propose the following
Exercise 1. a) Let $\alpha \in R$ and $k\alpha \in R$ for some $k \in \mathbb{R}$, then $k = \pm 1$, ± 2, or $\pm \frac{1}{2}$.
b) Show that all possibilities above are indeed realized. (Hint: consider root systems of rank 1).

It is worth to mention that for any $m \in \mathbb{Z}_+$ there is only one indecomposable non-reduced root system of rank m. (See examples below).

d) The simply-laced root systems are also called ADE-systems. They appear in many classification problems, see e.g. [HHH].

Main Example 1. Let $\{e_i\}_{0 \le i \le n}$ be the standard basis in $\mathbb{R}^{n+1}$. Put

$$R = \{e_i - e_j \mid i \neq j\}.$$

Exercise 2. Show that R is a degenerate indecomposable reduced simply laced root system of rank n. The standard notation for this system is $\mathbf{A}_n$.

Main Example 2. Let $R \subset \mathbb{R}^n$, $n \geq 2$, be the set $\{\pm e_i \pm e_j \mid i \neq j\}$.

Exercise 3. Show that R is a root system of rank n satisfying all requirements. The standard notation for this system is $\mathbf{D}_n$.

Main Example 3. Let $R \subset \mathbb{R}^8$ be the set

$$\{\pm e_i \pm e_j \mid i \neq j\} \bigcup \{\sum_{i=1}^{8} \epsilon_i e_i \mid \epsilon_i = \pm 1 \,\&\, \prod_{i=1}^{8} \epsilon_i = 1\}.$$

Exercise 4. Show that R is a root system of rank 8 satisfying all requirements. The standard notation for this system is $\mathbf{E}_8$.

For the convenience of the reader we give here the picture of all root systems of rank 2

Figure 1.

1.1.2. Simple roots and Dynkin diagrams.

It turns out that a very powerful tool in studying root systems is the introduction of a linear order in R compatible with linear operations (i.e. a sum of positive roots must be positive and if α is positive, then $-\alpha$ is negative). Any such order can be obtained as follows. Choose any linear functional on $\mathbb{R}^n \supset R$ so that it do not vanish on R. Then, call a root α positive or negative according to the sign of $f(\alpha)$.

It is clear that R splits into positive and negative parts: $R = R_+ \sqcup R_-$.

Definition 1. A positive root is called **simple** if it is not a sum of two positive roots.

Exercise 3. Prove that any positive (resp. negative) root is a linear combination of simple roots with integer non-negative (resp. non-positive) coefficients.

Therefore, the number of simple roots cannot be less than the rank of the system.

Theorem 1. *The number of simple roots in a root system R is equal to the rank of R.*

The proof would follow if we show that simple roots are linearly independent. For this end we prove

Lemma 1. *If α and β are simple roots, then $(\alpha, \beta) \leq 0$. (In other words, the angle between two simple roots is right or obtuse).*

Proof. Suppose the contrary: $(\alpha, \beta) > 0$. We can assume that $|\alpha| \geq |\beta|$. Then the number $\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ must be 1 (because it is positive integer and less than 2). Therefore, $s_\alpha \beta = \beta - \alpha \in R$. Consider two cases: $\beta - \alpha$ can be positive or negative. In the first case $\beta = (\beta - \alpha) + \alpha$ which contradicts the simplicity of β. In the second case $\alpha - \beta$ is positive and $\alpha = (\alpha - \beta) + \beta$ which contradicts the simplicity of α. $\square$

Come back to the theorem. Suppose that there are more simple roots than the rank of R. Then simple roots are linearly dependent: $\sum_{k \in K} c_k \alpha_k = 0$. Since all simple roots are positive, the coefficiens $\{c_k\}$ can not be of the same sign. So, we can rewrite the equation above in the form

$$\Sigma_1 := \sum_{k_1 \in K_1} c_{k_1} \alpha_k = \sum_{k_2 \in K_2} c_{k_2} \alpha_k =: \Sigma_2$$

with strictly positive coefficients c_k. But from Lemma 1 we conclude that

$$(\Sigma_1, \Sigma_2) = \sum_{k_i \in K_i} c_{k_1} c_{k_2} (\alpha_{k_1}, \alpha_{k_2}) \leq 0.$$

Hence, $|\Sigma_1|^2 = |\Sigma_2|^2 = (\Sigma_1, \Sigma_2) = 0$, a contradiction. $\square$

We shall denote by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. Recall, that this set depends on the order we introduced. We shall see soon (theorem 3) that this dependence is immaterial.

The remarkable fact is that the root system R can be uniquely reconstructed from the geometry of Π. Thus, the classification of root systems can be given in terms of corresponding sets of simple roots. The information encoded in Π can be written in two convenient ways.

1. Via the so-called **Cartan matrix** A with elements $A_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

2. Via the **Dynkin graph** or **Dynkin diagram** whose vertices correspond to simple roots and vertices i and j are joint by $A_{i,j}A_{j,i}$ edges. When the roots have different length (i.e. when multiple edges exist), one put an arrow pointing to the shortes root (or marks black the vertices corresponding to short roots).

Example. The Dynkin diagrams corresponding to root systems of ADE-type are drawn on Figure 2.

Figure 3.

Theorem 2. *The characteristic properties of Cartan matrices are:*
a) all diagonal entries $A_{i,i}$ are equal to 2;
b) the off-diagonal entries are non-positive integers satisfying

$$A_{i,j} = 0 \Longleftrightarrow A_{j,i} = 0;$$

c) the quadratic form $A(x) = \sum_{i,j} A_{i,j} x_i x_j$ is positive definite.

Exercise 4. Show that the condition c) of theorem 2 implies that Dynkin graphs cannot contain the following fragments:
a) cycles,
b) branching points with more than 3 branches,
c) more than one branching point in one connected component,
d) more than one multiple edge in one connected component.

Exercise 5. Check directly that the root systems of type $\mathbf{A}_3$ and $\mathbf{D}_3$ are isomorphic as suggested by Fig. 2 (compare the corresponding Dynkin diagrams).

1.1.3. The Weyl group.

We assume now that our root system $R \subset \mathbb{R}^n$ is non-degenerate and reduced. Consider the group W of linear transformations of $\mathbb{R}^n$ generated by reflections s_α, $\alpha \in R$. It is finite because it has a faithful realization by transformations of the finite set R.

The union of all mirrors $\bigcup\limits_{\alpha \in R} M_\alpha$ split $\mathbb{R}^n$ into convex connected components which are called **Weyl chambers**. Note, that to determine a Weyl chamber containing a given vector x is the same as to fix the sign of all inner products (x, α), $\alpha \in R_+$.

In particular, there is a chamber where all these inner products are positive. It is called a **dominant chamber** and is denoted by C_+. The elements of C_+ are called **dominant vectors**.

We shall call a vector $x \in \mathbb{R}^n$ **singular** if it belongs to a mirror and **regular** otherwise.

Theorem 3. *a) The Weyl group W is generated by reflections $\{s_i\}, 1 \le i \le n$, corresponding to simple roots.*
b) The group W acts on the set of Weyl chambers in a transitive way.

Lemma 2. *The dominant chamber C_+ is bounded by mirrors M_i, $1 \le i \le n$, corresponding to simple roots.*

Proof. Let M_α be any mirror. We can assume that $\alpha \in R_+$, hence $\alpha = \sum_{i=1}^n k_i \alpha_i$ with $k_i \ge 0$. Therefore, one cannot cross M_α, i.e. change the sign of (x, α) without crossing one of M_i. Therefore $C_+ \bigcap M_\alpha = \emptyset$ unless $\alpha = \alpha_i$ for some i. $\square$

Let now W_0 denote the subgroup of W generated by simple reflections s_i, $1 \le i \le n$. I claim that any reflection s_β, $\beta \in R$, is conjugated to a simple reflection by an element of W_0 and, consequently, belongs to W_0.

Indeed, let C be any chamber and $x \in C$. Choose a point $x_+ \in C_+$ and join it with x by a path which intersects consequitively the mirrors $M_{\beta_1}, M_{\beta_2}, \dots, M_{\beta_k}$ and passes through the chambers $C_0 = C_+, C_1, C_2, \dots, C_k$. We prove by induction that all S_{β_i} belong to W_0.

For $i = 1$ it is true by Lemma 2. Suppose, it is true for $i < j$ and consider M_{β_j}. The transformation s_{β_i} interchanges the chamber C_{i-1} and C_i, separated by the mirror M_{β_i}. Hence, $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{j-1}}$ sends C_{j-1} to C_0 and the mirror M_{β_j} to a mirror adjacent to C_0. Therefore, s_{β_j} is conjugated to a simple reflection by an element of W_0 and, consequently, belongs to W_0.

Note, that the same argument shows that W acts transitively on the set of Weyl chambers.

For the future use we formulate here

Theorem 3. *The group W is determined by the generators s_i, $1 \le i \le n$, and relations*

$$(1) \quad (s_i s_j)^{m(i,j)} = 1, \quad \text{where } m(i,j) = 2,3,4,6,1 \quad \text{if} \quad \frac{4(\alpha_i, \alpha_j)^2}{|\alpha_i|^2 |\alpha_j|^2} = 0,1,2,3,4.$$

Sketch of the proof. Suppose that $s_{i_2} s_{i_2} \cdots s_{i_k} = e$. Consider a piecewise linear loop starting from a point $x_0 \in C_+$ and going through vertices $x_1 = s_{i_k} x_0, x_2 = s_{i_{k-1}} x_1$, etc till $x_k = x_0$. We can deform this loop to a point so that the deformed loop never touched any triple intersections $M_\alpha \bigcap M_\beta \bigcap M_\gamma$ and crosses transversally the double intersections and single mirrors.

Then the elementary moves are removing of double intersection of a path with a mirror M_α:

Figure 3.

and shrinking of a subloop around a double intersection $M_\alpha \bigcap M_\beta$:

Figure 4.

The first move correspond to the relation (1) for $\alpha = \beta$: $s_\alpha^2 = e$, while the second – to the relation (1) for different α and β.

The latter relation can be proved directly by considering the root system of rank 2 generated by the pair α, β. Indeed, the product $s_\alpha s_\beta$ is a rotation on an angle which is twice the angle between α and β.

□

Theorem 4. *The stabilizer in W of any vector $x \in \mathbb{R}^n$ is generated by reflections in those mirrors which contain x.*

We leave to the reader to find a transparent proof of this important statement. In fact one can modify the argument ud in the proof of Theorem 2, to show that an element $w = s_{i_2} s_{i_2} \cdots s_{i_k}$ maps a point $x_0 \in C_+$ to some point $x_k \in C_+$ iff $w = e$.

1.1.4. Length and invariants.

Each element of the Weyl group has the form

$$w = s_{i_1} s_{i_2} \cdots s_{i_l} \tag{2}$$

where s_i are simple reflections. If the length l of this expression is minimal, it is called **length** of w and (1) is called **reduced decomposition** of w.

Lemma 3. *The length of $w \in W$ can be defined also by the following equivalent ways.*

a) $l(w) = \#\left(w(R_+) \bigcap R_-\right)$.

b) For any regular $x \in \mathbb{R}^n$ we have $l(w) =$ the number of mirrors which separate x and $w(x)$.

Exercise 5. Let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be an element of W. We define the sequence of mirrors by $M_1 = M_{\alpha_k}$, $M_2 = s_{i_{k-i}} M_1$, etc.

Show that mirrors M_i are all different iff the decomposition (1) is reduced.

Corollary. There is a unique element $w_0 \in W$ which has the maximal length $\# R_+$. For any Weyl chamber C we have $w_0(C) = -C$. In particular, $w_0(C_+)$ coincides with $C_- := -C_+$.

Now, we consider the expression

$$|W|_q := \sum_{w \in W} q^{l(w)}$$

which we shall call the q-order (or graduate order) of W.

Remark. This definition is a particular case of the following. Let V be a $\mathbb{Z}_+$-filtered vector space, i.e. a family of subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \ldots$$

is given, such that all factors V_i / V_{i-1} are finite-dimensional. Introduce the notation $\mathrm{gr}^i(V) := V_i / V_{i-1}$. Then the space

$$\mathrm{gr} V := \bigoplus_{i \in \mathbb{Z}_+} \mathrm{gr}^i(V)$$

is called the $\mathbb{Z}_+$-graded space associated with this filtration. The **filtered dimension** of V and the **graded dimension** of gr V are defined as a formal series

$$\mathrm{Dim}_q V = \sum_{i \in \mathbb{Z}_+} q^{\dim \mathrm{gr}^i(V)}.$$

In many examples the q-dimension is not only formal series but an analytic function of q in some appropriate domain of complex field.

Theorem 4. *For any root system R of rank n there exist integers $e_1, e_2, \ldots, e_n$ called* **exponents** *of R such that*

$$|W(R)|_q = \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^{e_i}) = \prod_{i=1}^{n} [1 + e_i]_q.$$

These numbers have many remarkable properties. We mention here some of them (the proof see below):

a) The algebra of W-invariant polynomials on $\mathbb{R}^n$ is freely generated by homogeneous polynomials $P_1, \ldots, P_n$ such that $\deg P_i = e_i + 1$.

b) The algebra of W-invariant elements in the Grassmann algebra $\wedge(\mathbb{R}^n)$ is freely generated by homogeneous elements $Q_1, \ldots, Q_n$ such that $\deg Q_i = 2e_i + 1$.

c) The exponents satisfy the following relations

1) $\prod_{i=1}^{n} (e_i + 1) = |W|$; 2) $\sum_{i=1}^{n} e_i = |R_+|$; 3) $e_i + e_{n-i+1} = h$

where h is the order of the so-called **Coxeter element** $c = s_1 s_2 \cdots s_n \in W$.[1]

To x simplify the notation, denote by $\mathcal{A}$ the algebra of polynomials on $\mathbb{R}^n$, by I the algebra of W-invariant polynomials, by J the ideal in $\mathcal{A}$ generated by homogeneous elements of positive degree in I. Introduce the inner product in $\mathcal{A}$ by

$$(P, Q) = P(\partial/\partial x)Q(x) \,\Big|_{x=0} . \tag{3}$$

Exercise 6. Show that the inner product (3) is symmetric and invariant under orthogonal transformations of $\mathbb{R}^n$ (in particular, under the W-action).

Let H denote the space of W-**harmonic polynomials** defined by the condition

$$H = J^{\perp}, \text{ or } \qquad P \in H \text{ iff } \quad Q(\partial/\partial x)P = 0 \text{ for all } Q \in J.$$

Theorem 5. *a) $\mathcal{A} = I \otimes H$;*

b) The graded dimension of H is equal to $|W|_q$.

This result admits an interesting detalization for the case $R = \mathbf{A}_n$. In this case $W \simeq S_{n+1}$. The irreducible representations π_lambda of S_{n+1} correspond to Young diagrams λ with $n+1$ boxes. Consider the isotypic component $\mathbb{C}[x_1, x_2, \ldots, x_n]_\lambda$. It turns out that its graded dimension can be given by the formula:

$$\mathrm{Dim}_q \mathcal{A}_\lambda = \dim \pi_\lambda \cdot \prod_{(i,j)\in\lambda} \frac{q^{i-1}}{1 - q^{h(i,j)}} \tag{4}$$

where $h(i,j)$ is the length of the **hook** of the box $(i,j) \in \lambda$.

Note also the formula $\dim \pi_\lambda = \prod_{k=1}^{n+1} \frac{k}{h_k}$ where h_k is the length of the **hook** of the k-th box of λ.

[1]In fact, this product depends on the choice of Π and on the numeration of simple roots, but all these Coxeter elements belong to the same conjugacy class in W, hence have the same order.

It is worth to mention that the analogous result holds for the Weyl group of systems $\mathbf{B}_n$ and $\mathbf{C}_n$. These root system are obtained from $\mathbf{D}_n$ by adding of vectors $\pm e_i$ or $\pm 2e_i$. (If we add both $\pm e_i$ and $\pm 2e_i$, we get an unreduced system $\mathbf{B}C_n$).

The Weyl groups for $\mathbf{B}_n$ and $\mathbf{C}_n$ are isomorphic to the group C_n of isometries of a unit cube I^n. Algebraically, $C_n \simeq S_n \ltimes \mathbb{Z}_2^n$. (Elements of S_n permute the coordinates and elements of $\mathbb{Z}_2^n$ change the signs of coordinates).

It is known, that unirreps of C_n are labelled by pairs $(\lambda,\,\mu)$ of Young diagrams with total amount of boxes equal to n. The following analogue of (4) holds:

$$\mathrm{Dim}_q \mathcal{A}_{\lambda,\mu} = \dim \pi_{\lambda,\mu} \cdot \prod_{(i,j)\in\lambda} \frac{q^{2i-2}}{1-q^{h(i,j)}} \cdot \prod_{(i,j)\in\lambda} \frac{q^{i-1}+zq^{j-1}}{1-q^{h(i,j)}} \tag{5}$$

1.1.5 Superanalogue.

Some facts and formulae of preceeding section have remarkable superanalogues. The odd analog of the polynomial algebra $\mathcal{A}$ is the **Grassmann** algebra $\mathcal{B} = \wedge(\theta_1,\,\theta_2,\,\dots,\,\theta_n)$ defined by relations

$$\theta_i\theta_j + \theta_j\theta_i = 0 \text{ for all } i, j.$$

The superanalogue is the A.Weil algebra $\mathcal{E} = \mathcal{A} \otimes \mathcal{B}$. It's bigraded dimension is a formal power series in two variables $q,\,z$.

It turns out that here again $\mathcal{E} = I \otimes H$. Moreover, for $\mathbf{A}_n$ we have

$$\mathrm{Dim}_{q,z} \mathcal{E}_\lambda = \dim \pi_\lambda \cdot \prod_{(i,j)\in\lambda} \frac{q^{i-1}+zq^{j-1}}{1-q^{h(i,j)}} \tag{4$'$}$$

and for $\mathbf{C}_n$

$$\mathrm{Dim}_{q,z} \mathcal{E}_{\lambda,\mu} = \dim \pi_{\lambda,\mu} \cdot \prod_{(i,j)\in\lambda} \frac{q^{2i-2}+zq^{2j-1}}{1-q^{h(i,j)}} \cdot \prod_{(k,l)\in\mu} \frac{q^{2k-1}+zq^{2l-2}}{1-q^{h(k,l}} \tag{5$'$}$$

Theorem.

1.1.6. Extended Dynkin diagrams and affine Weyl groups.

Let R be a reduced non-degenerate and undecomposable root system. Choose an order on R and fix the corresponding set R_+ of positive roots and the set $\Pi = \{\alpha_1,\,\dots,\,\alpha_n\}$ of simple roots.

Let θ be the maximal root and put $\alpha_0 = -\theta$. The set $\widetilde{\Pi} = \{\alpha_0,\,\alpha_1,\,\dots,\,\alpha_n\}$ turns out to be a set of simple roots for some infinite root system $\widetilde{R}$ in a pseudo Euclidian space $\mathbb{R}^{n+1,1}$. The correspondig graph (defined by the same rules as before) is called the **extended Dynkin diagram**.

Denote by W_{aff} the group of affine transformations of $\mathbb{R}^n$ generated by linear reflections $s_i := s_{\alpha_i}$, $1 \le i \le n$, and the affine reflection s_0 defined by

$$s_0(x) = x - 2\frac{(\alpha_0,\,x)+1}{(\alpha_0,\,\alpha_0)}.$$

Note that geometrically s_0 is the reflection in the affine hyperplane M_0 given by the equation $\alpha_0(x) + 1 = 0$.

Affine Weyl groups share many properties of ordinary Weyl groups. We list here the most important ones.

1. Consider all mirrors obtained from M_i, $0 \leq i \leq n$, by the action of W_{aff}. The complement in R^n to the union of all these mirrors splits into connected components called **affine Weyl chambers**.

The group W_{aff} acts on the set of these chambers in a simply transitive way.

Example. Consider the root system of type $\mathbf{A}_2$. The corresponding affine chambers form a regular triangular lattice in $\mathbb{R}^2$.

Figure 5.

2. The generators $\{s_0, s_1, \ldots, s_n\}$ of the group W_{aff} satisfy the same defining relations (1) as before. In particular, for the system $\widetilde{\mathbf{A}}_2$ we have:

$$s_0^2 = s_0^2 = s_0^2 = (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = 1.$$

Exercise 7. Find the q-order of the group W_{aff} for the system $\widetilde{\mathbf{A}}_n$.

Answer: $|W_{aff}|_q = \frac{1-q^{n+1}}{(1-q)^{n+1}}$.

1.1.1.

1.2. Semi-simple Lie algebras.

2. Family algebras

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