Chapter I.
Complex Differential Calculus and Pseudoconvexity

This introductory chapter is mainly a review of the basic tools and concepts which will be employed in the rest of the book: differential forms, currents, holomorphic and plurisubharmonic functions, holomorphic convexity and pseudoconvexity. Our study of holomorphic convexity is principally concentrated here on the case of domains in $\mathbb{C}^n$. The more powerful machinery needed for the study of general complex varieties (sheaves, positive currents, hermitian differential geometry) will be introduced in Chapters II to V. Although our exposition pretends to be almost self-contained, the reader is assumed to have at least a vague familiarity with a few basic topics, such as differential calculus, measure theory and distributions, holomorphic functions of one complex variable, ... Most of the necessary background can be found in the books of (Rudin, 1966) and (Warner, 1971); the basics of distribution theory can be found in Chapter I of (Hörmander 1963). On the other hand, the reader who has already some knowledge of complex analysis in several variables should probably bypass this chapter.

§1. Differential Calculus on Manifolds

§1.A. Differentiable Manifolds

The notion of manifold is a natural extension of the notion of submanifold defined by a set of equations in $\mathbb{R}^n$. However, as already observed by Riemann during the 19th century, it is important to define the notion of a manifold in a flexible way, without necessarily requiring that the underlying topological space is embedded in an affine space. The precise formal definition was first introduced by H. Weyl in (Weyl, 1913).

Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. We denote by C^k the class of functions which are k-times differentiable with continuous derivatives if $k \neq \omega$, and by C^ω the class of real analytic functions. A *differentiable manifold* M of real dimension m and of class C^k is a topological space (which we shall always assume Hausdorff and separable, i.e. possessing a countable basis of the topology), equipped with an atlas of class C^k with values in $\mathbb{R}^m$. An *atlas* of class C^k is a collection of homeomorphisms $\tau_\alpha : U_\alpha \longrightarrow V_\alpha$, $\alpha \in I$, called *differentiable charts*, such that $(U_\alpha)_{\alpha \in I}$ is an open covering of M and V_α an open subset of $\mathbb{R}^m$, and such that for all $\alpha, \beta \in I$ the *transition map*

$$(1.1) \quad \tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(U_\alpha \cap U_\beta) \longrightarrow \tau_\alpha(U_\alpha \cap U_\beta)$$

Fig. I-1 Charts and transition maps

is a C^k diffeomorphism from an open subset of V_β onto an open subset of V_α (see Fig. 1). Then the components $\tau_\alpha(x) = (x_1^\alpha, \ldots, x_m^\alpha)$ are called the *local coordinates* on U_α defined by the chart τ_α ; they are related by the transition relation $x^\alpha = \tau_{\alpha\beta}(x^\beta)$.

If $\Omega \subset M$ is open and $s \in \mathbb{N} \cup \{\infty, \omega\}$, $0 \le s \le k$, we denote by $C^s(\Omega, \mathbb{R})$ the set of functions f of class C^s on Ω, i.e. such that $f \circ \tau_\alpha^{-1}$ is of class C^s on $\tau_\alpha(U_\alpha \cap \Omega)$ for each α ; if Ω is not open, $C^s(\Omega, \mathbb{R})$ is the set of functions which have a C^s extension to some neighborhood of Ω.

A *tangent vector* ξ at a point $a \in M$ is by definition a differential operator acting on functions, of the type

$$C^1(\Omega, \mathbb{R}) \ni f \longmapsto \xi \cdot f = \sum_{1 \le j \le m} \xi_j \frac{\partial f}{\partial x_j}(a)$$

in any local coordinate system $(x_1, \ldots, x_m)$ on an open set $\Omega \ni a$. We then simply write $\xi = \sum \xi_j \, \partial/\partial x_j$. For every $a \in \Omega$, the n-tuple $(\partial/\partial x_j)_{1 \le j \le m}$ is therefore a basis of the *tangent space* to M at a, which we denote by $T_{M,a}$. The *differential* of a function f at a is the linear form on $T_{M,a}$ defined by

$$df_a(\xi) = \xi \cdot f = \sum \xi_j \, \partial f/\partial x_j(a), \qquad \forall \xi \in T_{M,a}.$$

In particular $dx_j(\xi) = \xi_j$ and we may write $df = \sum (\partial f/\partial x_j) dx_j$. Therefore $(dx_1, \ldots, dx_m)$ is the dual basis of $(\partial/\partial x_1, \ldots, \partial/\partial x_m)$ in the cotangent space $T^\star_{M,a}$. The disjoint unions $T_M = \bigcup_{x \in M} T_{M,x}$ and $T^\star_M = \bigcup_{x \in M} T^\star_{M,x}$ are called the *tangent* and *cotangent bundles* of M.

If ξ is a vector field of class C^s over Ω, that is, a map $x \mapsto \xi(x) \in T_{M,x}$ such that $\xi(x) = \sum \xi_j(x) \, \partial/\partial x_j$ has C^s coefficients, and if η is another vector field of class C^s with $s \ge 1$, the *Lie bracket* $[\xi, \eta]$ is the vector field such that

(1.2) $[\xi,\eta]\cdot f = \xi\cdot(\eta\cdot f) - \eta\cdot(\xi\cdot f).$

In coordinates, it is easy to check that

(1.3) $$[\xi,\eta] = \sum_{1\le j,k\le m}\Big(\xi_j\frac{\partial\eta_k}{\partial x_j} - \eta_j\frac{\partial\xi_k}{\partial x_j}\Big)\frac{\partial}{\partial x_k}.$$

§1.B. Differential Forms

A differential form u of degree p, or briefly a p-form over M, is a map u on M with values $u(x)\in\Lambda^pT^\star_{M,x}$. In a coordinate open set $\Omega\subset M$, a differential p-form can be written

$$u(x) = \sum_{|I|=p} u_I(x)\,dx_I,$$

where $I=(i_1,\ldots,i_p)$ is a multi-index with integer components, $i_1<\ldots<i_p$ and $dx_I := dx_{i_1}\wedge\ldots\wedge dx_{i_p}$. The notation $|I|$ stands for the number of components of I, and is read *length* of I. For all integers $p=0,1,\ldots,m$ and $s\in\mathbb{N}\cup\{\infty\}$, $s\le k$, we denote by $C^s(M,\Lambda^pT^\star_M)$ the space of differential p-forms of class C^s, i.e. with C^s coefficients u_I. Several natural operations on differential forms can be defined.

§1.B.1. Wedge Product. If $v(x)=\sum v_J(x)\,dx_J$ is a q-form, the *wedge product* of u and v is the form of degree $(p+q)$ defined by

(1.4) $$u\wedge v(x) = \sum_{|I|=p,|J|=q} u_I(x)v_J(x)\,dx_I\wedge dx_J.$$

§1.B.2. Contraction by a tangent vector. A p-form u can be viewed as an antisymmetric p-linear form on T_M. If $\xi=\sum\xi_j\,\partial/\partial x_j$ is a tangent vector, we define the *contraction* $\xi \lrcorner u$ to be the differential form of degree $p-1$ such that

(1.5) $(\xi\lrcorner u)(\eta_1,\ldots,\eta_{p-1}) = u(\xi,\eta_1,\ldots,\eta_{p-1})$

for all tangent vectors η_j. Then $(\xi,u)\longmapsto\xi\lrcorner u$ is bilinear and we find easily

$$\frac{\partial}{\partial x_j}\lrcorner dx_I = \begin{cases} 0 & \text{if } j\notin I,\\ (-1)^{l-1}dx_{I\smallsetminus\{j\}} & \text{if } j=i_l\in I.\end{cases}$$

A simple computation based on the above formula shows that contraction by a tangent vector is a *derivation*, i.e.

(1.6) $\xi\lrcorner(u\wedge v) = (\xi\lrcorner u)\wedge v + (-1)^{\deg u}u\wedge(\xi\lrcorner v).$

§1.B.3. Exterior derivative. This is the differential operator

$$d : C^s(M, \Lambda^p T_M^\star) \longrightarrow C^{s-1}(M, \Lambda^{p+1} T_M^\star)$$

defined in local coordinates by the formula

$$du = \sum_{|I|=p,\,1\le k\le m} \frac{\partial u_I}{\partial x_k}\, dx_k \wedge dx_I. \tag{1.7}$$

Alternatively, one can define du by its action on arbitrary vector fields $\xi_0, \ldots, \xi_p$ on M. The formula is as follows

$$\begin{aligned} du(\xi_0,\ldots,\xi_p) &= \sum_{0\le j\le p} (-1)^j \xi_j \cdot u(\xi_0,\ldots,\widehat{\xi_j},\ldots,\xi_p) \\ &\quad + \sum_{0\le j<k\le p} (-1)^{j+k} u([\xi_j,\xi_k],\xi_0,\ldots,\widehat{\xi_j},\ldots,\widehat{\xi_k},\ldots,\xi_p). \end{aligned} \tag{1.7'}$$

The reader will easily check that (1.7) actually implies (1.7′). The advantage of (1.7′) is that it does not depend on the choice of coordinates, thus du is intrinsically defined. The two basic properties of the exterior derivative (again left to the reader) are:

$$d(u\wedge v) = du\wedge v + (-1)^{\deg u} u \wedge dv, \qquad (\textit{Leibnitz' rule}) \tag{1.8}$$

$$d^2 = 0. \tag{1.9}$$

A form u is said to be *closed* if $du = 0$ and *exact* if u can be written $u = dv$ for some form v.

§1.B.4. De Rham Cohomology Groups. Recall that a cohomological complex $K^\bullet = \bigoplus_{p\in\mathbb{Z}}$ is a collection of modules K^p over some ring, equipped with differentials, i.e., linear maps $d^p : K^p \to K^{p+1}$ such that $d^{p+1}\circ d^p = 0$. The *cocycle, coboundary* and *cohomology modules* $Z^p(K^\bullet)$, $B^p(K^\bullet)$ and $H^p(K^\bullet)$ are defined respectively by

$$\begin{cases} Z^p(K^\bullet) = \operatorname{Ker} d^p : K^p \to K^{p+1}, & Z^p(K^\bullet) \subset K^p, \\ B^p(K^\bullet) = \operatorname{Im} d^{p-1} : K^{p-1} \to K^p, & B^p(K^\bullet) \subset Z^p(K^\bullet) \subset K^p, \\ H^p(K^\bullet) = Z^p(K^\bullet)/B^p(K^\bullet). \end{cases} \tag{1.10}$$

Now, let M be a differentiable manifold, say of class C^∞ for simplicity. The *De Rham complex* of M is defined to be the complex $K^p = C^\infty(M, \Lambda^p T_M^\star)$ of smooth differential forms, together with the exterior derivative $d^p = d$ as differential, and $K^p = \{0\}$, $d^p = 0$ for $p < 0$. We denote by $Z^p(M,\mathbb{R})$ the cocycles (closed p-forms) and by $B^p(M,\mathbb{R})$ the coboundaries (exact p-forms). By convention $B^0(M,\mathbb{R}) = \{0\}$. The *De Rham cohomology group* of M in degree p is

$$H^p_{\mathrm{DR}}(M,\mathbb{R}) = Z^p(M,\mathbb{R})/B^p(M,\mathbb{R}). \tag{1.11}$$

When no confusion with other types of cohomology groups may occur, we sometimes denote these groups simply by $H^p(M,\mathbb{R})$. The symbol $\mathbb{R}$ is used here to stress that we are considering real valued p-forms; of course one can introduce a similar group $H^p_{\mathrm{DR}}(M,\mathbb{C})$ for complex valued forms, i.e. forms with values in $\mathbb{C}\otimes\Lambda^pT^\star_M$. Then $H^p_{\mathrm{DR}}(M,\mathbb{C})=\mathbb{C}\otimes H^p_{\mathrm{DR}}(M,\mathbb{R})$ is the complexification of the real De Rham cohomology group. It is clear that $H^0_{\mathrm{DR}}(M,\mathbb{R})$ can be identified with the space of locally constant functions on M, thus

$$H^0_{\mathrm{DR}}(M,\mathbb{R})=\mathbb{R}^{\pi_0(X)},$$

where $\pi_0(X)$ denotes the set of connected components of M.

Similarly, we introduce the De Rham cohomology groups with compact support

$$(1.12)\qquad H^p_{\mathrm{DR},c}(M,\mathbb{R})=Z^p_c(M,\mathbb{R})/B^p_c(M,\mathbb{R}),$$

associated with the De Rham complex $K^p=C^\infty_c(M,\Lambda^pT^\star_M)$ of smooth differential forms with compact support.

§1.B.5. Pull-Back. If $F:M\longrightarrow M'$ is a differentiable map to another manifold M', $\dim_{\mathbb{R}}M'=m'$, and if $v(y)=\sum v_J(y)\,dy_J$ is a differential p-form on M', the pull-back $F^\star v$ is the differential p-form on M obtained after making the substitution $y=F(x)$ in v, i.e.

$$(1.13)\qquad F^\star v(x)=\sum v_I\big(F(x)\big)\,dF_{i_1}\wedge\ldots\wedge dF_{i_p}.$$

If we have a second map $G:M'\longrightarrow M''$ and if w is a differential form on M'', then $F^\star(G^\star w)$ is obtained by means of the substitutions $z=G(y)$, $y=F(x)$, thus

$$(1.14)\qquad F^\star(G^\star w)=(G\circ F)^\star w.$$

Moreover, we always have $d(F^\star v)=F^\star(dv)$. It follows that the pull-back $F^\star$ is closed if v is closed and exact if v is exact. Therefore $F^\star$ induces a morphism on the quotient spaces

$$(1.15)\qquad F^\star:H^p_{\mathrm{DR}}(M',\mathbb{R})\longrightarrow H^p_{\mathrm{DR}}(M,\mathbb{R}).$$

§1.C. Integration of Differential Forms

A manifold M is *orientable* if and only if there exists an atlas (τ_α) such that all transition maps $\tau_{\alpha\beta}$ preserve the orientation, i.e. have positive jacobian determinants. Suppose that M is oriented, that is, equipped with such an atlas. If $u(x)=f(x_1,\ldots,x_m)\,dx_1\wedge\ldots\wedge dx_m$ is a continuous form of maximum degree $m=\dim_{\mathbb{R}}M$, with compact support in a coordinate open set Ω, we set

$$(1.16)\qquad \int_M u = \int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\,dx_1\ldots dx_m.$$

By the change of variable formula, the result is independent of the choice of coordinates, provided we consider only coordinates corresponding to the given orientation. When u is an arbitrary form with compact support, the definition of $\int_M u$ is easily extended by means of a partition of unity with respect to coordinate open sets covering $\operatorname{Supp} u$. Let $F : M \longrightarrow M'$ be a diffeomorphism between oriented manifolds and v a volume form on M'. The change of variable formula yields

$$(1.17)\qquad \int_M F^\star v = \pm\int_{M'} v$$

according whether F preserves orientation or not.

We now state Stokes' formula, which is basic in many contexts. Let K be a compact subset of M with piecewise C^1 boundary. By this, we mean that for each point $a \in \partial K$ there are coordinates $(x_1,\ldots,x_m)$ on a neighborhood V of a, centered at a, such that

$$K\cap V = \big\{x\in V\,;\,x_1\le 0,\ldots,x_l\le 0\big\}$$

for some index $l \ge 1$. Then $\partial K\cap V$ is a union of smooth hypersurfaces with piecewise C^1 boundaries:

$$\partial K\cap V = \bigcup_{1\le j\le l}\big\{x\in V\,;\,x_1\le 0,\ldots,x_j=0,\ldots,x_l\le 0\big\}.$$

At points of ∂K where $x_j = 0$, then $(x_1,\ldots,\widehat{x_j},,\ldots,x_m)$ define coordinates on ∂K. We take the orientation of ∂K given by these coordinates or the opposite one, according to the sign $(-1)^{j-1}$. For any differential form u of class C^1 and degree $m-1$ on M, we then have

(1.18) Stokes' formula. $$\int_{\partial K} u = \int_K du.$$

The formula is easily checked by an explicit computation when u has compact support in V: indeed if $u = \sum_{1\le j\le n} u_j\,dx_1\wedge\ldots\widehat{dx_j}\ldots dx_m$ and $\partial_j K\cap V$ is the part of $\partial K\cap V$ where $x_j = 0$, a partial integration with respect to x_j yields

$$\begin{aligned}
&\int_{\partial_j K\cap V} u_j\,dx_1\wedge\ldots\widehat{dx_j}\ldots dx_m = \int_V \frac{\partial u_j}{\partial x_j}\,dx_1\wedge\ldots dx_m,\\
&\int_{\partial K\cap V} u = \sum_{1\le j\le m}(-1)^{j-1}\int_{\partial_j K\cap V} u_j\,dx_1\wedge\ldots\widehat{dx_j}\ldots\wedge dx_m = \int_V du.
\end{aligned}$$

The general case follows by a partition of unity. In particular, if u has compact support in M, we find $\int_M du = 0$ by choosing $K\supset \operatorname{Supp} u$.

§1.D. Homotopy Formula and Poincaré Lemma

Let u be a differential form on $[0,1]\times M$. For $(t,x)\in[0,1]\times M$, we write

$$u(t,x)=\sum_{|I|=p}u_I(t,x)\,dx_I+\sum_{|J|=p-1}\widetilde{u}_J(t,x)\,dt\wedge dx_J.$$

We define an operator

$$K:C^s([0,1]\times M,\Lambda^pT^\star_{[0,1]\times M})\longrightarrow C^s(M,\Lambda^{p-1}T^\star_M)$$

$$\text{(1.19)}\quad Ku(x)=\sum_{|J|=p-1}\Big(\int_0^1\widetilde{u}_J(t,x)\,dt\Big)dx_J$$

and say that Ku is the form obtained by integrating u along $[0,1]$. A computation of the operator $dK+Kd$ shows that all terms involving partial derivatives $\partial\widetilde{u}_J/\partial x_k$ cancel, hence

$$Kdu+dKu=\sum_{|I|=p}\Big(\int_0^1\frac{\partial u_I}{\partial t}(t,x)\,dt\Big)dx_I=\sum_{|I|=p}\big(u_I(1,x)-u_I(0,x)\big)dx_I,$$

$$\text{(1.20)}\qquad Kdu+dKu=i_1^\star u-i_0^\star u,$$

where $i_t:M\to[0,1]\times M$ is the injection $x\mapsto(t,x)$.

(1.20) Corollary. *Let $F,G:M\longrightarrow M'$ be C^∞ maps. Suppose that F,G are smoothly homotopic, i.e. that there exists a C^∞ map $H:[0,1]\times M\longrightarrow M'$ such that $H(0,x)=F(x)$ and $H(1,x)=G(x)$. Then*

$$F^\star=G^\star:H^p_{\rm DR}(M',\mathbb{R})\longrightarrow H^p_{\rm DR}(M,\mathbb{R}).$$

Proof. If v is a p-form on M', then

$$\begin{aligned}G^\star v-F^\star v&=(H\circ i_1)^\star v-(H\circ i_0)^\star v=i_1^\star(H^\star v)-i_0^\star(H^\star v)\\&=d(KH^\star v)+KH^\star(dv)\end{aligned}$$

by (1.20) applied to $u=H^\star v$. If v is closed, then $F^\star v$ and $G^\star v$ differ by an exact form, so they define the same class in $H^p_{\rm DR}(M,\mathbb{R})$. □

(1.21) Corollary. *If the manifold M is contractible, i.e. if there is a smooth homotopy $H:[0,1]\times M\to M$ from a constant map $F:M\to\{x_0\}$ to $G=\mathrm{Id}_X$, then $H^0_{\rm DR}(M,\mathbb{R})=\mathbb{R}$ and $H^p_{\rm DR}(M,\mathbb{R})=0$ for $p\ge1$.*

Proof. $F^\star$ is clearly zero in degree $p\ge1$, while $F^\star:H^0_{\rm DR}(M,\mathbb{R})\xrightarrow{\simeq}\mathbb{R}$ is induced by the evaluation map $u\mapsto u(x_0)$. The conclusion then follows from the equality $F^\star=G^\star=\mathrm{Id}$ on cohomology groups. □

(1.22) Poincaré lemma. *Let $\Omega \subset \mathbb{R}^m$ be a starshaped open set. If a form $v = \sum v_I dx_I \in C^s(\Omega, \Lambda^p T^\star_\Omega)$, $p \geq 1$, satisfies $dv = 0$, there exists a form $u \in C^s(\Omega, \Lambda^{p-1} T^\star_\Omega)$ such that $du = v$.*

Proof. Let $H(t,x) = tx$ be the homotopy between the identity map $\Omega \to \Omega$ and the constant map $\Omega \to \{0\}$. By the above formula

$$d(KH^\star v) = G^\star v - F^\star v = \begin{cases} v - v(0) & \text{if } p = 0, \\ v & \text{if } p \geq 1. \end{cases}$$

Hence $u = KH^\star v$ is the $(p-1)$-form we are looking for. An explicit computation based on (1.19) easily gives

$$(1.23) \quad u(x) = \sum_{\substack{|I|=p \\ 1 \leq k \leq p}} \Big(\int_0^1 t^{p-1} v_I(tx)\, dt \Big) (-1)^{k-1} x_{i_k} dx_{i_1} \wedge \ldots \widehat{dx_{i_k}} \ldots \wedge dx_{i_p}.$$

§2. Currents on Differentiable Manifolds

§2.A. Definition and Examples

Let M be a C^∞ differentiable manifold, $m = \dim_{\mathbb{R}} M$. All the manifolds considered in Sect. 2 will be assumed to be oriented. We first introduce a topology on the space of differential forms $C^s(M, \Lambda^p T^\star_M)$. Let $\Omega \subset M$ be a coordinate open set and u a p-form on M, written $u(x) = \sum u_I(x)\, dx_I$ on Ω. To every compact subset $L \subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$(2.1) \quad p^s_L(u) = \sup_{x \in L} \max_{|I|=p, |\alpha| \leq s} |D^\alpha u_I(x)|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ runs over $\mathbb{N}^m$ and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}$ is a derivation of order $|\alpha| = \alpha_1 + \cdots + \alpha_m$. This type of multi-index, which will always be denoted by Greek letters, should not be confused with multi-indices of the type $I = (i_1, \ldots, i_p)$ introduced in Sect. 1.

(2.2) Definition.

a) *We denote by $\mathcal{E}^p(M)$ $\big($resp. ${}^s\mathcal{E}^p(M)\big)$ the space $C^\infty(M, \Lambda^p T^\star_M)$ $\big($resp. the space $C^s(M, \Lambda^p T^\star_M)\big)$, equipped with the topology defined by all seminorms p^s_L when s, L, Ω vary (resp. when L, Ω vary).*

b) *If $K \subset M$ is a compact subset, $\mathcal{D}^p(K)$ will denote the subspace of elements $u \in \mathcal{E}^p(M)$ with support contained in K, together with the induced topology; $\mathcal{D}^p(M)$ will stand for the set of all elements with compact support, i.e. $\mathcal{D}^p(M) := \bigcup_K \mathcal{D}^p(K)$.*

c) *The spaces of C^s-forms ${}^s\mathcal{D}^p(K)$ and ${}^s\mathcal{D}^p(M)$ are defined similarly.*

Since our manifolds are assumed to be separable, the topology of $\mathcal{E}^p(M)$ can be defined by means of a countable set of seminorms p_L^s, hence $\mathcal{E}^p(M)$ (and likewise ${}^s\mathcal{E}^p(M)$) is a Fréchet space. The topology of ${}^s\mathcal{D}^p(K)$ is induced by any finite set of seminorms $p_{K_j}^s$ such that the compact sets K_j cover K ; hence ${}^s\mathcal{D}^p(K)$ is a Banach space. It should be observed however that $\mathcal{D}^p(M)$ is not a Fréchet space; in fact $\mathcal{D}^p(M)$ is dense in $\mathcal{E}^p(M)$ and thus non complete for the induced topology. According to (De Rham 1955) spaces of *currents* are defined as the topological duals of the above spaces, in analogy with the usual definition of distributions.

(2.3) Definition. *The space of currents of dimension p (or degree $m-p$) on M is the space $\mathcal{D}'_p(M)$ of linear forms T on $\mathcal{D}^p(M)$ such that the restriction of T to all subspaces $\mathcal{D}^p(K)$, $K \subset\subset M$, is continuous. The degree is indicated by raising the index, hence we set*

$$\mathcal{D}'^{\,m-p}(M) = \mathcal{D}'_p(M) := \textit{topological dual } \big(\mathcal{D}^p(M)\big)'.$$

The space ${}^s\mathcal{D}'_p(M) = {}^s\mathcal{D}'^{\,m-p}(M) := \big({}^s\mathcal{D}^p(M)\big)'$ is defined similarly and is called the space of currents of order s on M.

In the sequel, we let $\langle T, u\rangle$ be the pairing between a current T and a *test form* $u \in \mathcal{D}^p(M)$. It is clear that ${}^s\mathcal{D}'_p(M)$ can be identified with the subspace of currents $T \in \mathcal{D}'_p(M)$ which are continuous for the seminorm p_K^s on $\mathcal{D}^p(K)$ for every compact set K contained in a coordinate patch Ω. The *support* of T, denoted $\operatorname{Supp} T$, is the smallest closed subset $A \subset M$ such that the restriction of T to $\mathcal{D}^p(M \setminus A)$ is zero. The topological dual $\mathcal{E}'_p(M)$ can be identified with the set of currents of $\mathcal{D}'_p(M)$ with compact support: indeed, let T be a linear form on $\mathcal{E}^p(M)$ such that

$$|\langle T, u\rangle| \le C \max\{p_{K_j}^s(u)\}$$

for some $s \in \mathbb{N}$, $C \ge 0$ and a finite number of compact sets K_j ; it follows that $\operatorname{Supp} T \subset \bigcup K_j$. Conversely let $T \in \mathcal{D}'_p(M)$ with support in a compact set K. Let K_j be compact patches such that K is contained in the interior of $\bigcup K_j$ and $\psi \in \mathcal{D}(M)$ equal to 1 on K with $\operatorname{Supp}\psi \subset \bigcup K_j$. For $u \in \mathcal{E}^p(M)$, we define $\langle T, u\rangle = \langle T, \psi u\rangle$; this is independent of ψ and the resulting T is clearly continuous on $\mathcal{E}^p(M)$. The terminology used for the dimension and degree of a current is justified by the following two examples.

(2.4) Example. Let $Z \subset M$ be a closed oriented submanifold of M of dimension p and class C^1 ; Z may have a boundary ∂Z. The *current of integration* over Z, denoted $[Z]$, is defined by

$$\langle [Z], u\rangle = \int_Z u, \qquad u \in {}^0\mathcal{D}^p(M).$$

It is clear that $[Z]$ is a current of order 0 on M and that $\operatorname{Supp}[Z] = Z$. Its dimension is $p = \dim Z$.

(2.5) Example. If f is a differential form of degree q on M with $L^1_{\rm loc}$ coefficients, we can associate to f the current of dimension $m - q$:

$$\langle T_f, u\rangle = \int_M f \wedge u, \qquad u \in {}^0\mathcal{D}^{m-q}(M).$$

T_f is of degree q and of order 0. The correspondence $f \longmapsto T_f$ is injective. In the same way $L^1_{\rm loc}$ functions on $\mathbb{R}^m$ are identified to distributions, we will identify f with its image $T_f \in {}^0\mathcal{D}'^{\,q}(M) = {}^0\mathcal{D}'_{m-q}(M)$.

§2.B. Exterior Derivative and Wedge Product

§2.B.1. Exterior Derivative. Many of the operations available for differential forms can be extended to currents by simple duality arguments. Let $T \in {}^s\mathcal{D}'^{\,q}(M) = {}^s\mathcal{D}'_{m-p}(M)$. The *exterior derivative*

$$dT \in {}^{s+1}\mathcal{D}'^{\,q+1}(M) = {}^{s+1}\mathcal{D}'_{m-q-1}$$

is defined by

$$\text{(2.6)}\quad \langle dT, u\rangle = (-1)^{q+1}\,\langle T, du\rangle, \qquad u \in {}^{s+1}\mathcal{D}^{m-q-1}(M).$$

The continuity of the linear form dT on ${}^{s+1}\mathcal{D}^{m-q-1}(M)$ follows from the continuity of the map $d : {}^{s+1}\mathcal{D}^{m-q-1}(K) \longrightarrow {}^s\mathcal{D}^{m-q}(K)$. For all forms $f \in {}^1\mathcal{E}^q(M)$ and $u \in \mathcal{D}^{m-q-1}(M)$, Stokes' formula implies

$$0 = \int_M d(f \wedge u) = \int_M df \wedge u + (-1)^q\, f \wedge du,$$

thus in example (2.5) one actually has $dT_f = T_{df}$ as it should be. In example (2.4), another application of Stokes' formula yields $\int_Z du = \int_{\partial Z} u$, therefore $\langle [Z], du\rangle = \langle [\partial Z], u\rangle$ and

$$\text{(2.7)}\quad d[Z] = (-1)^{m-p+1}[\partial Z].$$

§2.B.2. Wedge Product. For $T \in {}^s\mathcal{D}'^{\,q}(M)$ and $g \in {}^s\mathcal{E}^r(M)$, the wedge product $T \wedge g \in {}^s\mathcal{D}'^{\,q+r}(M)$ is defined by

$$\text{(2.8)}\quad \langle T \wedge g, u\rangle = \langle T, g \wedge u\rangle, \qquad u \in {}^s\mathcal{D}^{m-q-r}(M).$$

This definition is licit because $u \mapsto g \wedge u$ is continuous in the C^s-topology. The relation

$$d(T \wedge g) = dT \wedge g + (-1)^{\deg T}\, T \wedge dg$$

is easily verified from the definitions.

(2.9) Proposition. *Let $(x_1, \ldots, x_m)$ be a coordinate system on an open subset $\Omega \subset M$. Every current $T \in {}^s\mathcal{D}'^q(M)$ of degree q can be written in a unique way*

$$T = \sum_{|I|=q} T_I \, dx_I \quad \text{on} \quad \Omega,$$

where T_I are distributions of order s on Ω, considered as currents of degree 0.

Proof. If the result is true, for all $f \in {}^s\mathcal{D}^0(\Omega)$ we must have

$$\langle T, f \, dx_{\complement I} \rangle = \langle T_I, dx_I \wedge f \, dx_{\complement I} \rangle = \varepsilon(I, \complement I) \, \langle T_I, f \, dx_1 \wedge \ldots \wedge dx_m \rangle,$$

where $\varepsilon(I, \complement I)$ is the signature of the permutation $(1, \ldots, m) \longmapsto (I, \complement I)$. Conversely, this can be taken as a definition of the coefficient T_I:

$$(2.10) \quad T_I(f) = \langle T_I, f \, dx_1 \wedge \ldots \wedge dx_m \rangle := \varepsilon(I, \complement I) \, \langle T, f \, dx_{\complement I} \rangle, \quad f \in {}^s\mathcal{D}^0(\Omega).$$

Then T_I is a distribution of order s and it is easy to check that $T = \sum T_I \, dx_I$. □

In particular, currents of order 0 on M can be considered as differential forms with measure coefficients. In order to unify the notations concerning forms and currents, we set

$$\langle T, u \rangle = \int_M T \wedge u$$

whenever $T \in {}^s\mathcal{D}'_p(M) = {}^s\mathcal{D}'^{m-p}(M)$ and $u \in {}^s\mathcal{E}^p(M)$ are such that $\operatorname{Supp} T \cap \operatorname{Supp} u$ is compact. This convention is made so that the notation becomes compatible with the identification of a form f to the current T_f.

§2.C. Direct and Inverse Images

§2.C.1. Direct Images. Assume now that M_1, M_2 are oriented differentiable manifolds of respective dimensions m_1, m_2, and that

$$(2.11) \quad F : M_1 \longrightarrow M_2$$

is a C^∞ map. The pull-back morphism

$$(2.12) \quad {}^s\mathcal{D}^p(M_2) \longrightarrow {}^s\mathcal{E}^p(M_1), \qquad u \longmapsto F^\star u$$

is continuous in the C^s topology and we have $\operatorname{Supp} F^\star u \subset F^{-1}(\operatorname{Supp} u)$, but in general $\operatorname{Supp} F^\star u$ is not compact. If $T \in {}^s\mathcal{D}'_p(M_1)$ is such that the restriction of F to $\operatorname{Supp} T$ is *proper*, i.e. if $\operatorname{Supp} T \cap F^{-1}(K)$ is compact for every compact subset $K \subset M_2$, then the linear form $u \longmapsto \langle T, F^\star u \rangle$ is well defined and continuous on ${}^s\mathcal{D}^p(M_2)$. There exists therefore a unique current denoted $F_\star T \in {}^s\mathcal{D}'_p(M_2)$, called *the direct image* of T by F, such that

$$(2.13)\quad \langle F_\star T, u\rangle = \langle T, F^\star u\rangle, \qquad \forall u \in {}^s\mathcal{D}^p(M_2).$$

We leave the straightforward proof of the following properties to the reader.

(2.14) Theorem. *For every $T \in {}^s\mathcal{D}'_p(M_1)$ such that $F_{\restriction \operatorname{Supp} T}$ is proper, the direct image $F_\star T \in {}^s\mathcal{D}'_p(M_2)$ is such that*

a) $\operatorname{Supp} F_\star T \subset F(\operatorname{Supp} T)$;

b) $d(F_\star T) = F_\star(dT)$;

c) $F_\star(T \wedge F^\star g) = (F_\star T) \wedge g, \quad \forall g \in {}^s\mathcal{E}^q(M_2, \mathbb{R})$;

d) *If $G : M_2 \longrightarrow M_3$ is a C^∞ map such that $(G \circ F)_{\restriction \operatorname{Supp} T}$ is proper, then*

$$G_\star(F_\star T) = (G \circ F)_\star T.$$

Fig. I-2 Local description of a submersion as a projection.

(2.15) Special case. Assume that F is a submersion, i.e. that F is surjective and that for every $x \in M_1$ the differential map $d_xF : T_{M_1,x} \longrightarrow T_{M_2,F(x)}$ is surjective. Let g be a differential form of degree q on M_1, with L^1_{loc} coefficients, such that $F_{\restriction \operatorname{Supp} g}$ is proper. We claim that $F_\star g \in {}^0\mathcal{D}'_{m_1-q}(M_2)$ is the form of degree $q - (m_1 - m_2)$ obtained from g by integration along the fibers of F, also denoted

$$F_\star g(y) = \int_{z \in F^{-1}(y)} g(z).$$

In fact, this assertion is equivalent to the following generalized form of Fubini's theorem:

$$\int_{M_1} g \wedge F^\star u = \int_{y\in M_2} \Big(\int_{z\in F^{-1}(y)} g(z) \Big) \wedge u(y), \qquad \forall u \in {}^0\mathcal{D}^{m_1-q}(M_2).$$

By using a partition of unity on M_1 and the constant rank theorem, the verification of this formula is easily reduced to the case where $M_1 = A \times M_2$ and $F = \mathrm{pr}_2$, cf. Fig. 2. The fibers $F^{-1}(y) \simeq A$ have to be oriented in such a way that the orientation of M_1 is the product of the orientation of A and M_2. Let us write $r = \dim A = m_1 - m_2$ and let $z = (x,y) \in A \times M_2$ be any point of M_1. The above formula becomes

$$\int_{A\times M_2} g(x,y) \wedge u(y) = \int_{y\in M_2} \Big(\int_{x\in A} g(x,y) \Big) \wedge u(y),$$

where the direct image of g is computed from $g = \sum g_{I,J}(x,y)\, dx_I \wedge dy_J$, $|I| + |J| = q$, by the formula

$$\begin{aligned} \text{(2.16)} \qquad F_\star g(y) &= \int_{x\in A} g(x,y) \\ &= \sum_{|J|=q-r} \Big(\int_{x\in A} g_{(1,\ldots,r),J}(x,y)\, dx_1 \wedge \ldots \wedge dx_r \Big) dy_J. \end{aligned}$$

In this situation, we see that $F_\star g$ has L^1_{loc} coefficients on M_2 if g is L^1_{loc} on M_1, and that the map $g \longmapsto F_\star g$ is continuous in the C^s topology.

(2.17) Remark. If $F : M_1 \longrightarrow M_2$ is a diffeomorphism, then we have $F_\star g = \pm (F^{-1})^\star g$ according whether F preserves the orientation or not. In fact formula (1.17) gives

$$\langle F_\star g, u\rangle = \int_{M_1} g \wedge F^\star u = \pm \int_{M_2} (F^{-1})^\star (g \wedge F^\star u) = \pm \int_{M_2} (F^{-1})^\star g \wedge u.$$

§**2.C.2. Inverse Images.** Assume that $F : M_1 \longrightarrow M_2$ is a submersion. As a consequence of the continuity statement after (2.16), one can always define the inverse image $F^\star T \in {}^s\mathcal{D}'^q(M_1)$ of a current $T \in {}^s\mathcal{D}'^q(M_2)$ by

$$\langle F^\star T, u\rangle = \langle T, F_\star u\rangle, \qquad u \in {}^s\mathcal{D}^{q+m_1-m_2}(M_1).$$

Then $\dim F^\star T = \dim T + m_1 - m_2$ and Th. 2.14 yields the formulas:

$$\text{(2.18)} \quad d(F^\star T) = F^\star(dT), \qquad F^\star(T\wedge g) = F^\star T \wedge F^\star g, \qquad \forall g \in {}^s\mathcal{D}^\bullet(M_2).$$

Take in particular $T = [Z]$, where Z is an oriented C^1-submanifold of M_2. Then $F^{-1}(Z)$ is a submanifold of M_1 and has a natural orientation given by the isomorphism

$$T_{M_1,x}/T_{F^{-1}(Z),x} \longrightarrow T_{M_2,F(x)}/T_{Z,F(x)},$$

induced by d_xF at every point $x \in Z$. We claim that

(2.19) $F^\star[Z] = [F^{-1}(Z)]$.

Indeed, we have to check that $\int_Z F_\star u = \int_{F^{-1}(Z)} u$ for every $u \in {}^s\mathcal{D}^\bullet(M_1)$. By using a partition of unity on M_1, we may again assume $M_1 = A \times M_2$ and $F = \mathrm{pr}_2$. The above equality can be written

$$\int_{y\in Z} F_\star u(y) = \int_{(x,y)\in A\times Z} u(x,y).$$

This follows precisely from (2.16) and Fubini's theorem.

§**2.C.3. Weak Topology.** The weak topology on $\mathcal{D}'_p(M)$ is the topology defined by the collection of seminorms $T \longmapsto |\langle T, f\rangle|$ for all $f \in \mathcal{D}^p(M)$. With respect to the weak topology, all the operations

(2.20) $T \longmapsto dT, \qquad T \longmapsto T\wedge g, \qquad T \longmapsto F_\star T, \qquad T \longmapsto F^\star T$

defined above are continuous. A set $B \subset \mathcal{D}'_p(M)$ is bounded for the weak topology (weakly bounded for short) if and only if $\langle T, f\rangle$ is bounded when T runs over B, for every fixed $f \in \mathcal{D}^p(M)$. The standard Banach-Alaoglu theorem implies that every weakly bounded closed subset $B \subset \mathcal{D}'_p(M)$ is weakly compact.

§2.D. Tensor Products, Homotopies and Poincaré Lemma

§**2.D.1. Tensor Products.** If S, T are currents on manifolds M, M' there exists a unique current on $M \times M'$, denoted $S \otimes T$ and defined in a way analogous to the tensor product of distributions, such that for all $u \in \mathcal{D}^\bullet(M)$ and $v \in \mathcal{D}^\bullet(M')$

(2.21) $\langle S\otimes T, \mathrm{pr}_1^\star u \wedge \mathrm{pr}_2^\star v\rangle = (-1)^{\deg T \deg u}\langle S, u\rangle\,\langle T, v\rangle$.

One verifies easily that $d(S\otimes T) = dS\otimes T + (-1)^{\deg S} S \otimes dT$.

§**2.D.2. Homotopy Formula.** Assume that $H : [0,1]\times M_1 \longrightarrow M_2$ is a C^∞ homotopy from $F(x) = H(0,x)$ to $G(x) = H(1,x)$ and that $T \in \mathcal{D}'_\bullet(M_1)$ is a current such that $H_{\restriction [0,1]\times \mathrm{Supp}\, T}$ is proper. If $[0,1]$ is considered as the current of degree 0 on $\mathbb{R}$ associated to its characteristic function, we find $d[0,1] = \delta_0 - \delta_1$, thus

$$\begin{aligned} d\big(H_\star([0,1]\otimes T)\big) &= H_\star(\delta_0\otimes T - \delta_1 \otimes T + [0,1]\otimes dT) \\ &= F_\star T - G_\star T + H_\star([0,1]\otimes dT). \end{aligned}$$

Therefore we obtain the *homotopy formula*

(2.22) $F_\star T - G_\star T = d\big(H_\star([0,1]\otimes T)\big) - H_\star([0,1]\otimes dT)$.

When T is closed, i.e. $dT = 0$, we see that $F_\star T$ and $G_\star T$ are cohomologous on M_2, i.e. they differ by an exact current dS.

§2.D.3. Regularization of Currents. Let $\rho \in C^\infty(\mathbb{R}^m)$ be a function with support in $B(0,1)$, such that $\rho(x)$ depends only on $|x| = (\sum |x_i|^2)^{1/2}$, $\rho \geq 0$ and $\int_{\mathbb{R}^m} \rho(x)\,dx = 1$. We associate to ρ the family of functions (ρ_ε) such that

$$(2.23)\quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^m}\,\rho\Big(\frac{x}{\varepsilon}\Big), \qquad \operatorname{Supp}\rho_\varepsilon \subset B(0,\varepsilon), \qquad \int_{\mathbb{R}^m} \rho_\varepsilon(x)\,dx = 1.$$

We shall refer to this construction by saying that (ρ_ε) is a *family of smoothing kernels*. For every current $T = \sum T_I\,dx_I$ on an open subset $\Omega \subset \mathbb{R}^m$, the family of smooth forms

$$T \star \rho_\varepsilon = \sum_I (T_I \star \rho_\varepsilon)\,dx_I,$$

defined on $\Omega_\varepsilon = \{x \in \mathbb{R}^m\ ;\ d(x, \complement\Omega) > \varepsilon\}$, converges weakly to T as ε tends to 0. Indeed, $\langle T \star \rho_\varepsilon, f\rangle = \langle T, \rho_\varepsilon \star f\rangle$ and $\rho_\varepsilon \star f$ converges to f in $\mathcal{D}^p(\Omega)$ with respect to all seminorms p_K^s.

§2.D.4. Poincaré Lemma for Currents. Let $T \in {}^s\mathcal{D}'^{\,q}(\Omega)$ be a closed current on an open set $\Omega \subset \mathbb{R}^m$. We first show that T is cohomologous to a smooth form. In fact, let $\psi \in C^\infty(\mathbb{R}^m)$ be a cut-off function such that $\operatorname{Supp}\psi \subset \overline{\Omega}$, $0 < \psi \leq 1$ and $|d\psi| \leq 1$ on Ω. For any vector $v \in B(0,1)$ we set

$$F_v(x) = x + \psi(x)v.$$

Since $x \mapsto \psi(x)v$ is a contraction, F_v is a diffeomorphism of $\mathbb{R}^m$ which leaves $\complement\Omega$ invariant pointwise, so $F_v(\Omega) = \Omega$. This diffeomorphism is homotopic to the identity through the homotopy $H_v(t,x) = F_{tv}(x) : [0,1] \times \Omega \longrightarrow \Omega$ which is proper for every v. Formula (2.22) implies

$$(F_v)_\star T - T = d\big((H_v)_\star([0,1] \otimes T)\big).$$

After averaging with a smoothing kernel $\rho_\varepsilon(v)$ we get $\Theta - T = dS$ where

$$\Theta = \int_{B(0,\varepsilon)} (F_v)_\star T\,\rho_\varepsilon(v)\,dv, \qquad S = \int_{B(0,\varepsilon)} (H_v)_\star([0,1] \otimes T)\,\rho_\varepsilon(v)\,dv.$$

Then S is a current of the same order s as T and Θ is smooth. Indeed, for $u \in \mathcal{D}^p(\Omega)$ we have

$$\langle \Theta, u\rangle = \langle T, u_\varepsilon\rangle \quad \text{where} \quad u_\varepsilon(x) = \int_{B(0,\varepsilon)} F_v^\star u(x)\,\rho_\varepsilon(v)\,dv\ ;$$

we can make a change of variable $z = F_v(x) \Leftrightarrow v = \psi(x)^{-1}(z - x)$ in the last integral and perform derivatives on ρ_ε to see that each seminorm $p_K^t(u_\varepsilon)$ is controlled by the sup norm of u. Thus Θ and all its derivatives are currents of order 0, so Θ is smooth. Now we have $d\Theta = 0$ and by the usual Poincaré lemma (1.22) applied to Θ we obtain

(2.24) Theorem. *Let $\Omega \subset \mathbb{R}^m$ be a starshaped open subset and $T \in {}^s\mathcal{D}'^{\,q}(\Omega)$ a current of degree $q \geq 1$ and order s such that $dT = 0$. There exists a current $S \in {}^s\mathcal{D}'^{\,q-1}(\Omega)$ of degree $q-1$ and order $\leq s$ such that $dS = T$ on Ω.* □

§3. Holomorphic Functions and Complex Manifolds

§3.A. Cauchy Formula in One Variable

We start by recalling a few elementary facts in one complex variable theory. Let $\Omega \subset \mathbb{C}$ be an open set and let $z = x + \mathrm{i}y$ be the complex variable, where $x, y \in \mathbb{R}$. If f is a function of class C^1 on Ω, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z}$$

with the usual notations

$$(3.1)\qquad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big).$$

The function f is holomorphic on Ω if df is $\mathbb{C}$-linear, that is, $\partial f/\partial\overline{z} = 0$.

(3.2) Cauchy formula. *Let $K \subset \mathbb{C}$ be a compact set with piecewise C^1 boundary ∂K. Then for every $f \in C^1(K, \mathbb{C})$*

$$f(w) = \frac{1}{2\pi\mathrm{i}}\int_{\partial K}\frac{f(z)}{z-w}\,dz - \int_K \frac{1}{\pi(z-w)}\,\frac{\partial f}{\partial \overline{z}}\,d\lambda(z), \qquad w \in K^\circ$$

where $d\lambda(z) = \frac{\mathrm{i}}{2}dz \wedge d\overline{z} = dx \wedge dy$ is the Lebesgue measure on $\mathbb{C}$.

Proof. Assume for simplicity $w = 0$. As the function $z \mapsto 1/z$ is locally integrable at $z = 0$, we get

$$\begin{aligned}
\int_K \frac{1}{\pi z}\,\frac{\partial f}{\partial \overline{z}}\,d\lambda(z) &= \lim_{\varepsilon\to 0}\int_{K\smallsetminus D(0,\varepsilon)} \frac{1}{\pi z}\,\frac{\partial f}{\partial \overline{z}}\,\frac{\mathrm{i}}{2}dz \wedge d\overline{z} \\
&= \lim_{\varepsilon\to 0}\int_{K\smallsetminus D(0,\varepsilon)} d\Big[\frac{1}{2\pi\mathrm{i}}\,f(z)\,\frac{dz}{z}\Big] \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\partial K} f(z)\,\frac{dz}{z} - \lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_{\partial D(0,\varepsilon)} f(z)\,\frac{dz}{z}
\end{aligned}$$

by Stokes' formula. The last integral is equal to $\frac{1}{2\pi}\int_0^{2\pi} f(\varepsilon e^{\mathrm{i}\theta})\,d\theta$ and converges to $f(0)$ as ε tends to 0. □

When f is holomorphic on Ω, we get the usual Cauchy formula

(3.3) $$f(w) = \frac{1}{2\pi\mathrm{i}} \int_{\partial K} \frac{f(z)}{z-w}\, dz, \qquad w \in K^\circ,$$

from which many basic properties of holomorphic functions can be derived: power and Laurent series expansions, Cauchy residue formula, ... Another interesting consequence is:

(3.4) Corollary. *The L^1_{loc} function $E(z) = 1/\pi z$ is a fundamental solution of the operator $\partial/\partial\overline{z}$ on $\mathbb{C}$, i.e. $\partial E/\partial\overline{z} = \delta_0$ (Dirac measure at 0). As a consequence, if v is a distribution with compact support in $\mathbb{C}$, then the convolution $u = (1/\pi z) \star v$ is a solution of the equation $\partial u/\partial\overline{z} = v$.*

Proof. Apply (3.2) with $w = 0$, $f \in \mathcal{D}(\mathbb{C})$ and $K \supset \operatorname{Supp} f$, so that $f = 0$ on the boundary ∂K and $f(0) = \langle 1/\pi z, -\partial f/\partial\overline{z}\rangle$. □

(3.5) Remark. It should be observed that this formula cannot be used to solve the equation $\partial u/\partial\overline{z} = v$ when $\operatorname{Supp} v$ is not compact; moreover, if $\operatorname{Supp} v$ is compact, a solution u with compact support need not always exist. Indeed, we have a necessary condition

$$\langle v, z^n\rangle = -\langle u, \partial z^n/\partial\overline{z}\rangle = 0$$

for all integers $n \geq 0$. Conversely, when the necessary condition $\langle v, z^n\rangle = 0$ is satisfied, the canonical solution $u = (1/\pi z) \star v$ has compact support: this is easily seen by means of the power series expansion $(w-z)^{-1} = \sum z^n w^{-n-1}$, if we suppose that $\operatorname{Supp} v$ is contained in the disk $|z| < R$ and that $|w| > R$.

§3.B. Holomorphic Functions of Several Variables

Let $\Omega \subset \mathbb{C}^n$ be an open set. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if f is continuous and separately holomorphic with respect to each variable, i.e. $z_j \mapsto f(\ldots, z_j, \ldots)$ is holomorphic when $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ are fixed. The set of holomorphic functions on Ω is a ring and will be denoted $\mathcal{O}(\Omega)$. We first extend the Cauchy formula to the case of polydisks. The open polydisk $D(z_0, R)$ of center $(z_{0,1}, \ldots, z_{0,n})$ and (multi)radius $R = (R_1, \ldots, R_n)$ is defined as the product of the disks of center $z_{0,j}$ and radius $R_j > 0$ in each factor $\mathbb{C}$:

(3.6) $$D(z_0, R) = D(z_{0,1}, R_1) \times \ldots \times D(z_{0,n}, R_n) \subset \mathbb{C}^n.$$

The *distinguished boundary* of $D(z_0, R)$ is by definition the product of the boundary circles

(3.7) $$\Gamma(z_0, R) = \Gamma(z_{0,1}, R_1) \times \ldots \times \Gamma(z_{0,n}, R_n).$$

It is important to observe that the distinguished boundary is smaller than the topological boundary $\partial D(z_0, R) = \bigcup_j \{z \in \overline{D}(z_0, R)\,;\, |z_j - z_{0,j}| = R_j\}$ when $n \geq 2$. By induction on n, we easily get the

(3.8) Cauchy formula on polydisks. *If $\overline{D}(z_0, R)$ is a closed polydisk contained in Ω and $f \in \mathcal{O}(\Omega)$, then for all $w \in D(z_0, R)$ we have*

$$f(w) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\Gamma(z_0,R)} \frac{f(z_1, \ldots, z_n)}{(z_1 - w_1) \ldots (z_n - w_n)} \, dz_1 \ldots dz_n. \qquad \square$$

The expansion $(z_j - w_j)^{-1} = \sum (w_j - z_{0,j})^{\alpha_j} (z_j - z_{0,j})^{-\alpha_j - 1}$, $\alpha_j \in \mathbb{N}$, $1 \le j \le n$, shows that f can be expanded as a convergent power series $f(w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (w - z_0)^\alpha$ over the polydisk $D(z_0, R)$, with the standard notations $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and with

$$(3.9) \quad a_\alpha = \frac{1}{(2\pi \mathrm{i})^n} \int_{\Gamma(z_0,R)} \frac{f(z_1, \ldots, z_n) \, dz_1 \ldots dz_n}{(z_1 - z_{0,1})^{\alpha_1 + 1} \ldots (z_n - z_{0,n})^{\alpha_n + 1}} = \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

As a consequence, f is holomorphic over Ω if and only if f is $\mathbb{C}$-analytic. Arguments similar to the one variable case easily yield the

(3.10) Analytic continuation theorem. *If Ω is connected and if there exists a point $z_0 \in \Omega$ such that $f^{(\alpha)}(z_0) = 0$ for all $\alpha \in \mathbb{N}^n$, then $f = 0$ on Ω.* $\square$

Another consequence of (3.9) is the *Cauchy inequality*

$$(3.11) \quad |f^{(\alpha)}(z_0)| \le \frac{\alpha!}{R^\alpha} \sup_{\Gamma(z_0,R)} |f|, \qquad \overline{D}(z_0, R) \subset \Omega,$$

From this, it follows that every bounded holomorphic function on $\mathbb{C}^n$ is constant (Liouville's theorem), and more generally, every holomorphic function F on $\mathbb{C}^n$ such that $|F(z)| \le A(1 + |z|)^B$ with suitable constants $A, B \ge 0$ is in fact a polynomial of total degree $\le B$.

We endow $\mathcal{O}(\Omega)$ with the topology of uniform convergence on compact sets $K \subset\subset \Omega$, that is, the topology induced by $C^0(\Omega, \mathbb{C})$. Then $\mathcal{O}(\Omega)$ is closed in $C^0(\Omega, \mathbb{C})$. The Cauchy inequalities (3.11) show that all derivations D^α are continuous operators on $\mathcal{O}(\Omega)$ and that any sequence $f_j \in \mathcal{O}(\Omega)$ that is uniformly bounded on all compact sets $K \subset\subset \Omega$ is locally equicontinuous. By Ascoli's theorem, we obtain

(3.12) Montel's theorem. *Every locally uniformly bounded sequence (f_j) in $\mathcal{O}(\Omega)$ has a convergent subsequence $(f_{j(\nu)})$.*

In other words, bounded subsets of the Fréchet space $\mathcal{O}(\Omega)$ are relatively compact (a Fréchet space possessing this property is called a Montel space).

§3.C. Differential Calculus on Complex Analytic Manifolds

A *complex analytic manifold* X of dimension $\dim_{\mathbb{C}} X = n$ is a differentiable manifold equipped with a holomorphic atlas (τ_α) with values in $\mathbb{C}^n$; this means by definition that the transition maps $\tau_{\alpha\beta}$ are holomorphic. The tangent spaces $T_{X,x}$ then have a natural complex vector space structure, given by the coordinate isomorphisms

$$d\tau_\alpha(x) : T_{X,x} \longrightarrow \mathbb{C}^n, \qquad U_\alpha \ni x\,;$$

the induced complex structure on $T_{X,x}$ is indeed independent of α since the differentials $d\tau_{\alpha\beta}$ are $\mathbb{C}$-linear isomorphisms. We denote by $T_X^{\mathbb{R}}$ the underlying real tangent space and by $J \in \mathrm{End}(T_X^{\mathbb{R}})$ the *almost complex structure*, i.e. the operator of multiplication by $\mathrm{i} = \sqrt{-1}$. If $(z_1, \ldots, z_n)$ are complex analytic coordinates on an open subset $\Omega \subset X$ and $z_k = x_k + \mathrm{i}y_k$, then $(x_1, y_1, \ldots, x_n, y_n)$ define real coordinates on Ω, and $T^{\mathbb{R}}_{X\restriction\Omega}$ admits $(\partial/\partial x_1$, $\partial/\partial y_1$, …, $\partial/\partial x_n$, $\partial/\partial y_n)$ as a basis; the almost complex structure is given by $J(\partial/\partial x_k) = \partial/\partial y_k$, $J(\partial/\partial y_k) = -\partial/\partial x_k$. The complexified tangent space $\mathbb{C}\otimes T_X = \mathbb{C}\otimes_{\mathbb{R}} T_X^{\mathbb{R}} = T_X^{\mathbb{R}} \oplus \mathrm{i}T_X^{\mathbb{R}}$ splits into conjugate complex subspaces which are the eigenspaces of the complexified endomorphism $\mathrm{Id}\otimes J$ associated to the eigenvalues i and $-$i. These subspaces have respective bases

$$(3.13)\qquad \frac{\partial}{\partial z_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} - \mathrm{i}\frac{\partial}{\partial y_k}\Big), \qquad \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + \mathrm{i}\frac{\partial}{\partial y_k}\Big), \qquad 1 \le k \le n$$

and are denoted $T^{1,0}X$ (*holomorphic vectors* or *vectors of type* $(1,0)$) and $T^{0,1}X$ (*antiholomorphic vectors* or *vectors of type* $(0,1)$). The subspaces $T^{1,0}X$ and $T^{0,1}X$ are canonically isomorphic to the complex tangent space T_X (with complex structure J) and its conjugate $\overline{T_X}$ (with conjugate complex structure $-J$), via the $\mathbb{C}$-linear embeddings

$$\begin{aligned} T_X &\longrightarrow T_X^{1,0} \subset \mathbb{C}\otimes T_X, & \overline{T_X} &\longrightarrow T_X^{0,1} \subset \mathbb{C}\otimes T_X \\ \xi &\longmapsto \tfrac{1}{2}(\xi - \mathrm{i}J\xi), & \xi &\longmapsto \tfrac{1}{2}(\xi + \mathrm{i}J\xi). \end{aligned}$$

We thus have a canonical decomposition $\mathbb{C}\otimes T_X = T_X^{1,0} \oplus T_X^{0,1} \simeq T_X \oplus \overline{T_X}$, and by duality a decomposition

$$\mathrm{Hom}_{\mathbb{R}}(T_X^{\mathbb{R}}; \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes T_X; \mathbb{C}) \simeq T_X^\star \oplus \overline{T_X^\star}$$

where $T_X^\star$ is the space of $\mathbb{C}$-linear forms and $\overline{T_X^\star}$ the space of conjugate $\mathbb{C}$-linear forms. With these notations, (dx_k, dy_k) is a basis of $\mathrm{Hom}_{\mathbb{R}}(T_{\mathbb{R}}X, \mathbb{C})$, (dz_j) a basis of $T_X^\star$, $(d\overline{z}_j)$ a basis of $\overline{T_X^\star}$, and the differential of a function $f \in C^1(\Omega, \mathbb{C})$ can be written

$$(3.14)\qquad df = \sum_{k=1}^n \frac{\partial f}{\partial x_k}\,dx_k + \frac{\partial f}{\partial y_k}\,dy_k = \sum_{k=1}^n \frac{\partial f}{\partial z_k}\,dz_k + \frac{\partial f}{\partial \overline{z}_k}\,d\overline{z}_k.$$

The function f is holomorphic on Ω if and only if df is $\mathbb{C}$-linear, i.e. if and only if f satisfies the *Cauchy-Riemann equations* $\partial f/\partial\overline{z}_k = 0$ on Ω, $1 \le k \le n$. We still denote here by $\mathcal{O}(X)$ the algebra of holomorphic functions on X.

Now, we study the basic rules of complex differential calculus. The complexified exterior algebra $\mathbb{C} \otimes_{\mathbb{R}} \Lambda^\bullet_{\mathbb{R}}(T^{\mathbb{R}}_X)^\star = \Lambda^\bullet_{\mathbb{C}}(\mathbb{C} \otimes T_X)^\star$ is given by

$$\Lambda^k(\mathbb{C} \otimes T_X)^\star = \Lambda^k\big(T_X \oplus \overline{T_X}\big)^\star = \bigoplus_{p+q=k} \Lambda^{p,q}T^\star_X, \qquad 0 \le k \le 2n$$

where the exterior products are taken over $\mathbb{C}$, and where the components $\Lambda^{p,q}T^\star_X$ are defined by

$$\Lambda^{p,q}T^\star_X = \Lambda^p T^\star_X \otimes \Lambda^q \overline{T^\star_X}. \tag{3.15}$$

A complex differential form u on X is said to be of *bidegree* or *type* (p,q) if its value at every point lies in the component $\Lambda^{p,q}T^\star_X$; we shall denote by $C^s(\Omega, \Lambda^{p,q}T^\star_X)$ the space of differential forms of bidegree (p,q) and class C^s on any open subset Ω of X. If Ω is a coordinate open set, such a form can be written

$$u(z) = \sum_{|I|=p,|J|=q} u_{I,J}(z)\, dz_I \wedge d\overline{z}_J, \qquad u_{I,J} \in C^s(\Omega, \mathbb{C}).$$

This writing is usually much more convenient than the expression in terms of the real basis $(dx_I \wedge dy_J)_{|I|+|J|=k}$ which is not compatible with the splitting of $\Lambda^k T^\star_{\mathbb{C}} X$ in its (p,q) components. Formula (3.14) shows that the exterior derivative d splits into $d = d' + d''$, where

$$d' : C^\infty(X, \Lambda^{p,q}T^\star_X) \longrightarrow C^\infty(X, \Lambda^{p+1,q}T^\star_X),$$
$$d'' : C^\infty(X, \Lambda^{p,q}T^\star_X) \longrightarrow C^\infty(X, \Lambda^{p,q+1}T^\star_X),$$

$$d'u = \sum_{I,J} \sum_{1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k}\, dz_k \wedge dz_I \wedge d\overline{z}_J, \tag{3.16$'$}$$

$$d''u = \sum_{I,J} \sum_{1\le k\le n.} \frac{\partial u_{I,J}}{\partial \overline{z}_k}\, d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J. \tag{3.16$''$}$$

The identity $d^2 = (d' + d'')^2 = 0$ is equivalent to

$$d'^2 = 0, \qquad d'd'' + d''d' = 0, \qquad d''^2 = 0, \tag{3.17}$$

since these three operators send (p,q)-forms in $(p+2,q)$, $(p+1,q+1)$ and $(p,q+2)$-forms, respectively. In particular, the operator d'' defines for each $p = 0, 1, \ldots, n$ a complex, called the *Dolbeault complex*

$$C^\infty(X, \Lambda^{p,0}T^\star_X) \xrightarrow{d''} \cdots \longrightarrow C^\infty(X, \Lambda^{p,q}T^\star_X) \xrightarrow{d''} C^\infty(X, \Lambda^{p,q+1}T^\star_X)$$

and corresponding *Dolbeault cohomology groups*

$$(3.18)\quad H^{p,q}(X,\mathbb{C})=\frac{\operatorname{Ker} d''^{\,p,q}}{\operatorname{Im} d''^{\,p,q-1}},$$

with the convention that the image of d'' is zero for $q=0$. The cohomology group $H^{p,0}(X,\mathbb{C})$ consists of $(p,0)$-forms $u=\sum_{|I|=p}u_I(z)\,dz_I$ such that $\partial u_I/\partial\overline{z}_k=0$ for all I,k, i.e. such that all coefficients u_I are holomorphic. Such a form is called a *holomorphic p-form* on X.

Let $F:X_1\longrightarrow X_2$ be a holomorphic map between complex manifolds. The pull-back $F^\star u$ of a (p,q)-form u of bidegree (p,q) on X_2 is again homogeneous of bidegree (p,q), because the components F_k of F in any coordinate chart are holomorphic, hence $F^\star dz_k=dF_k$ is $\mathbb{C}$-linear. In particular, the equality $dF^\star u=F^\star du$ implies

$$(3.19)\quad d'F^\star u=F^\star d'u,\qquad d''F^\star u=F^\star d''u.$$

Note that these commutation relations are no longer true for a non holomorphic change of variable. As in the case of the De Rham cohomology groups, we get a pull-back morphism

$$F^\star : H^{p,q}(X_2,\mathbb{C})\longrightarrow H^{p,q}(X_1,\mathbb{C}).$$

The rules of complex differential calculus can be easily extended to currents. We use the following notation.

(3.20) Definition. *There are decompositions*

$$\mathcal{D}^k(X,\mathbb{C})=\bigoplus_{p+q=k}\mathcal{D}^{p,q}(X,\mathbb{C}),\qquad \mathcal{D}'_k(X,\mathbb{C})=\bigoplus_{p+q=k}\mathcal{D}'_{p,q}(X,\mathbb{C}).$$

The space $\mathcal{D}'_{p,q}(X,\mathbb{C})$ is called the space of currents of bidimension (p,q) and bidegree $(n-p,n-q)$ on X, and is also denoted $\mathcal{D}'^{\,n-p,n-q}(X,\mathbb{C})$.

§3.D. Newton and Bochner-Martinelli Kernels

The *Newton kernel* is the elementary solution of the usual Laplace operator $\Delta=\sum\partial^2/\partial x_j^2$ in $\mathbb{R}^m$. We first recall a construction of the Newton kernel.

Let $d\lambda=dx_1\ldots dx_m$ be the Lebesgue measure on $\mathbb{R}^m$. We denote by $B(a,r)$ the euclidean open ball of center a and radius r in $\mathbb{R}^m$ and by $S(a,r)=\partial B(a,r)$ the corresponding sphere. Finally, we set $\alpha_m=\operatorname{Vol}\big(B(0,1)\big)$ and $\sigma_{m-1}=m\alpha_m$ so that

$$(3.21)\quad \operatorname{Vol}\big(B(a,r)\big)=\alpha_m r^m,\qquad \operatorname{Area}\big(S(a,r)\big)=\sigma_{m-1}r^{m-1}.$$

The second equality follows from the first by derivation. An explicit computation of the integral $\int_{\mathbb{R}^m}e^{-|x|^2}d\lambda(x)$ in polar coordinates shows that $\alpha_m=\pi^{m/2}/(m/2)!$ where $x!=\Gamma(x+1)$ is the Euler Gamma function. The *Newton kernel* is then given by:

$$\text{(3.22)}\quad \begin{cases} N(x) = \dfrac{1}{2\pi}\log|x| & \text{if } m = 2, \\ N(x) = -\dfrac{1}{(m-2)\sigma_{m-1}}\,|x|^{2-m} & \text{if } m \neq 2. \end{cases}$$

The function $N(x)$ is locally integrable on $\mathbb{R}^m$ and satisfies $\Delta N = \delta_0$. When $m = 2$, this follows from Cor. 3.4 and the fact that $\Delta = 4\partial^2/\partial z\partial\overline{z}$. When $m \neq 2$, this can be checked by computing the weak limit

$$\begin{aligned} \lim_{\varepsilon\to 0} \Delta(|x|^2+\varepsilon^2)^{1-m/2} &= \lim_{\varepsilon\to 0} m(2-m)\varepsilon^2(|x|^2+\varepsilon^2)^{-1-m/2} \\ &= m(2-m)\, I_m\, \delta_0 \end{aligned}$$

with $I_m = \int_{\mathbb{R}^m}(|x|^2+1)^{-1-m/2}\,d\lambda(x)$. The last equality is easily seen by performing the change of variable $y = \varepsilon x$ in the integral

$$\int_{\mathbb{R}^m} \varepsilon^2(|x|^2+\varepsilon^2)^{-1-m/2} f(x)\,d\lambda(x) = \int_{\mathbb{R}^m} (|y|^2+1)^{-1-m/2} f(\varepsilon y)\,d\lambda(y),$$

where f is an arbitrary test function. Using polar coordinates, we find that $I_m = \sigma_{m-1}/m$ and our formula follows.

The *Bochner-Martinelli kernel* is the $(n, n-1)$-differential form on $\mathbb{C}^n$ with L^1_{loc} coefficients defined by

$$\text{(3.23)}\quad \begin{aligned} k_{\mathrm{BM}}(z) &= c_n \sum_{1\le j\le n} (-1)^j \frac{\overline{z}_j\, dz_1\wedge\ldots dz_n\wedge d\overline{z}_1\wedge\ldots\widehat{d\overline{z}_j}\ldots\wedge d\overline{z}_n}{|z|^{2n}}, \\ c_n &= (-1)^{n(n-1)/2}\frac{(n-1)!}{(2\pi\mathrm{i})^n}. \end{aligned}$$

(3.24) Lemma. *$d''k_{\mathrm{BM}} = \delta_0$ on $\mathbb{C}^n$.*

Proof. Since the Lebesgue measure on $\mathbb{C}^n$ is

$$d\lambda(z) = \bigwedge_{1\le j\le n}\frac{\mathrm{i}}{2}dz_j\wedge d\overline{z}_j = \Big(\frac{\mathrm{i}}{2}\Big)^n(-1)^{\frac{n(n-1)}{2}}\,dz_1\wedge\ldots dz_n\wedge d\overline{z}_1\wedge\ldots d\overline{z}_n,$$

we find

$$\begin{aligned} d''k_{\mathrm{BM}} &= -\frac{(n-1)!}{\pi^n}\sum_{1\le j\le n}\frac{\partial}{\partial\overline{z}_j}\Big(\frac{\overline{z}_j}{|z|^{2n}}\Big)d\lambda(z) \\ &= -\frac{1}{n(n-1)\alpha_{2n}}\sum_{1\le j\le n}\frac{\partial^2}{\partial z_j\partial\overline{z}_j}\Big(\frac{1}{|z|^{2n-2}}\Big)d\lambda(z) \\ &= \Delta N(z)d\lambda(z) = \delta_0. \end{aligned}$$

□

We let $K_{\mathrm{BM}}(z,\zeta)$ be the pull-back of k_{BM} by the map $\pi : \mathbb{C}^n\times\mathbb{C}^n \to \mathbb{C}^n$, $(z,\zeta) \longmapsto z-\zeta$. Then Formula (2.19) implies

(3.25) $\quad d''K_{\mathrm{BM}} = \pi^\star\delta_0 = [\Delta],$

where $[\Delta]$ denotes the current of integration on the diagonal $\Delta \subset \mathbb{C}^n \times \mathbb{C}^n$.

(3.26) Koppelman formula. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with piecewise C^1 boundary. Then for every (p,q)-form v of class C^1 on $\overline{\Omega}$ we have*

$$v(z) = \int_{\partial\Omega} K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) + d''_z\int_\Omega K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) + \int_\Omega K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta)$$

on Ω, where $K^{p,q}_{\mathrm{BM}}(z,\zeta)$ denotes the component of $K_{\mathrm{BM}}(z,\zeta)$ of type (p,q) in z and $(n-p,n-q-1)$ in ζ.

Proof. Given $w \in \mathcal{D}^{n-p,n-q}(\Omega)$, we consider the integral

$$\int_{\partial\Omega\times\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z).$$

It is well defined since K_{BM} has no singularities on $\partial\Omega \times \operatorname{Supp} v \subset\subset \partial\Omega\times\Omega$. Since $w(z)$ vanishes on $\partial\Omega$ the integral can be extended as well to $\partial(\Omega\times\Omega)$. As $K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z)$ is of total bidegree $(2n,2n-1)$, its differential d' vanishes. Hence Stokes' formula yields

$$\begin{aligned}\int_{\partial\Omega\times\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) &= \int_{\Omega\times\Omega} d''\big(K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z)\big)\\ &= \int_{\Omega\times\Omega} d''K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) - K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta)\wedge w(z)\\ &\qquad - (-1)^{p+q}\int_{\Omega\times\Omega} K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge d''w(z).\end{aligned}$$

By (3.25) we have

$$\int_{\Omega\times\Omega} d''K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) = \int_{\Omega\times\Omega}[\Delta]\wedge v(\zeta)\wedge w(z) = \int_\Omega v(z)\wedge w(z)$$

Denoting $\langle\ ,\ \rangle$ the pairing between currents and test forms on Ω, the above equality is thus equivalent to

$$\begin{aligned}\langle\int_{\partial\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta), w(z)\rangle &= \langle v(z) - \int_\Omega K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta), w(z)\rangle\\ &\qquad - (-1)^{p+q}\langle\int_\Omega K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta), d''w(z)\rangle,\end{aligned}$$

which is itself equivalent to the Koppelman formula by integrating $d''v$ by parts. $\square$

(3.27) Corollary. *Let $v \in {}^s\mathcal{D}^{p,q}(\mathbb{C}^n)$ be a form of class C^s with compact support such that $d''v = 0$, $q \ge 1$. Then the $(p, q-1)$-form*

$$u(z) = \int_{\mathbb{C}^n} K_{\mathrm{BM}}^{p,q-1}(z,\zeta) \wedge v(\zeta)$$

is a C^s solution of the equation $d''u = v$. Moreover, if $(p,q) = (0,1)$ and $n \ge 2$ then u has compact support, thus the Dolbeault cohomology group with compact support $H_c^{0,1}(\mathbb{C}^n, \mathbb{C})$ vanishes for $n \ge 2$.

Proof. Apply the Koppelman formula on a sufficiently large ball $\overline{\Omega} = \overline{B}(0,R)$ containing $\operatorname{Supp} v$. Then the formula immediately gives $d''u = v$. Observe that the coefficients of $K_{\mathrm{BM}}(z,\zeta)$ are $O(|z-\zeta|^{-(2n-1)})$, hence $|u(z)| = O(|z|^{-(2n-1)})$ at infinity. If $q = 1$, then u is holomorphic on $\mathbb{C}^n \smallsetminus \overline{B}(0,R)$. Now, this complement is a union of complex lines when $n \ge 2$, hence $u = 0$ on $\mathbb{C}^n \smallsetminus \overline{B}(0,R)$ by Liouville's theorem. □

(3.28) Hartogs extension theorem. *Let Ω be an open set in $\mathbb{C}^n$, $n \ge 2$, and let $K \subset \Omega$ be a compact subset such that $\Omega \smallsetminus K$ is connected. Then every holomorphic function $f \in \mathcal{O}(\Omega \smallsetminus K)$ extends into a function $\widetilde{f} \in \mathcal{O}(\Omega)$.*

Proof. Let $\psi \in \mathcal{D}(\Omega)$ be a cut-off function equal to 1 on a neighborhood of K. Set $f_0 = (1-\psi)f \in C^\infty(\Omega)$, defined as 0 on K. Then $v = d''f_0 = -f\,d''\psi$ can be extended by 0 outside Ω, and can thus be seen as a smooth $(0,1)$-form with compact support in $\mathbb{C}^n$, such that $d''v = 0$. By Cor. 3.27, there is a smooth function u with compact support in $\mathbb{C}^n$ such that $d''u = v$. Then $\widetilde{f} = f_0 - u \in \mathcal{O}(\Omega)$. Now u is holomorphic outside $\operatorname{Supp}\psi$, so u vanishes on the unbounded component G of $\mathbb{C}^n \smallsetminus \operatorname{Supp}\psi$. The boundary ∂G is contained in $\partial \operatorname{Supp}\psi \subset \Omega \smallsetminus K$, so $\widetilde{f} = (1-\psi)f - u$ coincides with f on the non empty open set $\Omega \cap G \subset \Omega \smallsetminus K$. Therefore $\widetilde{f} = f$ on the connected open set $\Omega \smallsetminus K$. □

A refined version of the Hartogs extension theorem due to Bochner will be given in Exercise 8.13. It shows that f need only be given as a C^1 function on $\partial\Omega$, satisfying the tangential Cauchy-Riemann equations (a so-called *CR-function*). Then f extends as a holomorphic function $\widetilde{f} \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$, provided that $\partial\Omega$ is connected.

§3.E. The Dolbeault-Grothendieck Lemma

We are now in a position to prove the Dolbeault-Grothendieck lemma (Dolbeault 1953), which is the analogue for d'' of the Poincaré lemma. The proof given below makes use of the Bochner-Martinelli kernel. Many other proofs can be given, e.g. by using a reduction to the one dimensional case in combination with the Cauchy formula (3.2), see Exercise 8.5 or (Hörmander 1966).

(3.29) Dolbeault-Grothendieck lemma. *Let Ω be a neighborhood of 0 in $\mathbb{C}^n$ and $v \in {}^s\mathcal{E}^{p,q}(\Omega,\mathbb{C})$, [resp. $v \in {}^s\mathcal{D}'^{p,q}(\Omega,\mathbb{C})$], such that $d''v = 0$, where $1 \le s \le \infty$.*

a) *If $q = 0$, then $v(z) = \sum_{|I|=p} v_I(z)\,dz_I$ is a holomorphic p-form, i.e. a form whose coefficients are holomorphic functions.*

b) *If $q \ge 1$, there exists a neighborhood $\omega \subset \Omega$ of 0 and a form u in ${}^s\mathcal{E}^{p,q-1}(\omega,\mathbb{C})$ [resp. a current $u \in {}^s\mathcal{D}'^{p,q-1}(\omega,\mathbb{C})$] such that $d''u = v$ on ω.*

Proof. We assume that Ω is a ball $B(0,r) \subset \mathbb{C}^n$ and take for simplicity $r > 1$ (possibly after a dilation of coordinates). We then set $\omega = B(0,1)$. Let $\psi \in \mathcal{D}(\Omega)$ be a cut-off function equal to 1 on ω. The Koppelman formula (3.26) applied to the form ψv on Ω gives

$$\psi(z)v(z) = d''_z\int_\Omega K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge\psi(\zeta)v(\zeta) + \int_\Omega K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''\psi(\zeta)\wedge v(\zeta).$$

This formula is valid even when v is a current, because we may regularize v as $v \star \rho_\varepsilon$ and take the limit. We introduce on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$ the kernel

$$K(z,w,\zeta) = c_n\sum_{j=1}^n \frac{(-1)^j(w_j-\overline{\zeta}_j)}{((z-\zeta)\cdot(w-\overline{\zeta}))^n}\bigwedge_k(dz_k - d\zeta_k)\wedge\bigwedge_{k\ne j}(dw_k - d\overline{\zeta}_k).$$

By construction, $K_{\mathrm{BM}}(z,\zeta)$ is the result of the substitution $w = \overline{z}$ in $K(z,w,\zeta)$, i.e. $K_{\mathrm{BM}} = h^\star K$ where $h(z,\zeta) = (z,\overline{z},\zeta)$. We denote by $K^{p,q}$ the component of K of bidegree $(p,0)$ in z, $(q,0)$ in w and $(n-p,n-q-1)$ in ζ. Then $K^{p,q}_{\mathrm{BM}} = h^\star K^{p,q}$ and we find

$$v = d''u_0 + g^\star v_1 \qquad \text{on } \omega,$$

where $g(z) = (z,\overline{z})$ and

$$u_0(z) = \int_\Omega K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge\psi(\zeta)v(\zeta),$$
$$v_1(z,w) = \int_\Omega K^{p,q}(z,w,\zeta)\wedge d''\psi(\zeta)\wedge v(\zeta).$$

By definition of $K^{p,q}(z,w,\zeta)$, v_1 is holomorphic on the open set

$$U = \big\{(z,w) \in \omega\times\omega\,;\ \forall\zeta\notin\omega,\ \mathrm{Re}(z-\zeta)\cdot(w-\overline{\zeta}) > 0\big\},$$

which contains the "conjugate-diagonal" points $(z,\overline{z})$ as well as the points $(z,0)$ and $(0,w)$ in $\omega\times\omega$. Moreover U clearly has convex slices $(\{z\}\times\mathbb{C}^n)\cap U$ and $(\mathbb{C}^n\times\{w\})\cap U$. In particular U is starshaped with respect to w, i.e.

$$(z,w) \in U \Longrightarrow (z,tw) \in U, \qquad \forall t \in [0,1].$$

As u_1 is of type $(p,0)$ in z and $(q,0)$ in w, we get $d''_z(g^\star v_1) = g^\star d_w v_1 = 0$, hence $d_w v_1 = 0$. For $q = 0$ we have $K_{\rm BM}^{p,q-1} = 0$, thus $u_0 = 0$, and v_1 does not depend on w, thus v is holomorphic on ω. For $q \ge 1$, we can use the homotopy formula (1.23) with respect to w (considering z as a parameter) to get a holomorphic form $u_1(z,w)$ of type $(p,0)$ in z and $(q-1,0)$ in w, such that $d_w u_1(z,w) = v_1(z,w)$. Then we get $d''g^\star u_1 = g^\star d_w u_1 = g^\star v_1$, hence

$$v = d''(u_0 + g^\star u_1) \quad \text{on } \omega.$$

Finally, the coefficients of u_0 are obtained as linear combinations of convolutions of the coefficients of ψv with $L^1_{\rm loc}$ functions of the form $\overline{\zeta}_j|\zeta|^{-2n}$. Hence u_0 is of class C^s (resp. is a current of order s), if v is. □

(3.30) Corollary. *The operator d'' is hypoelliptic in bidegree $(p,0)$, i.e. if a current $f \in \mathcal{D}'^{p,0}(X,\mathbb{C})$ satisfies $d''f \in \mathcal{E}^{p,1}(X,\mathbb{C})$, then $f \in \mathcal{E}^{p,0}(X,\mathbb{C})$.*

Proof. The result is local, so we may assume that $X = \Omega$ is a neighborhood of 0 in $\mathbb{C}^n$. The $(p,1)$-form $v = d''f \in \mathcal{E}^{p,1}(X,\mathbb{C})$ satisfies $d''v = 0$, hence there exists $u \in \mathcal{E}^{p,0}(\widetilde{\Omega},\mathbb{C})$ such that $d''u = d''f$. Then $f - u$ is holomorphic and $f = (f-u) + u \in \mathcal{E}^{p,0}(\widetilde{\Omega},\mathbb{C})$. □

§4. Subharmonic Functions

A *harmonic* (resp. *subharmonic*) function on an open subset of $\mathbb{R}^m$ is essentially a function (or distribution) u such that $\Delta u = 0$ (resp. $\Delta u \ge 0$). A fundamental example of subharmonic function is given by the Newton kernel N, which is actually harmonic on $\mathbb{R}^m \smallsetminus \{0\}$. Subharmonic functions are an essential tool of harmonic analysis and potential theory. Before giving their precise definition and properties, we derive a basic integral formula involving the Green kernel of the Laplace operator on the ball.

§4.A. Construction of the Green Kernel

The *Green kernel* $G_\Omega(x,y)$ of a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$ is the solution of the following *Dirichlet boundary problem* for the Laplace operator Δ on Ω:

(4.1) Definition. *The Green kernel of a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$ is a function $G_\Omega(x,y) : \overline{\Omega} \times \overline{\Omega} \to [-\infty, 0]$ with the following properties:*

a) $G_\Omega(x,y)$ *is* C^∞ *on* $\overline{\Omega} \times \overline{\Omega} \smallsetminus \mathrm{Diag}_\Omega$ $\quad(\mathrm{Diag}_\Omega =$ *diagonal*$)$;

b) $G_\Omega(x,y) = G_\Omega(y,x)$;

c) $G_\Omega(x,y) < 0$ *on* $\Omega \times \Omega$ *and* $G_\Omega(x,y) = 0$ *on* $\partial\Omega \times \Omega$;

d) $\Delta_x G_\Omega(x,y) = \delta_y$ *on* Ω *for every fixed* $y \in \Omega$.

It can be shown that G_Ω always exists and is unique. The uniqueness is an easy consequence of the maximum principle (see Th. 4.14 below). In the case where $\Omega = B(0,r)$ is a ball (the only case we are going to deal with), the existence can be shown through explicit calculations. In fact the Green kernel $G_r(x,y)$ of $B(0,r)$ is

$$(4.2)\quad G_r(x,y) = N(x-y) - N\Big(\frac{|y|}{r}\Big(x - \frac{r^2}{|y|^2}\,y\Big)\Big), \qquad x,y \in \overline{B}(0,r).$$

A substitution of the explicit value of $N(x)$ yields:

$$G_r(x,y) = \frac{1}{4\pi}\log\frac{|x-y|^2}{r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2} \qquad \text{if } m=2, \quad \text{otherwise}$$

$$G_r(x,y) = \frac{-1}{(m-2)\sigma_{m-1}}\Big(|x-y|^{2-m} - \big(r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2\big)^{1-m/2}\Big).$$

(4.3) Theorem. *The above defined function* G_r *satisfies all four properties* (4.1 a–d) *on* $\Omega = B(0,r)$*, thus* G_r *is the Green kernel of* $B(0,r)$*.*

Proof. The first three properties are immediately verified on the formulas, because

$$r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2 = |x-y|^2 + \frac{1}{r^2}\big(r^2 - |x|^2\big)\big(r^2 - |y|^2\big).$$

For property d), observe that $r^2 y/|y|^2 \notin \overline{B}(0,r)$ whenever $y \in B(0,r) \smallsetminus \{0\}$. The second Newton kernel in the right hand side of (4.1) is thus harmonic in x on $B(0,r)$, and

$$\Delta_x G_r(x,y) = \Delta_x N(x-y) = \delta_y \quad \text{on} \quad B(0,r). \qquad \square$$

§4.B. Green-Riesz Representation Formula and Dirichlet Problem

§4.B.1. Green-Riesz Formula. For all smooth functions u, v on a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$, we have

$$(4.4)\qquad \int_\Omega (u\,\Delta v - v\,\Delta u)\,d\lambda = \int_{\partial\Omega}\Big(u\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial u}{\partial \nu}\Big)\,d\sigma$$

where $\partial/\partial\nu$ is the derivative along the outward normal unit vector ν of $\partial\Omega$ and $d\sigma$ the euclidean area measure. Indeed

$$(-1)^{j-1}\,dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m{}_{\restriction \partial\Omega} = \nu_j\,d\sigma,$$

for the wedge product of $\langle \nu, dx\rangle$ with the left hand side is $\nu_j\,d\lambda$. Therefore

$$\frac{\partial v}{\partial \nu}\, d\sigma = \sum_{j=1}^{m} \frac{\partial v}{\partial x_j}\, \nu_j\, d\sigma = \sum_{j=1}^{m} (-1)^{j-1} \frac{\partial v}{\partial x_j}\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m.$$

Formula (4.4) is then an easy consequence of Stokes' theorem. Observe that (4.4) is still valid if v is a distribution with singular support relatively compact in Ω. For $\Omega = B(0,r)$, $u \in C^2\big(\overline{B}(0,r), \mathbb{R}\big)$ and $v(y) = G_r(x,y)$, we get the *Green-Riesz representation formula*:

$$(4.5) \quad u(x) = \int_{B(0,r)} \Delta u(y)\, G_r(x,y)\, d\lambda(y) + \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y)$$

where $P_r(x,y) = \partial G_r(x,y)/\partial\nu(y)$, $(x,y) \in B(0,r) \times S(0,r)$. The function $P_r(x,y)$ is called the *Poisson kernel.* It is smooth and satisfies $\Delta_x P_r(x,y) = 0$ on $B(0,r)$ by (4.1 d). A simple computation left to the reader yields:

$$(4.6) \quad P_r(x,y) = \frac{1}{\sigma_{m-1} r}\, \frac{r^2 - |x|^2}{|x-y|^m}.$$

Formula (4.5) for $u \equiv 1$ shows that $\int_{S(0,r)} P_r(x,y)\, d\sigma(y) = 1$. When x in $B(0,r)$ tends to $x_0 \in S(0,r)$, we see that $P_r(x,y)$ converges uniformly to 0 on every compact subset of $S(0,r) \smallsetminus \{x_0\}$; it follows that the measure $P_r(x,y)\, d\sigma(y)$ converges weakly to δ_{x_0} on $S(0,r)$.

§4.B.2. Solution of the Dirichlet Problem. For any bounded measurable function v on $S(a,r)$ we define

$$(4.7) \quad P_{a,r}[v](x) = \int_{S(a,r)} v(y)\, P_r(x-a, y-a)\, d\sigma(y), \qquad x \in B(a,r).$$

If $u \in C^0\big(\overline{B}(a,r), \mathbb{R}\big) \cap C^2\big(B(a,r), \mathbb{R}\big)$ is harmonic, i.e. $\Delta u = 0$ on $B(a,r)$, then (4.5) gives $u = P_{a,r}[u]$ on $B(a,r)$, i.e. the Poisson kernel reproduces harmonic functions. Suppose now that $v \in C^0\big(S(a,r), \mathbb{R}\big)$ is given. Then $P_r(x-a, y-a)\, d\sigma(y)$ converges weakly to δ_{x_0} when x tends to $x_0 \in S(a,r)$, so $P_{a,r}[v](x)$ converges to $v(x_0)$. It follows that the function u defined by

$$\begin{cases} u = P_{a,r}[v] & \text{on} \quad B(a,r), \\ u = v & \text{on} \quad S(a,r) \end{cases}$$

is continuous on $\overline{B}(a,r)$ and harmonic on $B(a,r)$; thus u is the solution of the Dirichlet problem with boundary values v.

§4.C. Definition and Basic Properties of Subharmonic Functions

§4.C.1. Definition. Mean Value Inequalities. If u is a Borel function on $\overline{B}(a,r)$ which is bounded above or below, we consider the mean values of u over the ball or sphere:

$$(4.8) \qquad \mu_B(u\,;a,r) = \frac{1}{\alpha_m r^m}\int_{B(a,r)} u(x)\,d\lambda(x),$$

$$(4.8') \qquad \mu_S(u\,;a,r) = \frac{1}{\sigma_{m-1} r^{m-1}}\int_{S(a,r)} u(x)\,d\sigma(x).$$

As $d\lambda = dr\,d\sigma$ these mean values are related by

$$(4.9) \qquad \begin{aligned}\mu_B(u\,;a,r) &= \frac{1}{\alpha_m r^m}\int_0^r \sigma_{m-1} t^{m-1}\,\mu_S(u\,;a,t)\,dt\\ &= m\int_0^1 t^{m-1}\,\mu_S(u\,;a,rt)\,dt.\end{aligned}$$

Now, apply formula (4.5) with $x = 0$. We get $P_r(0,y) = 1/\sigma_{m-1}r^{m-1}$ and $G_r(0,y) = (|y|^{2-m} - r^{2-m})/(2-m)\sigma_{m-1} = -(1/\sigma_{m-1})\int_{|y|}^r t^{1-m}dt$, thus

$$\begin{aligned}\int_{B(0,r)} \Delta u(y)\,G_r(0,y)\,d\lambda(y) &= -\frac{1}{\sigma_{m-1}}\int_0^r \frac{dt}{t^{m-1}}\int_{|y|<t}\Delta u(y)\,d\lambda(y)\\ &= -\frac{1}{m}\int_0^r \mu_B(\Delta u\,;0,t)\,t\,dt\end{aligned}$$

thanks to the Fubini formula. By translating $S(0,r)$ to $S(a,r)$, (4.5) implies the *Gauss formula*

$$(4.10) \qquad \mu_S(u\,;a,r) = u(a) + \frac{1}{m}\int_0^r \mu_B(\Delta u\,;a,t)\,t\,dt.$$

Let Ω be an open subset of $\mathbb{R}^m$ and $u \in C^2(\Omega,\mathbb{R})$. If $a \in \Omega$ and $\Delta u(a) > 0$ (resp. $\Delta u(a) < 0$), Formula (4.10) shows that $\mu_S(u\,;a,r) > u(a)$ (resp. $\mu_S(u\,;a,r) < u(a)$) for r small enough. In particular, u is harmonic (i.e. $\Delta u = 0$) if and only if u satisfies the *mean value equality*

$$\mu_S(u\,;a,r) = u(a), \qquad \forall \overline{B}(a,r) \subset \Omega.$$

Now, observe that if (ρ_ε) is a family of radially symmetric smoothing kernels associated with $\rho(x) = \widetilde{\rho}(|x|)$ and if u is a Borel locally bounded function, an easy computation yields

$$(4.11) \qquad \begin{aligned}u\star\rho_\varepsilon(a) &= \int_{B(0,1)} u(a+\varepsilon x)\,\rho(x)\,d\lambda\\ &= \sigma_{m-1}\int_0^1 \mu_S(u\,;a,\varepsilon t)\,\widetilde{\rho}(t)\,t^{m-1}\,dt.\end{aligned}$$

Thus, if u is a Borel locally bounded function satisfying the mean value equality on Ω, (4.11) shows that $u\star\rho_\varepsilon = u$ on Ω_ε, in particular u must be smooth. Similarly, if we replace the mean value equality by an inequality, the relevant regularity property to be required for u is just semicontinuity.

(4.12) Theorem and definition. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. The following various forms of mean value inequalities are equivalent:*

a) $u(x) \le P_{a,r}[u](x), \quad \forall \overline{B}(a,r) \subset \Omega, \quad \forall x \in B(a,r)$;

b) $u(a) \le \mu_S\,(u\,; a, r), \quad \forall \overline{B}(a,r) \subset \Omega$;

c) $u(a) \le \mu_B(u\,; a, r), \quad \forall \overline{B}(a,r) \subset \Omega$;

d) *for every $a \in \Omega$, there exists a sequence (r_ν) decreasing to* 0 *such that*

$$u(a) \le \mu_B(u\,; a, r_\nu) \qquad \forall \nu \ ;$$

e) *for every $a \in \Omega$, there exists a sequence (r_ν) decreasing to* 0 *such that*

$$u(a) \le \mu_S(u\,; a, r_\nu) \qquad \forall \nu.$$

A function u satisfying one of the above properties is said to be subharmonic on Ω. The set of subharmonic functions will be denoted by $\mathrm{Sh}(\Omega)$.

By (4.10) we see that a function $u \in C^2(\Omega, \mathbb{R})$ is subharmonic if and only if $\Delta u \ge 0$: in fact $\mu_S(u\,; a, r) < u(a)$ for r small if $\Delta u(a) < 0$. It is also clear on the definitions that every (locally) convex function on Ω is subharmonic.

Proof. We have obvious implications

$$\text{a)} \Longrightarrow \text{b)} \Longrightarrow \text{c)} \Longrightarrow \text{d)} \Longrightarrow \text{e)},$$

the second and last ones by (4.10) and the fact that $\mu_B(u\,; a, r_\nu) \le \mu_S(u\,; a, t)$ for at least one $t \in \,]0, r_\nu[$. In order to prove e) $\Longrightarrow$ a), we first need a suitable version of the maximum principle.

(4.13) Lemma. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function satisfying property* 4.12 e). *If u attains its supremum at a point $x_0 \in \Omega$, then u is constant on the connected component of x_0 in Ω.*

Proof. We may assume that Ω is connected. Let

$$W = \{x \in \Omega \ ; \ u(x) < u(x_0)\}.$$

W is open by the upper semicontinuity, and distinct from Ω since $x_0 \notin W$. We want to show that $W = \emptyset$. Otherwise W has a non empty connected component W_0, and W_0 has a boundary point $a \in \Omega$. We have $a \in \Omega \smallsetminus W$, thus $u(a) = u(x_0)$. By assumption 4.12 e), we get $u(a) \le \mu_S(u\,; a, r_\nu)$ for some sequence $r_\nu \to 0$. For r_ν small enough, W_0 intersects $\Omega \smallsetminus \overline{B}(a, r_\nu)$ and $B(a, r_\nu)$; as W_0 is connected, we also have $S(a, r_\nu) \cap W_0 \ne \emptyset$. Since $u \le u(x_0)$ on the sphere $S(a, r_\nu)$ and $u < u(x_0)$ on its open subset $S(a, r_\nu) \cap W_0$, we get $u(a) \le \mu_S(u\,; a, r) < u(x_0)$, a contradiction. $\square$

(4.14) Maximum principle. *If u is subharmonic in Ω (in the sense that u satisfies the weakest property* 4.12 e)), *then*

$$\sup_{\Omega} u = \limsup_{\Omega \ni z \to \partial\Omega \cup \{\infty\}} u(z),$$

and $\sup_K u = \sup_{\partial K} u(z)$ *for every compact subset $K \subset \Omega$.*

Proof. We have of course $\limsup_{z \to \partial\Omega \cup \{\infty\}} u(z) \le \sup_\Omega u$. If the inequality is strict, this means that the supremum is achieved on some compact subset $L \subset \Omega$. Thus, by the upper semicontinuity, there is $x_0 \in L$ such that $\sup_\Omega u = \sup_L u = u(x_0)$. Lemma 4.13 shows that u is constant on the connected component Ω_0 of x_0 in Ω, hence

$$\sup_{\Omega} u = u(x_0) = \limsup_{\Omega_0 \ni z \to \partial\Omega_0 \cup \{\infty\}} u(z) \le \limsup_{\Omega \ni z \to \partial\Omega \cup \{\infty\}} u(z),$$

contradiction. The statement involving a compact subset K is obtained by applying the first statement to $\Omega' = K^\circ$. □

Proof of (4.12) e) $\Longrightarrow$ a) Let u be an upper semicontinuous function satisfying 4.12 e) and $\overline{B}(a,r) \subset \Omega$ an arbitrary closed ball. One can find a decreasing sequence of continuous functions $v_k \in C^0\big(S(a,r), \mathbb{R}\big)$ such that $\lim v_k = u$. Set $h_k = P_{a,r}[v_k] \in C^0\big(\overline{B}(a,r), \mathbb{R}\big)$. As h_k is harmonic on $B(a,r)$, the function $u - h_k$ satisfies 4.12 e) on $B(a,r)$. Furthermore $\limsup_{x \to \xi \in S(a,r)} u(x) - h_k(x) \le u(\xi) - v_k(\xi) \le 0$, so $u - h_k \le 0$ on $B(a,r)$ by Th. 4.14. By monotone convergence, we find $u \le P_{a,r}[u]$ on $B(a,r)$ when k tends to $+\infty$. □

§**4.C.2. Basic Properties.** Here is a short list of the most basic properties.

(4.15) Theorem. *For any decreasing sequence (u_k) of subharmonic functions, the limit $u = \lim u_k$ is subharmonic.*

Proof. A decreasing limit of upper semicontinuous functions is again upper semicontinuous, and the mean value inequalities 4.12 remain valid for u by Lebesgue's monotone convergence theorem. □

(4.16) Theorem. *Let $u_1, \ldots, u_p \in \mathrm{Sh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is non decreasing in each t_j. If χ is extended by continuity into a function $[-\infty, +\infty[^p \longrightarrow [-\infty, +\infty[$, then*

$$\chi(u_1, \ldots, u_p) \in \mathrm{Sh}(\Omega).$$

In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p}) \in \mathrm{Sh}(\Omega)$.

Proof. Every convex function is continuous, hence $\chi(u_1, \ldots, u_p)$ is upper semicontinuous. One can write

$$\chi(t) = \sup_{i\in I} A_i(t)$$

where $A_i(t) = a_1t_1 + \cdots + a_pt_p + b$ is the family of affine functions that define supporting hyperplanes of the graph of χ. As $\chi(t_1, \ldots, t_p)$ is non-decreasing in each t_j, we have $a_j \ge 0$, thus

$$\sum_{1\le j\le p} a_ju_j(x) + b \le \mu_B\big(\sum a_ju_j + b\,; x, r\big) \le \mu_B\big(\chi(u_1, \ldots, u_p)\,; x, r\big)$$

for every ball $\overline{B}(x, r) \subset \Omega$. If one takes the supremum of this inequality over all the A_i 's, it follows that $\chi(u_1, \ldots, u_p)$ satisfies the mean value inequality 4.12 c). In the last example, the function $\chi(t_1, \ldots, t_p) = \log(e^{t_1} + \cdots + e^{t_p})$ is convex because

$$\sum_{1\le j,k\le p} \frac{\partial^2\chi}{\partial t_j\partial t_k}\,\xi_j\xi_k = e^{-\chi}\sum \xi_j^2\,e^{t_j} - e^{-2\chi}\big(\sum \xi_j\,e^{t_j}\big)^2$$

and $\big(\sum \xi_j\,e^{t_j}\big)^2 \le \big(\sum \xi_j^2\,e^{t_j}\big)\,e^{\chi}$ by the Cauchy-Schwarz inequality. □

(4.17) Theorem. *If Ω is connected and $u \in \mathrm{Sh}(\Omega)$, then either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.*

Proof. Note that a subharmonic function is always locally bounded above. Let W be the set of points $x \in \Omega$ such that u is integrable in a neighborhood of x. Then W is open by definition and $u > -\infty$ almost everywhere on W. If $x \in \overline{W}$, one can choose $a \in W$ such that $|a - x| < r = \frac{1}{2}d(x, \complement\Omega)$ and $u(a) > -\infty$. Then $B(a, r)$ is a neighborhood of x, $\overline{B}(a, r) \subset \Omega$ and $\mu_B(u\,; a, r) \ge u(a) > -\infty$. Therefore $x \in W$, W is also closed. We must have $W = \Omega$ or $W = \emptyset$; in the last case $u \equiv -\infty$ by the mean value inequality. □

(4.18) Theorem. *Let $u \in \mathrm{Sh}(\Omega)$ be such that $u \not\equiv -\infty$ on each connected component of Ω. Then*

a) $r \longmapsto \mu_S(u\,; a, r)$, $r \longmapsto \mu_B(u\,; a, r)$ *are non decreasing functions in the interval* $]0, d(a, \complement\Omega)[$, *and* $\mu_B(u\,; a, r) \le \mu_S(u\,; a, r)$.

b) *For any family (ρ_ε) of smoothing kernels, $u \star \rho_\varepsilon \in \mathrm{Sh}(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon, \mathbb{R})$, the family $(u \star \rho_\varepsilon)$ is non decreasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.*

Proof. We first verify statements a) and b) when $u \in C^2(\Omega, \mathbb{R})$. Then $\Delta u \ge 0$ and $\mu_S(u\,; a, r)$ is non decreasing in virtue of (4.10). By (4.9), we find that $\mu_B(u\,; a, r)$ is also non decreasing and that $\mu_B(u\,; a, r) \le \mu_S(u\,; a, r)$. Furthermore, Formula (4.11) shows that $\varepsilon \longmapsto u \star \rho_\varepsilon(a)$ is non decreasing (provided that ρ_ε is radially symmetric).

In the general case, we first observe that property 4.12 c) is equivalent to the inequality

$$u \le u \star \mu_r \quad \text{on} \quad \Omega_r, \qquad \forall r > 0,$$

where μ_r is the probability measure of uniform density on $B(0,r)$. This inequality implies $u\star\rho_\varepsilon \le u\star\rho_\varepsilon\star\mu_r$ on $(\Omega_r)_\varepsilon = \Omega_{r+\varepsilon}$, thus $u\star\rho_\varepsilon \in C^\infty(\Omega_\varepsilon,\mathbb{R})$ is subharmonic on Ω_ε. It follows that $u\star\rho_\varepsilon\star\rho_\eta$ is non decreasing in η ; by symmetry, it is also non decreasing in ε, and so is $u\star\rho_\varepsilon = \lim_{\eta\to 0} u\star\rho_\varepsilon\star\rho_\eta$. We have $u\star\rho_\varepsilon \ge u$ by (4.19) and $\limsup_{\varepsilon\to 0} u\star\rho_\varepsilon \le u$ by the upper semicontinuity. Hence $\lim_{\varepsilon\to 0} u\star\rho_\varepsilon = u$. Property a) for u follows now from its validity for $u\star\rho_\varepsilon$ and from the monotone convergence theorem. □

(4.19) Corollary. *If $u \in \mathrm{Sh}(\Omega)$ is such that $u \not\equiv -\infty$ on each connected component of Ω, then Δu computed in the sense of distribution theory is a positive measure.*

Indeed $\Delta(u\star\rho_\varepsilon) \ge 0$ as a function, and $\Delta(u\star\rho_\varepsilon)$ converges weakly to Δu in $\mathcal{D}'(\Omega)$. Corollary 4.19 has a converse, but the correct statement is slightly more involved than for the direct property:

(4.20) Theorem. *If $v \in \mathcal{D}'(\Omega)$ is such that Δv is a positive measure, there exists a unique function $u \in \mathrm{Sh}(\Omega)$ locally integrable such that v is the distribution associated to u.*

We must point out that u need not coincide everywhere with v, even when v is a locally integrable upper semicontinuous function: for example, if v is the characteristic function of a compact subset $K \subset \Omega$ of measure 0, the subharmonic representant of v is $u = 0$.

Proof. Set $v_\varepsilon = v\star\rho_\varepsilon \in C^\infty(\Omega_\varepsilon,\mathbb{R})$. Then $\Delta v_\varepsilon = (\Delta v)\star\rho_\varepsilon \ge 0$, thus $v_\varepsilon \in \mathrm{Sh}(\Omega_\varepsilon)$. Arguments similar to those in the proof of Th. 4.18 show that (v_ε) is non decreasing in ε. Then $u := \lim_{\varepsilon\to 0} v_\varepsilon \in \mathrm{Sh}(\Omega)$ by Th. 4.15. Since v_ε converges weakly to v, the monotone convergence theorem shows that

$$\langle v, f\rangle = \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\, f\, d\lambda = \int_\Omega u\, f\, d\lambda, \qquad \forall f \in \mathcal{D}(\Omega), \quad f \ge 0,$$

which concludes the existence part. The uniqueness of u is clear from the fact that u must satisfy $u = \lim u\star\rho_\varepsilon = \lim v\star\rho_\varepsilon$. □

The most natural topology on the space $\mathrm{Sh}(\Omega)$ of subharmonic functions is the topology induced by the vector space topology of $L^1_{\mathrm{loc}}(\Omega)$ (Fréchet topology of convergence in L^1 norm on every compact subset of Ω).

(4.21) Proposition. *The convex cone $\mathrm{Sh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$, and it has the property that every bounded subset is relatively compact.*

Proof. Let (u_j) be a sequence in $\mathrm{Sh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$. If $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ then $\Delta u_j \to \Delta u$ in the weak topology of distributions, hence $\Delta u \ge 0$ and u can be represented by a subharmonic function thanks to Th. 4.20. Now, suppose that $\|u_j\|_{L^1(K)}$ is uniformly bounded for every compact subset K of Ω. Let $\mu_j = \Delta u_j \geq 0$. If $\psi \in \mathcal{D}(\Omega)$ is a test function equal to 1 on a neighborhood ω of K and such that $0 \leq \psi \leq 1$ on Ω, we find

$$\mu_j(K) \leq \int_\Omega \psi\, \Delta u_j\, d\lambda = \int_\Omega \Delta\psi\, u_j\, d\lambda \leq C\|u_j\|_{L^1(K')},$$

where $K' = \operatorname{Supp}\psi$, hence the sequence of measures (μ_j) is uniformly bounded in mass on every compact subset of Ω. By weak compactness, there is a subsequence (μ_{j_ν}) which converges weakly to a positive measure μ on Ω. We claim that $f\star(\psi\mu_{j_\nu})$ converges to $f\star(\psi\mu)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$ for every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. In fact, this is clear if $f \in C^\infty(\mathbb{R}^m)$, and in general we use an approximation of f by a smooth function g together with the estimate

$$\|(f-g)\star(\psi\mu_{j_\nu})\|_{L^1(A)} \leq \|(f-g)\|_{L^1(A+K')}\mu_{j_\nu}(K'), \qquad \forall A \subset\subset \mathbb{R}^m$$

to get the conclusion. We apply this when $f = N$ is the Newton kernel. Then $h_j = u_j - N\star(\psi\mu_j)$ is harmonic on ω and bounded in $L^1(\omega)$. As $h_j = h_j \star \rho_\varepsilon$ for any smoothing kernel ρ_ε, we see that all derivatives $D^\alpha h_j = h_j \star (D^\alpha\rho_\varepsilon)$ are in fact uniformly locally bounded in ω. Hence, after extracting a new subsequence, we may suppose that h_{j_ν} converges uniformly to a limit h on ω. Then $u_{j_\nu} = h_{j_\nu} + N\star(\psi\mu_{j_\nu})$ converges to $u = h + N\star(\psi\mu)$ in $L^1_{\mathrm{loc}}(\omega)$, as desired. □

We conclude this subsection by stating a generalized version of the Green-Riesz formula.

(4.22) Proposition. *Let $u \in \mathrm{Sh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$ and $\overline{B}(0,r) \subset \Omega$.*

a) *The Green-Riesz formula still holds true for such an u, namely, for every $x \in B(0,r)$*

$$u(x) = \int_{B(0,r)} \Delta u(y)\, G_r(x,y)\, d\lambda(y) + \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y).$$

b) (Harnack inequality)
If $u \geq 0$ on $\overline{B}(0,r)$, then for all $x \in B(0,r)$

$$0 \leq u(x) \leq \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y) \leq \frac{r^{m-2}(r+|x|)}{(r-|x|)^{m-1}}\, \mu_S(u\,;0,r).$$

If $u \leq 0$ on $\overline{B}(0,r)$, then for all $x \in B(0,r)$

$$u(x) \leq \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y) \leq \frac{r^{m-2}(r-|x|)}{(r+|x|)^{m-1}}\, \mu_S(u\,;0,r) \leq 0.$$

Proof. We know that a) holds true if u is of class C^2. In general, we replace u by $u \star \rho_\varepsilon$ and take the limit. We only have to check that

$$\int_{B(0,r)} \mu \star \rho_\varepsilon(y)\, G_r(x,y)\, d\lambda(y) = \lim_{\varepsilon \to 0} \int_{B(0,r)} \mu(y)\, G_r(x,y)\, d\lambda(y)$$

for the positive measure $\mu = \Delta u$. Let us denote by $\widetilde{G}_x(y)$ the function such that

$$\widetilde{G}_x(y) = \begin{cases} G_r(x,y) & \text{if } x \in B(0,r) \\ 0 & \text{if } x \notin B(0,r). \end{cases}$$

Then

$$\begin{aligned} \int_{B(0,r)} \mu \star \rho_\varepsilon(y)\, G_r(x,y)\, d\lambda(y) &= \int_{\mathbb{R}^m} \mu \star \rho_\varepsilon(y)\, \widetilde{G}_x(y)\, d\lambda(y) \\ &= \int_{\mathbb{R}^m} \mu(y)\, \widetilde{G}_x \star \rho_\varepsilon(y)\, d\lambda(y). \end{aligned}$$

However $\widetilde{G}_x$ is continuous on $\mathbb{R}^m \smallsetminus \{x\}$ and subharmonic in a neighborhood of x, hence $\widetilde{G}_x \star \rho_\varepsilon$ converges uniformly to $\widetilde{G}_x$ on every compact subset of $\mathbb{R}^m \smallsetminus \{x\}$, and converges pointwise monotonically in a neighborhood of x. The desired equality follows by the monotone convergence theorem. Finally, b) is a consequence of a), for the integral involving Δu is nonpositive and

$$\frac{1}{\sigma_{m-1} r^{m-1}} \frac{r^{m-2}(r - |x|)}{(r + |x|)^{m-1}} \le P_r(x,y) \le \frac{1}{\sigma_{m-1} r^{m-1}} \frac{r^{m-2}(r + |x|)}{(r - |x|)^{m-1}}$$

by (4.6) combined with the obvious inequality $(r - |x|)^m \le |x - y|^m \le (r + |x|)^m$. □

§4.C.3. Upper Envelopes and Choquet's Lemma. Let $\Omega \subset \mathbb{R}^n$ and let $(u_\alpha)_{\alpha \in I}$ be a family of upper semicontinuous functions $\Omega \longrightarrow [-\infty, +\infty[$. We assume that (u_α) is locally uniformly bounded above. Then the upper envelope

$$u = \sup u_\alpha$$

need not be upper semicontinuous, so we consider its *upper semicontinuous regularization*:

$$u^\star(z) = \lim_{\varepsilon \to 0} \sup_{B(z,\varepsilon)} u \ge u(z).$$

It is easy to check that $u^\star$ is the smallest upper semicontinuous function which is $\ge u$. Our goal is to show that $u^\star$ can be computed with a countable subfamily of (u_α). Let $B(z_j, \varepsilon_j)$ be a countable basis of the topology of Ω. For each j, let (z_{jk}) be a sequence of points in $B(z_j, \varepsilon_j)$ such that

$$\sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u,$$

and for each pair (j,k), let $\alpha(j,k,l)$ be a sequence of indices $\alpha \in I$ such that $u(z_{jk}) = \sup_l u_{\alpha(j,k,l)}(z_{jk})$. Set

$$v = \sup_{j,k,l} u_{\alpha(j,k,l)}.$$

Then $v \le u$ and $v^\star \le u^\star$. On the other hand

$$\sup_{B(z_j,\varepsilon_j)} v \ge \sup_k v(z_{jk}) \ge \sup_{k,l} u_{\alpha(j,k,l)}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u.$$

As every ball $B(z,\varepsilon)$ is a union of balls $B(z_j,\varepsilon_j)$, we easily conclude that $v^\star \ge u^\star$, hence $v^\star = u^\star$. Therefore:

(4.23) Choquet's lemma. *Every family (u_α) has a countable subfamily $(v_j) = (u_{\alpha(j)})$ such that its upper envelope v satisfies $v \le u \le u^\star = v^\star$.* □

(4.24) Proposition. *If all u_α are subharmonic, the upper regularization $u^\star$ is subharmonic and equal almost everywhere to u.*

Proof. By Choquet's lemma we may assume that (u_α) is countable. Then $u = \sup u_\alpha$ is a Borel function. As each u_α satisfies the mean value inequality on every ball $\overline{B}(z,r) \subset \Omega$, we get

$$u(z) = \sup u_\alpha(z) \le \sup \mu_B(u_\alpha\,;\,z,r) \le \mu_B(u\,;\,z,r).$$

The right-hand side is a continuous function of z, so we infer

$$u^\star(z) \le \mu_B(u\,;\,z,r) \le \mu_B(u^\star\,;\,z,r)$$

and $u^\star$ is subharmonic. By the upper semicontinuity of $u^\star$ and the above inequality we find $u^\star(z) = \lim_{r\to 0} \mu_B(u\,;\,z,r)$, thus $u^\star = u$ almost everywhere by Lebesgue's lemma. □

§5. Plurisubharmonic Functions

§5.A. Definition and Basic Properties

Plurisubharmonic functions have been introduced independently by (Lelong 1942) and (Oka 1942) for the study of holomorphic convexity. They are the complex counterparts of subharmonic functions.

(5.1) Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if*

a) *u is upper semicontinuous* ;

b) *for every complex line* $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ *is subharmonic on* $\Omega \cap L$.

The set of plurisubharmonic functions on Ω *is denoted by* $\mathrm{Psh}(\Omega)$.

An equivalent way of stating property b) is: for all $a \in \Omega$, $\xi \in \mathbb{C}^n$, $|\xi| < d(a, \complement\Omega)$, then

$$(5.2) \quad u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{\mathrm{i}\theta}\,\xi)\,d\theta.$$

An integration of (5.2) over $\xi \in S(0,r)$ yields $u(a) \le \mu_S(u\,; a, r)$, therefore

$$(5.3) \quad \mathrm{Psh}(\Omega) \subset \mathrm{Sh}(\Omega).$$

The following results have already been proved for subharmonic functions and are easy to extend to the case of plurisubharmonic functions:

(5.4) Theorem. *For any decreasing sequence of plurisubharmonic functions* $u_k \in \mathrm{Psh}(\Omega)$, *the limit* $u = \lim u_k$ *is plurisubharmonic on* Ω.

(5.5) Theorem. *Let* $u \in \mathrm{Psh}(\Omega)$ *be such that* $u \not\equiv -\infty$ *on every connected component of* Ω. *If* (ρ_ε) *is a family of smoothing kernels, then* $u \star \rho_\varepsilon$ *is* C^∞ *and plurisubharmonic on* Ω_ε, *the family* $(u \star \rho_\varepsilon)$ *is non decreasing in* ε *and* $\lim_{\varepsilon \to 0} u \star \rho_\varepsilon = u$.

(5.6) Theorem. *Let* $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ *and* $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ *be a convex function such that* $\chi(t_1, \ldots, t_p)$ *is non decreasing in each* t_j. *Then* $\chi(u_1, \ldots, u_p)$ *is plurisubharmonic on* Ω. *In particular* $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ *are plurisubharmonic on* Ω.

(5.7) Theorem. *Let* $\{u_\alpha\} \subset \mathrm{Psh}(\Omega)$ *be locally uniformly bounded from above and* $u = \sup u_\alpha$. *Then the regularized upper envelope* $u^\star$ *is plurisubharmonic and is equal to* u *almost everywhere.*

Proof. By Choquet's lemma, we may assume that (u_α) is countable. Then u is a Borel function which clearly satisfies (5.2), and thus $u \star \rho_\varepsilon$ also satisfies (5.2). Hence $u \star \rho_\varepsilon$ is plurisubharmonic. By Proposition 4.24, $u^\star = u$ almost everywhere and $u^\star$ is subharmonic, so

$$u^\star = \lim u^\star \star \rho_\varepsilon = \lim u \star \rho_\varepsilon$$

is plurisubharmonic. □

If $u \in C^2(\Omega, \mathbb{R})$, the subharmonicity of restrictions of u to complex lines, $\mathbb{C} \ni w \longmapsto u(a + w\xi)$, $a \in \Omega$, $\xi \in \mathbb{C}^n$, is equivalent to

$$\frac{\partial^2}{\partial w\partial\overline{w}}u(a+w\xi)=\sum_{1\le j,k\le n}\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}(a+w\xi)\,\xi_j\overline{\xi}_k\ge 0.$$

Therefore, u is plurisubharmonic on Ω if and only if the hermitian form $\sum\partial^2 u/\partial z_j\partial\overline{z}_k(a)\,\xi_j\overline{\xi}_k$ is semipositive at every point $a\in\Omega$. This equivalence is still true for arbitrary plurisubharmonic functions, under the following form:

(5.8) Theorem. *If $u\in\mathrm{Psh}(\Omega)$, $u\not\equiv-\infty$ on every connected component of Ω, then for all $\xi\in\mathbb{C}^n$*

$$Hu(\xi):=\sum_{1\le j,k\le n}\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\xi}_k\in\mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v\in\mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi\in\mathbb{C}^n$, there exists a unique function $u\in\mathrm{Psh}(\Omega)$ locally integrable on Ω such that v is the distribution associated to u.

Proof. If $u\in\mathrm{Psh}(\Omega)$, then $Hu(\xi)=\text{weak}\ \lim H(u\star\rho_\varepsilon)(\xi)\ge 0$. Conversely, $Hv\ge 0$ implies $H(v\star\rho_\varepsilon)=(Hv)\star\rho_\varepsilon\ge 0$, thus $v\star\rho_\varepsilon\in\mathrm{Psh}(\Omega)$, and also $\Delta v\ge 0$, hence $(v\star\rho_\varepsilon)$ is non decreasing in ε and $u=\lim_{\varepsilon\to 0}v\star\rho_\varepsilon\in\mathrm{Psh}(\Omega)$ by Th. 5.4. □

(5.9) Proposition. *The convex cone $\mathrm{Psh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$, and it has the property that every bounded subset is relatively compact.*

§5.B. Relations with Holomorphic Functions

In order to get a better geometric insight, we assume more generally that u is a C^2 function on a complex n-dimensional manifold X. The *complex Hessian* of u at a point $a\in X$ is the hermitian form on T_X defined by

$$(5.10)\qquad Hu_a=\sum_{1\le j,k\le n}\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}(a)\,dz_j\otimes d\overline{z}_k.$$

If $F:X\longrightarrow Y$ is a holomorphic mapping and if $v\in C^2(Y,\mathbb{R})$, we have $d'd''(v\circ F)=F^\star d'd''v$. In equivalent notations, a direct calculation gives for all $\xi\in T_{X,a}$

$$H(v\circ F)_a(\xi)=\sum_{j,k,l,m}\frac{\partial^2 v\big(F(a)\big)}{\partial z_l\partial\overline{z}_m}\,\frac{\partial F_l(a)}{\partial z_j}\xi_j\,\overline{\frac{\partial F_m(a)}{\partial z_k}\xi_k}=Hv_{F(a)}\big(F'(a).\xi\big).$$

In particular Hu_a does not depend on the choice of coordinates $(z_1,\ldots,z_n)$ on X, and $Hv_a\ge 0$ on Y implies $H(v\circ F)_a\ge 0$ on X. Therefore, the notion of plurisubharmonic function makes sense on any complex manifold.

(5.11) Theorem. *If $F : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ F \in \mathrm{Psh}(X)$.*

Proof. It is enough to prove the result when $X = \Omega_1 \subset \mathbb{C}^n$ and $X = \Omega_2 \subset \mathbb{C}^p$ are open subsets . The conclusion is already known when v is of class C^2, and it can be extended to an arbitrary upper semicontinuous function v by using Th. 5.4 and the fact that $v = \lim v \star \rho_\varepsilon$. □

(5.12) Example. By (3.22) we see that $\log|z|$ is subharmonic on $\mathbb{C}$, thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in \mathcal{O}(X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in \mathcal{O}(X)$ and $\alpha_j \geq 0$ (apply Th. 5.6 with $u_j = \alpha_j \log|f_j|$).

§5.C. Convexity Properties

The close analogy of plurisubharmonicity with the concept of convexity strongly suggests that there are deeper connections between these notions. We describe here a few elementary facts illustrating this philosophy. Another interesting connection between plurisubharmonicity and convexity will be seen in § 7.B (Kiselman's minimum principle).

(5.13) Theorem. *If $\Omega = \omega + \mathrm{i}\omega'$ where ω, ω' are open subsets of $\mathbb{R}^n$, and if $u(z)$ is a plurisubharmonic function on Ω that depends only on $x = \mathrm{Re}\, z$, then $\omega \ni x \longmapsto u(x)$ is convex.*

Proof. This is clear when $u \in C^2(\Omega, \mathbb{R})$, for $\partial^2 u/\partial z_j \partial \overline{z}_k = \frac{1}{4}\, \partial^2 u/\partial x_j \partial x_k$. In the general case, write $u = \lim u \star \rho_\varepsilon$ and observe that $u \star \rho_\varepsilon(z)$ depends only on x. □

(5.14) Corollary. *If u is a plurisubharmonic function in the open polydisk $D(a,R) = \prod D(a_j, R_j) \subset \mathbb{C}^n$, then*

$$\begin{aligned}
\mu(u\,; r_1, \ldots, r_n) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} u(a_1 + r_1 e^{\mathrm{i}\theta_1}, \ldots, a_n + r_n e^{\mathrm{i}\theta_n})\, d\theta_1 \ldots d\theta_n, \\
m(u\,; r_1, \ldots, r_n) &= \sup_{z \in D(a,r)} u(z_1, \ldots, z_n), \qquad r_j < R_j
\end{aligned}$$

are convex functions of $(\log r_1, \ldots, \log r_n)$ that are non decreasing in each variable.

Proof. That μ is non decreasing follows from the subharmonicity of u along every coordinate axis. Now, it is easy to verify that the functions

$$\widetilde{\mu}(z_1,\ldots,z_n) = \frac{1}{(2\pi)^n}\int_0^{2\pi} u(a_1+e^{z_1}e^{\mathrm{i}\theta_1},\ldots,a_n+e^{z_n}e^{\mathrm{i}\theta_n})\,d\theta_1\ldots d\theta_n,$$
$$\widetilde{m}(z_1,\ldots,z_n) = \sup_{|w_j|\le 1} u(a_1+e^{z_1}w_1,\ldots,a_n+e^{z_n}w_n)$$

are upper semicontinuous, satisfy the mean value inequality, and depend only on $\mathrm{Re}\, z_j \in \,]0,\log R_j[$. Therefore $\widetilde{\mu}$ and $\widetilde{M}$ are convex. Cor. 5.14 follows from the equalities

$$\mu(u\,;r_1,\ldots,r_n) = \widetilde{\mu}(\log r_1,\ldots,\log r_n),$$
$$m(u\,;r_1,\ldots,r_n) = \widetilde{m}(\log r_1,\ldots,\log r_n). \qquad \square$$

§5.D. Pluriharmonic Functions

Pluriharmonic functions are the counterpart of harmonic functions in the case of functions of complex variables:

(5.15) Definition. *A function u is said to be pluriharmonic if u and $-u$ are plurisubharmonic.*

A pluriharmonic function is harmonic (in particular smooth) in any $\mathbb{C}$-analytic coordinate system, and is characterized by the condition $Hu = 0$, i.e. $d'd''u = 0$ or

$$\partial^2 u/\partial z_j \partial \overline{z}_k = 0 \quad \text{for all} \ \ j,k.$$

If $f \in \mathcal{O}(X)$, it follows that the functions $\mathrm{Re}\, f$, $\mathrm{Im}\, f$ are pluriharmonic. Conversely:

(5.16) Theorem. *If the De Rham cohomology group $H^1_{\mathrm{DR}}(X,\mathbb{R})$ is zero, every pluriharmonic function u on X can be written $u = \mathrm{Re}\, f$ where f is a holomorphic function on X.*

Proof. By hypothesis $H^1_{\mathrm{DR}}(X,\mathbb{R}) = 0$, $u \in C^\infty(X)$ and $d(d'u) = d''d'u = 0$, hence there exists $g \in C^\infty(X)$ such that $dg = d'u$. Then dg is of type $(1,0)$, i.e. $g \in \mathcal{O}(X)$ and

$$d(u - 2\,\mathrm{Re}\, g) = d(u - g - \overline{g}) = (d'u - dg) + (d''u - d\overline{g}) = 0.$$

Therefore $u = \mathrm{Re}(2g + C)$, where C is a locally constant function. $\square$

§5.E. Global Regularization of Plurisubharmonic Functions

We now study a very efficient regularization and patching procedure for continuous plurisubharmonic functions, essentially due to (Richberg 1968). The main idea is contained in the following lemma:

(5.17) Lemma. *Let $u_\alpha \in \mathrm{Psh}(\Omega_\alpha)$ where $\Omega_\alpha \subset\subset X$ is a locally finite open covering of X. Assume that for every index β*

$$\limsup_{\zeta \to z} u_\beta(\zeta) < \max_{\Omega_\alpha \ni z} \{u_\alpha(z)\}$$

at all points $z \in \partial\Omega_\beta$. Then the function

$$u(z) = \max_{\Omega_\alpha \ni z} \; u_\alpha(z)$$

is plurisubharmonic on X.

Proof. Fix $z_0 \in X$. Then the indices β such that $z_0 \in \partial\Omega_\beta$ or $z_0 \notin \overline{\Omega}_\beta$ do not contribute to the maximum in a neighborhood of z_0. Hence there is a a finite set I of indices α such that $\Omega_\alpha \ni z_0$ and a neighborhood $V \subset \bigcap_{\alpha\in I} \Omega_\alpha$ on which $u(z) = \max_{\alpha\in I} u_\alpha(z)$. Therefore u is plurisubharmonic on V. □

The above patching procedure produces functions which are in general only continuous. When smooth functions are needed, one has to use a regularized max function. Let $\theta \in C^\infty(\mathbb{R},\mathbb{R})$ be a nonnegative function with support in $[-1,1]$ such that $\int_{\mathbb{R}} \theta(h)\,dh = 1$ and $\int_{\mathbb{R}} h\theta(h)\,dh = 0$.

(5.18) Lemma. *For arbitrary $\eta = (\eta_1,\ldots,\eta_p) \in \,]0,+\infty[^p$, the function*

$$M_\eta(t_1,\ldots,t_p) = \int_{\mathbb{R}^n} \max\{t_1+h_1,\ldots,t_p+h_p\} \prod_{1\le j\le n} \theta(h_j/\eta_j)\,dh_1\ldots dh_p$$

possesses the following properties:

a) *$M_\eta(t_1,\ldots,t_p)$ is non decreasing in all variables, smooth and convex on $\mathbb{R}^n$;*

b) $\max\{t_1,\ldots,t_p\} \le M_\eta(t_1,\ldots,t_p) \le \max\{t_1+\eta_1,\ldots,t_p+\eta_p\}$;

c) $M_\eta(t_1,\ldots,t_p) = M_{(\eta_1,\ldots,\widehat{\eta_j},\ldots,\eta_p)}(t_1,\ldots,\widehat{t_j},\ldots,t_p)$ *if $t_j + \eta_j \le \max_{k\ne j}\{t_k - \eta_k\}$;*

d) $M_\eta(t_1+a,\ldots,t_p+a) = M_\eta(t_1,\ldots,t_p) + a, \qquad \forall a \in \mathbb{R}$;

e) *if $u_1,\ldots,u_p$ are plurisubharmonic and satisfy $H(u_j)_z(\xi) \ge \gamma_z(\xi)$ where $z \mapsto \gamma_z$ is a continuous hermitian form on T_X, then $u = M_\eta(u_1,\ldots,u_p)$ is plurisubharmonic and satisfies $Hu_z(\xi) \ge \gamma_z(\xi)$.*

Proof. The change of variables $h_j \mapsto h_j - t_j$ shows that M_η is smooth. All properties are immediate consequences of the definition, except perhaps e).

That $M_\eta(u_1,\ldots,u_p)$ is plurisubharmonic follows from a) and Th. 5.6. Fix a point z_0 and $\varepsilon>0$. All functions $u'_j(z)=u_j(z)-\gamma_{z_0}(z-z_0)+\varepsilon|z-z_0|^2$ are plurisubharmonic near z_0. It follows that

$$M_\eta(u'_1,\ldots,u'_p)=u-\gamma_{z_0}(z-z_0)+\varepsilon|z-z_0|^2$$

is also plurisubharmonic near z_0. Since $\varepsilon>0$ was arbitrary, e) follows. □

(5.19) Corollary. *Let $u_\alpha\in C^\infty(\overline{\Omega}_\alpha)\cap\mathrm{Psh}(\Omega_\alpha)$ where $\Omega_\alpha\subset\subset X$ is a locally finite open covering of X. Assume that $u_\beta(z)<\max\{u_\alpha(z)\}$ at every point $z\in\partial\Omega_\beta$, when α runs over the indices such that $\Omega_\alpha\ni z$. Choose a family (η_α) of positive numbers so small that $u_\beta(z)+\eta_\beta\le\max_{\Omega_\alpha\ni z}\{u_\alpha(z)-\eta_\alpha\}$ for all β and $z\in\partial\Omega_\beta$. Then the function defined by*

$$\widetilde{u}(z)=M_{(\eta_\alpha)}\big(u_\alpha(z)\big)\qquad\text{for }\alpha\text{ such that }\ \Omega_\alpha\ni z$$

is smooth and plurisubharmonic on X. □

(5.20) Definition. *A function $u\in\mathrm{Psh}(X)$ is said to be strictly plurisubharmonic if $u\in L^1_{\mathrm{loc}}(X)$ and if for every point $x_0\in X$ there exists a neighborhood Ω of x_0 and $c>0$ such that $u(z)-c|z|^2$ is plurisubharmonic on Ω, i.e. $\sum(\partial^2u/\partial z_j\partial\overline{z}_k)\xi_j\overline{\xi}_k\ge c|\xi|^2$ (as distributions on Ω) for all $\xi\in\mathbb{C}^n$.*

(5.21) Theorem (Richberg 1968)**.** *Let $u\in\mathrm{Psh}(X)$ be a continuous function which is strictly plurisubharmonic on an open subset $\Omega\subset X$, with $Hu\ge\gamma$ for some continuous positive hermitian form γ on Ω. For any continuous function $\lambda\in C^0(\Omega)$, $\lambda>0$, there exists a plurisubharmonic function $\widetilde{u}$ in $C^0(X)\cap C^\infty(\Omega)$ such that $u\le\widetilde{u}\le u+\lambda$ on Ω and $\widetilde{u}=u$ on $X\smallsetminus\Omega$, which is strictly plurisubharmonic on Ω and satisfies $H\widetilde{u}\ge(1-\lambda)\gamma$. In particular, $\widetilde{u}$ can be chosen strictly plurisubharmonic on X if u has the same property.*

Proof. Let (Ω_α) be a locally finite open covering of Ω by relatively compact open balls contained in coordinate patches of X. Choose concentric balls $\Omega''_\alpha\subset\Omega'_\alpha\subset\Omega_\alpha$ of respective radii $r''_\alpha<r'_\alpha<r_\alpha$ and center $z=0$ in the given coordinates $z=(z_1,\ldots,z_n)$ near $\overline{\Omega}_\alpha$, such that Ω''_α still cover Ω. We set

$$u_\alpha(z)=u\star\rho_{\varepsilon_\alpha}(z)+\delta_\alpha(r'^2_\alpha-|z|^2)\qquad\text{on }\ \overline{\Omega}_\alpha.$$

For $\varepsilon_\alpha<\varepsilon_{\alpha,0}$ and $\delta_\alpha<\delta_{\alpha,0}$ small enough, we have $u_\alpha\le u+\lambda/2$ and $Hu_\alpha\ge(1-\lambda)\gamma$ on $\overline{\Omega}_\alpha$. Set

$$\eta_\alpha=\delta_\alpha\,\min\{r'^2_\alpha-r''^2_\alpha,(r^2_\alpha-r'^2_\alpha)/2\}.$$

Choose first $\delta_\alpha<\delta_{\alpha,0}$ such that $\eta_\alpha<\min_{\overline{\Omega}_\alpha}\lambda/2$, and then $\varepsilon_\alpha<\varepsilon_{\alpha,0}$ so small that $u\le u\star\rho_{\varepsilon_\alpha}<u+\eta_\alpha$ on $\overline{\Omega}_\alpha$. As $\delta_\alpha(r'^2-|z|^2)$ is $\le-2\eta_\alpha$ on $\partial\Omega_\alpha$

and $> \eta_\alpha$ on $\overline{\Omega}''_\alpha$, we have $u_\alpha < u - \eta_\alpha$ on $\partial\Omega_\alpha$ and $u_\alpha > u + \eta_\alpha$ on $\overline{\Omega}''_\alpha$, so that the condition required in Corollary 5.19 is satisfied. We define

$$\widetilde{u} = \begin{cases} u & \text{on } \; X \smallsetminus \Omega, \\ M_{(\eta_\alpha)}(u_\alpha) & \text{on } \; \Omega. \end{cases}$$

By construction, $\widetilde{u}$ is smooth on Ω and satisfies $u \le \widetilde{u} \le u+\lambda$, $Hu \ge (1-\lambda)\gamma$ thanks to 5.18 (b,e). In order to see that $\widetilde{u}$ is plurisubharmonic on X, observe that $\widetilde{u}$ is the uniform limit of $\widetilde{u}_\alpha$ with

$$\widetilde{u}_\alpha = \max\big\{u\,,\, M_{(\eta_1\ldots\eta_\alpha)}(u_1\ldots u_\alpha)\big\} \quad \text{on} \quad \bigcup_{1\le\beta\le\alpha} \Omega_\beta$$

and $\widetilde{u}_\alpha = u$ on the complement. □

§5.F. Polar and Pluripolar Sets.

Polar and pluripolar sets are sets of $-\infty$ poles of subharmonic and plurisubharmonic functions. Although these functions possess a large amount of flexibility, pluripolar sets have some properties which remind their loose relationship with holomorphic functions.

(5.22) Definition. *A set $A \subset \Omega \subset \mathbb{R}^m$ (resp. $A \subset X$, $\dim_{\mathbb{C}} X = n$) is said to be polar (resp. pluripolar) if for every point $x \in \Omega$ there exist a connected neighborhood W of x and $u \in \mathrm{Sh}(W)$ (resp. $u \in \mathrm{Psh}(W)$), $u \not\equiv -\infty$, such that $A \cap W \subset \{x \in W\ ;\ u(x) = -\infty\}$.*

Theorem 4.17 implies that a polar or pluripolar set is of zero Lebesgue measure. Now, we prove a simple extension theorem.

(5.23) Theorem. *Let $A \subset \Omega$ be a closed polar set and $v \in \mathrm{Sh}(\Omega \smallsetminus A)$ such that v is bounded above in a neighborhood of every point of A. Then v has a unique extension $\widetilde{v} \in \mathrm{Sh}(\Omega)$.*

Proof. The uniqueness is clear because A has zero Lebesgue measure. On the other hand, every point of A has a neighborhood W such that

$$A \cap W \subset \{x \in W\ ;\ u(x) = -\infty\}, \qquad u \in \mathrm{Sh}(W), \quad u \not\equiv -\infty.$$

After shrinking W and subtracting a constant to u, we may assume $u \le 0$. Then for every $\varepsilon > 0$ the function $v_\varepsilon = v + \varepsilon u \in \mathrm{Sh}(W \smallsetminus A)$ can be extended as an upper semicontinuous on W by setting $v_\varepsilon = -\infty$ on $A \cap W$. Moreover, v_ε satisfies the mean value inequality $v_\varepsilon(a) \le \mu_S(v_\varepsilon\,; a, r)$ if $a \in W \smallsetminus A$, $r < d(a, A \cup \complement W)$, and also clearly if $a \in A$, $r < d(a, \complement W)$. Therefore $v_\varepsilon \in \mathrm{Sh}(W)$ and $\widetilde{v} = (\sup v_\varepsilon)^\star \in \mathrm{Sh}(W)$. Clearly $\widetilde{v}$ coincides with v on $W \smallsetminus A$. A similar proof gives:

(5.24) Theorem. *Let A be a closed pluripolar set in a complex analytic manifold X. Then every function $v \in \mathrm{Psh}(X \smallsetminus A)$ that is locally bounded above near A extends uniquely into a function $\widetilde{v} \in \mathrm{Psh}(X)$.* □

(5.25) Corollary. *Let $A \subset X$ be a closed pluripolar set. Every holomorphic function $f \in \mathcal{O}(X \smallsetminus A)$ that is locally bounded near A extends to a holomorphic function $\widetilde{f} \in \mathcal{O}(X)$.*

Proof. Apply Th. 5.24 to $\pm \mathrm{Re}\, f$ and $\pm \mathrm{Im}\, f$. It follows that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ have pluriharmonic extensions to X, in particular f extends to $\widetilde{f} \in C^\infty(X)$. By density of $X \smallsetminus A$, $d''\widetilde{f} = 0$ on X. □

(5.26) Corollary. *Let $A \subset \Omega$ (resp. $A \subset X$) be a closed (pluri)polar set. If Ω (resp. X) is connected, then $\Omega \smallsetminus A$ (resp. $X \smallsetminus A$) is connected.*

Proof. If $\Omega \smallsetminus A$ (resp. $X \smallsetminus A$) is a disjoint union $\Omega_1 \cup \Omega_2$ of non empty open subsets, the function defined by $f \equiv 0$ on Ω_1, $f \equiv 1$ on Ω_2 would have a harmonic (resp. holomorphic) extension through A, a contradiction. □

§6. Domains of Holomorphy and Stein Manifolds

§6.A. Domains of Holomorphy in $\mathbb{C}^n$. Examples

Loosely speaking, a domain of holomorphy is an open subset Ω in $\mathbb{C}^n$ such that there is no part of $\partial\Omega$ across which all functions $f \in \mathcal{O}(\Omega)$ can be extended. More precisely:

(6.1) Definition. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. Ω is said to be a domain of holomorphy if for every connected open set $U \subset \mathbb{C}^n$ which meets $\partial\Omega$ and every connected component V of $U \cap \Omega$ there exists $f \in \mathcal{O}(\Omega)$ such that $f_{\restriction V}$ has no holomorphic extension to U.*

Under the hypotheses made on U, we have $\emptyset \neq \partial V \cap U \subset \partial\Omega$. In order to show that Ω is a domain of holomorphy, it is thus sufficient to find for every $z_0 \in \partial\Omega$ a function $f \in \mathcal{O}(\Omega)$ which is unbounded near z_0.

(6.2) Examples. Every open subset $\Omega \subset \mathbb{C}$ is a domain of holomorphy (for any $z_0 \in \partial\Omega$, $f(z) = (z - z_0)^{-1}$ cannot be extended at z_0). In $\mathbb{C}^n$, every *convex* open subset is a domain of holomorphy: if $\mathrm{Re}\langle z - z_0, \xi_0\rangle = 0$ is a supporting hyperplane of $\partial\Omega$ at z_0, the function $f(z) = (\langle z - z_0, \xi_0\rangle)^{-1}$ is holomorphic on Ω but cannot be extended at z_0.

(6.3) Hartogs figure. Assume that $n \geq 2$. Let $\omega \subset \mathbb{C}^{n-1}$ be a connected open set and $\omega' \subsetneq \omega$ an open subset. Consider the open sets in $\mathbb{C}^n$:

$$\begin{aligned} &\Omega = \big((D(R) \smallsetminus \overline{D}(r)) \times \omega\big) \cup \big(D(R) \times \omega'\big) && \text{(Hartogs figure)},\\ &\widetilde{\Omega} = D(R) \times \omega && \text{(filled Hartogs figure)}. \end{aligned}$$

where $0 \le r < R$ and $D(r) \subset \mathbb{C}$ denotes the open disk of center 0 and radius r in $\mathbb{C}$.

Then every function $f \in \mathcal{O}(\Omega)$ can be extended to $\widetilde{\Omega} = \omega \times D(R)$ by means of the Cauchy formula:

$$\widetilde{f}(z_1, z') = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta_1|=\rho} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1, \qquad z \in \widetilde{\Omega}, \quad \max\{|z_1|, r\} < \rho < R.$$

In fact $\widetilde{f} \in \mathcal{O}(D(R) \times \omega)$ and $\widetilde{f} = f$ on $D(R) \times \omega'$, so we must have $\widetilde{f} = f$ on Ω since Ω is connected. It follows that Ω is not a domain of holomorphy. Let us quote two interesting consequences of this example.

(6.4) Corollary (Riemann's extension theorem)**.** *Let X be a complex analytic manifold, and S a closed submanifold of codimension ≥ 2. Then every $f \in \mathcal{O}(X \smallsetminus S)$ extends holomorphically to X.*

Proof. This is a local result. We may choose coordinates $(z_1, \ldots, z_n)$ and a polydisk $D(R)^n$ in the corresponding chart such that $S \cap D(R)^n$ is given by equations $z_1 = \ldots = z_p = 0$, $p = \operatorname{codim} S \ge 2$. Then, denoting $\omega = D(R)^{n-1}$ and $\omega' = \omega \smallsetminus \{z_2 = \ldots = z_p = 0\}$, the complement $D(R)^n \smallsetminus S$ can be written as the Hartogs figure

$$D(R)^n \smallsetminus S = \big((D(R) \smallsetminus \{0\}) \times \omega\big) \cup \big(D(R) \times \omega'\big).$$

It follows that f can be extended to $\widetilde{\Omega} = D(R)^n$. $\square$

§6.B. Holomorphic Convexity and Pseudoconvexity

Let X be a complex manifold. We first introduce the notion of holomorphic hull of a compact set $K \subset X$. This can be seen somehow as the complex analogue of the notion of (affine) convex hull for a compact set in a real vector space. It is shown that domains of holomorphy in $\mathbb{C}^n$ are characterized a property of holomorphic convexity. Finally, we prove that holomorphic convexity implies pseudoconvexity – a complex analogue of the geometric notion of convexity.

(6.5) Definition. *Let X be a complex manifold and let K be a compact subset of X. Then the holomorphic hull of K in X is defined to be*

$$\widehat{K} = \widehat{K}_{\mathcal{O}(X)} = \big\{z \in X\,;\, |f(z)| \le \sup_K |f|,\ \forall f \in \mathcal{O}(X)\big\}.$$

(6.6) Elementary properties.

Fig. I-3 Hartogs figure

a) $\widehat{K}$ is a closed subset of X containing K. Moreover we have

$$\sup_{\widehat{K}} |f| = \sup_{K} |f|, \qquad \forall f \in \mathcal{O}(X),$$

hence $\widehat{\widehat{K}} = \widehat{K}$.

b) If $h : X \to Y$ is a holomorphic map and $K \subset X$ is a compact set, then $h(\widehat{K}_{\mathcal{O}(X)}) \subset \widehat{h(K)}_{\mathcal{O}(Y)}$. In particular, if $X \subset Y$, then $\widehat{K}_{\mathcal{O}(X)} \subset \widehat{K}_{\mathcal{O}(Y)} \cap X$. This is immediate from the definition.

c) $\widehat{K}$ contains the union of K with all relatively compact connected components of $X \smallsetminus K$ (thus $\widehat{K}$ "fills the holes" of K). In fact, for every connected component U of $X \smallsetminus K$ we have $\partial U \subset \partial K$, hence if $\overline{U}$ is compact the maximum principle yields

$$\sup_{\overline{U}} |f| = \sup_{\partial U} |f| \le \sup_{K} |f|, \qquad \text{for all } f \in \mathcal{O}(X).$$

d) More generally, suppose that there is a holomorphic map $h : U \to X$ defined on a relatively compact open set U in a complex manifold S, such that h extends as a continuous map $h : \overline{U} \to X$ and $h(\partial U) \subset K$. Then $h(\overline{U}) \subset \widehat{K}$. Indeed, for $f \in \mathcal{O}(X)$, the maximum principle again yields

$$\sup_{\overline{U}} |f \circ h| = \sup_{\partial U} |f \circ h| \le \sup_{K} |f|.$$

This is especially useful when U is the unit disk in $\mathbb{C}$.

e) Suppose that $X = \Omega \subset \mathbb{C}^n$ is an open set. By taking $f(z) = \exp(A(z))$ where A is an arbitrary affine function, we see that $\widehat{K}_{\mathcal{O}(\Omega)}$ is contained in the intersection of all affine half-spaces containing K. Hence $\widehat{K}_{\mathcal{O}(\Omega)}$ is contained in the affine convex hull $\widehat{K}_{\mathrm{aff}}$. As a consequence $\widehat{K}_{\mathcal{O}(\Omega)}$ is always bounded and $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is a compact set. However, when Ω is arbitrary, $\widehat{K}_{\mathcal{O}(\Omega)}$ is not always compact; for example, in case $\Omega = \mathbb{C}^n \smallsetminus \{0\}$, $n \geq 2$, then $\mathcal{O}(\Omega) = \mathcal{O}(\mathbb{C}^n)$ and the holomorphic hull of $K = S(0,1)$ is the non compact set $\widehat{K} = \overline{B}(0,1) \smallsetminus \{0\}$.

(6.7) Definition. *A complex manifold X is said to be holomorphically convex if the holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ of every compact set $K \subset X$ is compact.*

(6.8) Remark. *A complex manifold X is holomorphically convex if and only if there is an exhausting sequence of holomorphically compact subsets $K_\nu \subset X$, i.e. compact sets such that*

$$X = \bigcup K_\nu, \qquad \widehat{K}_\nu = K_\nu, \qquad K_\nu^\circ \supset K_{\nu-1}.$$

Indeed, if X is holomorphically convex, we may define K_ν inductively by $K_0 = \emptyset$ and $K_{\nu+1} = (K'_\nu \cup L_\nu)^\wedge_{\mathcal{O}(X)}$, where K'_ν is a neighborhood of K_ν and L_ν a sequence of compact sets of X such that $X = \bigcup L_\nu$. The converse is obvious: if such a sequence (K_ν) exists, then every compact subset $K \subset X$ is contained in some K_ν, hence $\widehat{K} \subset \widehat{K}_\nu = K_\nu$ is compact. □

We now concentrate on domains of holomorphy in $\mathbb{C}^n$. We denote by d and $B(z,r)$ the distance and the open balls associated to an arbitrary norm on $\mathbb{C}^n$, and we set for simplicity $B = B(0,1)$.

(6.9) Proposition. *If Ω is a domain of holomorphy and $K \subset \Omega$ is a compact subset, then $d(\widehat{K}, \complement\Omega) = d(K, \complement\Omega)$ and $\widehat{K}$ is compact.*

Proof. Let $f \in \mathcal{O}(\Omega)$. Given $r < d(K, \complement\Omega)$, we denote by M the supremum of $|f|$ on the compact subset $K + r\overline{B} \subset \Omega$. Then for every $z \in K$ and $\xi \in \overline{B}$, the function

$$(6.10) \quad \mathbb{C} \ni t \longmapsto f(z + t\xi) = \sum_{k=0}^{+\infty} \frac{1}{k!} D^k f(z)(\xi)^k t^k$$

is analytic in the disk $|t| < r$ and bounded by M. The Cauchy inequalities imply

$$|D^k f(z)(\xi)^k| \leq M k!\, r^{-k}, \qquad \forall z \in K, \quad \forall \xi \in \overline{B}.$$

As the left hand side is an analytic fuction of z in Ω, the inequality must also hold for $z \in \widehat{K}$, $\xi \in \overline{B}$. Every $f \in \mathcal{O}(\Omega)$ can thus be extended to any ball $B(z,r)$, $z \in \widehat{K}$, by means of the power series (6.10). Hence $B(z,r)$ must be contained in Ω, and this shows that $d(\widehat{K}, \complement\Omega) \geq r$. As $r < d(K, \complement\Omega)$ was arbitrary, we get $d(\widehat{K}, \complement\Omega) \geq d(K, \complement\Omega)$ and the converse inequality is clear, so $d(\widehat{K}, \complement\Omega) = d(K, \complement\Omega)$. As $\widehat{K}$ is bounded and closed in Ω, this shows that $\widehat{K}$ is compact. $\square$

(6.11) Theorem. *Let Ω be an open subset of $\mathbb{C}^n$. The following properties are equivalent:*

a) *Ω is a domain of holomorphy;*

b) *Ω is holomorphically convex;*

c) *For every countable subset $\{z_j\}_{j\in\mathbb{N}} \subset \Omega$ without accumulation points in Ω and every sequence of complex numbers (a_j), there exists an interpolation function $F \in \mathcal{O}(\Omega)$ such that $F(z_j) = a_j$.*

d) *There exists a function $F \in \mathcal{O}(\Omega)$ which is unbounded on any neighborhood of any point of $\partial\Omega$.*

Proof. d) $\Longrightarrow$ a) is obvious and a) $\Longrightarrow$ b) is a consequence of Prop. 6.9.

c) $\Longrightarrow$ d). If $\Omega = \mathbb{C}^n$ there is nothing to prove. Otherwise, select a dense sequence (ζ_j) in $\partial\Omega$ and take $z_j \in \Omega$ such that $d(z_j, \zeta_j) < 2^{-j}$. Then the interpolation function $F \in \mathcal{O}(\Omega)$ such that $F(z_j) = j$ satisfies d).

b) $\Longrightarrow$ c). Let $K_\nu \subset \Omega$ be an exhausting sequence of holomorphically convex compact sets as in Remark 6.8. Let $\nu(j)$ be the unique index ν such that $z_j \in K_{\nu(j)+1} \smallsetminus K_{\nu(j)}$. By the definition of a holomorphic hull, we can find a function $g_j \in \mathcal{O}(\Omega)$ such that

$$\sup_{K_{\nu(j)}} |g_j| < |g_j(z_j)|.$$

After multiplying g_j by a constant, we may assume that $g_j(z_j) = 1$. Let $P_j \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial equal to 1 at z_j and to 0 at $z_0, z_1, \ldots, z_{j-1}$. We set

$$F = \sum_{j=0}^{+\infty} \lambda_j P_j g_j^{m_j},$$

where $\lambda_j \in \mathbb{C}$ and $m_j \in \mathbb{N}$ are chosen inductively such that

$$\lambda_j = a_j - \sum_{0\leq k<j} \lambda_k P_k(z_j) g_k(z_j)^{m_k},$$

$$|\lambda_j P_j g_j^{m_j}| \leq 2^{-j} \quad \text{on} \quad K_{\nu(j)}\ ;$$

once λ_j has been chosen, the second condition holds as soon as m_j is large enough. Since $\{z_j\}$ has no accumulation point in Ω, the sequence $\nu(j)$ tends to $+\infty$, hence the series converges uniformly on compact sets. $\square$

We now show that a holomorphically convex manifold must satisfy some more geometric convexity condition, known as pseudoconvexity, which is most easily described in terms of the existence of plurisubharmonic exhaustion functions.

(6.12) Definition. *A function $\psi : X \longrightarrow [-\infty, +\infty[$ on a topological space X is said to be an exhaustion if all sublevel sets $X_c := \{z \in X\,;\, \psi(z) < c\}$, $c \in \mathbb{R}$, are relatively compact. Equivalently, ψ is an exhaustion if and only if ψ tends to $+\infty$ relatively to the filter of complements $X \smallsetminus K$ of compact subsets of X.*

A function ψ on an open set $\Omega \subset \mathbb{R}^n$ is thus an exhaustion if and only if $\psi(x) \to +\infty$ as $x \to \partial\Omega$ or $x \to \infty$. It is easy to check, cf. Exercise 8.8, that a connected open set $\Omega \subset \mathbb{R}^n$ is convex if and only if Ω has a locally convex exhaustion function. Since plurisubharmonic functions appear as the natural generalization of convex functions in complex analysis, we are led to the following definition.

(6.13) Definition. *Let X be a complex n-dimensional manifold. Then X is said to be*

a) *weakly pseudoconvex if there exists a smooth plurisubharmonic exhaustion function $\psi \in \mathrm{Psh}(X) \cap C^\infty(X)$;*
b) *strongly pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function $\psi \in \mathrm{Psh}(X) \cap C^\infty(X)$, i.e. $H\psi$ is positive definite at every point.*

(6.14) Theorem. *Every holomorphically convex manifold X is weakly pseudoconvex.*

Proof. Let (K_ν) be an exhausting sequence of holomorphically convex compact sets as in Remark 6.8. For every point $a \in L_\nu := K_{\nu+2} \smallsetminus K^\circ_{\nu+1}$, one can select $g_{\nu,a} \in \mathcal{O}(\Omega)$ such that $\sup_{K_\nu} |g_{\nu,a}| < 1$ and $|g_{\nu,a}(a)| > 1$. Then $|g_{\nu,a}(z)| > 1$ in a neighborhood of a ; by the Borel-Lebesgue lemma, one can find finitely many functions $(g_{\nu,a})_{a\in I_\nu}$ such that

$$\max_{a\in I_\nu}\big\{|g_{\nu,a}(z)|\big\} > 1 \quad\text{for}\quad z \in L_\nu, \qquad \max_{a\in I_\nu}\big\{|g_{\nu,a}(z)|\big\} < 1 \quad\text{for}\quad z \in K_\nu.$$

For a sufficiently large exponent $p(\nu)$ we get

$$\sum_{a\in I_\nu} |g_{\nu,a}|^{2p(\nu)} \geq \nu \quad\text{on}\quad L_\nu, \qquad \sum_{a\in I_\nu} |g_{\nu,a}|^{2p(\nu)} \leq 2^{-\nu} \quad\text{on}\quad K_\nu.$$

It follows that the series

$$\psi(z) = \sum_{\nu\in\mathbb{N}} \sum_{a\in I_\nu} |g_{\nu,a}(z)|^{2p(\nu)}$$

converges uniformly to a real analytic function $\psi \in \mathrm{Psh}(X)$ (see Exercise 8.11). By construction $\psi(z) \geq \nu$ for $z \in L_\nu$, hence ψ is an exhaustion. $\square$

(6.15) Example. The converse to Theorem 6.14 does not hold. In fact let $X = \mathbb{C}^2/\Gamma$ be the quotient of $\mathbb{C}^2$ by the free abelian group of rank 2 generated by the affine automorphisms

$$g_1(z,w) = (z+1, e^{\mathrm{i}\theta_1}w), \qquad g_2(z,w) = (z+\mathrm{i}, e^{\mathrm{i}\theta_2}w), \qquad \theta_1,\, \theta_2 \in \mathbb{R}.$$

Since Γ acts properly discontinuously on $\mathbb{C}^2$, the quotient has a structure of a complex (non compact) 2-dimensional manifold. The function $w \mapsto |w|^2$ is Γ-invariant, hence it induces a function $\psi((z,w)^\sim) = |w|^2$ on X which is in fact a plurisubharmonic exhaustion function. Therefore X is weakly pseudoconvex. On the other hand, any holomorphic function $f \in \mathcal{O}(X)$ corresponds to a Γ-invariant holomorphic function $\widetilde{f}(z,w)$ on $\mathbb{C}^2$. Then $z \mapsto \widetilde{f}(z,w)$ is bounded for w fixed, because $\widetilde{f}(z,w)$ lies in the image of the compact set $K \times \overline{D}(0,|w|)$, $K =$ unit square in $\mathbb{C}$. By Liouville's theorem, $\widetilde{f}(z,w)$ does not depend on z. Hence functions $f \in \mathcal{O}(X)$ are in one-to-one correspondence with holomorphic functions $\widetilde{f}(w)$ on $\mathbb{C}$ such that $\widetilde{f}(e^{\mathrm{i}\theta_j}w) = \widetilde{f}(w)$. By looking at the Taylor expansion at the origin, we conclude that $\widetilde{f}$ must be a constant if $\theta_1 \notin \mathbb{Q}$ or $\theta_1 \notin \mathbb{Q}$ (if $\theta_1, \theta_2 \in \mathbb{Q}$ and m is the least common denominator of θ_1, θ_2, then $\widetilde{f}$ is a power series of the form $\sum \alpha_k w^{mk}$). From this, it follows easily that X is holomorphically convex if and only if $\theta_1, \theta_2 \in \mathbb{Q}$.

§6.C. Stein Manifolds

The class of holomorphically convex manifolds contains two types of manifolds of a rather different nature:

- domains of holomorphy $X = \Omega \subset \mathbb{C}^n$;
- compact complex manifolds.

In the first case we have a lot of holomorphic functions, in fact the functions in $\mathcal{O}(\Omega)$ separate any pair of points of Ω. On the other hand, if X is compact and connected, the sets $\mathrm{Psh}(X)$ and $\mathcal{O}(X)$ consist of constant functions merely (by the maximum principle). It is therefore desirable to introduce a clear distinction between these two subclasses. For this purpose, (Stein 1951) introduced the class of manifolds which are now called Stein manifolds.

(6.16) Definition. *A complex manifold X is said to be a Stein manifold if*

a) *X is holomorphically convex;*

b) *$\mathcal{O}(X)$ locally separates points in X, i.e. every point $x \in X$ has a neighborhood V such that for any $y \in V \smallsetminus \{x\}$ there exists $f \in \mathcal{O}(X)$ with $f(y) \neq f(x)$.*

The second condition is automatic if $X = \Omega$ is an open subset of $\mathbb{C}^n$. Hence an open set $\Omega \subset \mathbb{C}^n$ is Stein if and only if Ω is a domain of holomorphy.

(6.17) Lemma. *If a complex manifold X satisfies the axiom* (6.16 b) *of local separation, there exists a smooth nonnegative strictly plurisubharmonic function $u \in \mathrm{Psh}(X)$.*

Proof. Fix $x_0 \in X$. We first show that there exists a smooth nonnegative function $u_0 \in \mathrm{Psh}(X)$ which is strictly plurisubharmonic on a neighborhood of x_0. Let $(z_1, \ldots, z_n)$ be local analytic coordinates centered at x_0, and if necessary, replace z_j by λz_j so that the closed unit ball $\overline{B} = \{\sum |z_j|^2 \leq 1\}$ is contained in the neighborhood $V \ni x_0$ on which (6.16 b) holds. Then, for every point $y \in \partial B$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(y) \neq f(x_0)$. Replacing f with $\lambda(f - f(x_0))$, we can achieve $f(x_0) = 0$ and $|f(y)| > 1$. By compactness of ∂B, we find finitely many functions $f_1, \ldots, f_N \in \mathcal{O}(X)$ such that $v_0 = \sum |f_j|^2$ satisfies $v_0(x_0) = 0$, while $v_0 \geq 1$ on ∂B. Now, we set

$$u_0(z) = \begin{cases} v_0(z) & \text{on } X \smallsetminus B, \\ M_\varepsilon\{v_0(z), (|z|^2 + 1)/3\} & \text{on } B. \end{cases}$$

where M_ε are the regularized max functions defined in 5.18. Then u_0 is smooth and plurisubharmonic, coincides with v_0 near ∂B and with $(|z|^2+1)/3$ on a neighborhood of x_0. We can cover X by countably many neighborhoods $(V_j)_{j \geq 1}$, for which we have a smooth plurisubharmonic functions $u_j \in \mathrm{Psh}(X)$ such that u_j is strictly plurisubharmonic on V_j. Then select a sequence $\varepsilon_j > 0$ converging to 0 so fast that $u = \sum \varepsilon_j u_j \in C^\infty(X)$. The function u is nonnegative and strictly plurisubharmonic everywhere on X. □

(6.18) Theorem. *Every Stein manifold is strongly pseudoconvex.*

Proof. By Th. 6.14, there is a smooth exhaustion function $\psi \in \mathrm{Psh}(X)$. If $u \geq 0$ is strictly plurisubharmonic, then $\psi' = \psi + u$ is a strictly plurisubharmonic exhaustion. □

The converse problem to know whether every strongly pseudoconvex manifold is actually a Stein manifold is known as the *Levi problem*, and was raised by (Levi 1910) in the case of domains $\Omega \subset \mathbb{C}^n$. In that case, the problem has been solved in the affirmative independently by (Oka 1953), (Norguet 1954) and (Bremermann 1954). The general solution of the Levi problem has been obtained by (Grauert 1958). Our proof will rely on the theory of L^2 estimates for d'', which will be available only in Chapter VIII.

(6.19) Remark. It will be shown later that Stein manifolds always have enough holomorphic functions to separate finitely many points, and one can

Fig. I-4 Hartogs figure with excrescence

even interpolate given values of a function and its derivatives of some fixed order at any discrete set of points. In particular, we might have replaced condition (6.16 b) by the stronger requirement that $\mathcal{O}(X)$ separates any pair of points. On the other hand, there are examples of manifolds satisfying the local separation condition (6.16 b), but not global separation. A simple example is obtained by attaching an excrescence inside a Hartogs figure, in such a way that the resulting map $\pi : X \to D = D(0,1)^2$ is not one-to-one (see Figure I-4 above); then $\mathcal{O}(X)$ coincides with $\pi^\star\mathcal{O}(D)$.

§6.D. Heredity Properties

Holomorphic convexity and pseudoconvexity are preserved under quite a number of natural constructions. The main heredity properties can be summarized in the following Proposition.

(6.20) Proposition. *Let $\mathcal{C}$ denote the class of holomorphically convex (resp. of Stein, or weakly pseudoconvex, strongly pseudoconvex manifolds).*

a) *If $X, Y \in \mathcal{C}$, then $X \times Y \in \mathcal{C}$.*

b) *If $X \in \mathcal{C}$ and S is a closed complex submanifold of X, then $S \in \mathcal{C}$.*

c) *If $(S_j)_{1\le j\le N}$ is a collection of (not necessarily closed) submanifolds of a complex manifold X such that $S = \bigcap S_j$ is a submanifold of X, and if $S_j \in \mathcal{C}$ for all j, then $S \in \mathcal{C}$.*

d) *If $F : X \to Y$ is a holomorphic map and $S \subset X$, $S' \subset Y$ are (not necessarily closed) submanifolds in the class $\mathcal{C}$, then $S \cap F^{-1}(S')$ is in $\mathcal{C}$, as long as it is a submanifold of X.*

e) *If X is a weakly (resp. strongly) pseudoconvex manifold and u is a smooth plurisubharmonic function on X, then the open set $\Omega = u^{-1}(]-\infty, c[$ is weakly (resp. strongly) pseudoconvex. In particular the sublevel sets*

$$X_c = \psi^{-1}(]-\infty, c[)$$

of a (strictly) plurisubharmonic exhaustion function are weakly (resp. strongly) pseudoconvex.

Proof. All properties are more or less immediate to check, so we only give the main facts.

a) For $K \subset X$, $L \subset Y$ compact, we have $(K \times L)^{\wedge}_{\mathcal{O}(X\times Y)} = \widehat{K}_{\mathcal{O}(X)} \times \widehat{K}_{\mathcal{O}(Y)}$, and if φ, ψ are plurisubharmonic exhaustions of X, Y, then $\varphi(x) + \psi(y)$ is a plurisubharmonic exhaustion of $X \times Y$.

b) For a compact set $K \subset S$, we have $\widehat{K}_{\mathcal{O}(S)} \subset \widehat{K}_{\mathcal{O}(X)} \cap S$, and if $\psi \in \mathrm{Psh}(X)$ is an exhaustion, then $\psi\restriction S \in \mathrm{Psh}(S)$ is an exhaustion (since S is closed).

c) $\bigcap S_j$ is a closed submanifold in $\prod S_j$ (equal to its intersection with the diagonal of X^N).

d) For a compact set $K \subset S \cap F^{-1}(S')$, we have

$$\widehat{K}_{\mathcal{O}(S\cap F^{-1}(S'))} \subset \widehat{K}_{\mathcal{O}(S)} \cap F^{-1}(\widehat{F(K)}_{\mathcal{O}(S')}),$$

and if φ, ψ are plurisubharmonic exhaustions of S, S', then $\varphi + \psi \circ F$ is a plurisubharmonic exhaustion of $S \cap F^{-1}(S')$.

e) $\varphi(z) := \psi(z) + 1/(c - u(z))$ is a (strictly) plurisubharmonic exhaustion function on Ω. □

§7. Pseudoconvex Open Sets in $\mathbb{C}^n$

§7.A. Geometric Characterizations of Pseudoconvex Open Sets

We first discuss some characterizations of pseudoconvex open sets in $\mathbb{C}^n$. We will need the following elementary criterion for plurisubharmonicity.

(7.1) Criterion. *Let $v : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. Then v is plurisubharmonic if and only if for every closed disk $\overline{\Delta} = z_0 + \overline{D}(1)\eta \subset \Omega$ and every polynomial $P \in \mathbb{C}[t]$ such that $v(z_0 + t\eta) \le \operatorname{Re} P(t)$ for $|t| = 1$, then $v(z_0) \le \operatorname{Re} P(0)$.*

Proof. The condition is necessary because $t \longmapsto v(z_0 + t\eta) - \operatorname{Re} P(t)$ is subharmonic in a neighborhood of $\overline{D}(1)$, so it satisfies the maximum principle on $D(1)$ by Th. 4.14. Let us prove now the sufficiency. The upper semicontinuity of v implies $v = \lim_{\nu\to+\infty} v_\nu$ on $\partial\Delta$ where (v_ν) is a strictly decreasing sequence of continuous functions on $\partial\Delta$. As trigonometric polynomials are dense in $C^0(S^1, \mathbb{R})$, we may assume $v_\nu(z_0 + e^{\mathrm{i}\theta}\eta) = \operatorname{Re} P_\nu(e^{\mathrm{i}\theta})$, $P_\nu \in \mathbb{C}[t]$. Then $v(z_0 + t\eta) \le \operatorname{Re} P_\nu(t)$ for $|t| = 1$, and the hypothesis implies

$$v(z_0) \le \operatorname{Re} P_\nu(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} P_\nu(e^{\mathrm{i}\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} v_\nu(z_0 + e^{\mathrm{i}\theta}\eta)\,d\theta.$$

Taking the limit when ν tends to $+\infty$ shows that v satisfies the mean value inequality (5.2). □

For any $z \in \Omega$ and $\xi \in \mathbb{C}^n$, we denote by

$$\delta_\Omega(z,\xi) = \sup\big\{r > 0\ ;\ z + D(r)\,\xi \subset \Omega\big\}$$

the distance from z to $\partial\Omega$ in the complex direction ξ.

(7.2) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. The following properties are equivalent:*

a) *Ω is strongly pseudoconvex* (*according to Def. 6.13* b);

b) *Ω is weakly pseudoconvex*;

c) *Ω has a plurisubharmonic exhaustion function ψ.*

d) $-\log\delta_\Omega(z,\xi)$ *is plurisubharmonic on $\Omega \times \mathbb{C}^n$* ;

e) $-\log d(z,\complement\Omega)$ *is plurisubharmonic on Ω.*

If one of these properties hold, Ω is said to be a pseudoconvex open set.

Proof. The implications a) $\Longrightarrow$ b) $\Longrightarrow$ c) are obvious. For the implication c) $\Longrightarrow$ d), we use Criterion 7.1. Consider a disk $\overline{\Delta} = (z_0,\xi_0) + \overline{D}(1)\,(\eta,\alpha)$ in $\Omega \times \mathbb{C}^n$ and a polynomial $P \in \mathbb{C}[t]$ such that

$$-\log\delta_\Omega(z_0 + t\eta, \xi_0 + t\alpha) \le \operatorname{Re} P(t) \quad \text{for} \quad |t| = 1.$$

We have to verify that the inequality also holds when $|t| < 1$. Consider the holomorphic mapping $h : \mathbb{C}^2 \longrightarrow \mathbb{C}^n$ defined by

$$h(t,w) = z_0 + t\eta + we^{-P(t)}(\xi_0 + t\alpha).$$

By hypothesis

$$h\big(\overline{D}(1)\times\{0\}\big) = \mathrm{pr}_1(\overline{\Delta}) \subset \Omega,$$
$$h\big(\partial D(1)\times D(1)\big) \subset \Omega \quad (\text{since } |e^{-P}| \le \delta_\Omega \ \text{ on } \partial\Delta),$$

and the desired conclusion is that $h\big(\overline{D}(1)\times D(1)\big) \subset \Omega$. Let J be the set of radii $r \ge 0$ such that $h\big(\overline{D}(1)\times\overline{D}(r)\big) \subset \Omega$. Then J is an open interval $[0,R[$, $R > 0$. If $R < 1$, we get a contradiction as follows. Let $\psi \in \mathrm{Psh}(\Omega)$ be an exhaustion function and

$$K = h\big(\partial D(1)\times\overline{D}(R)\big) \subset\subset \Omega, \qquad c = \sup_K \psi.$$

As $\psi\circ h$ is plurisubharmonic on a neighborhood of $\overline{D}(1)\times D(R)$, the maximum principle applied with respect to t implies

$$\psi \circ h(t,w) \le c \quad \text{on} \quad \overline{D}(1) \times D(R),$$

hence $h\big(\overline{D}(1) \times D(R)\big) \subset \Omega_c \subset\subset \Omega$ and $h\big(\overline{D}(1) \times \overline{D}(R+\varepsilon)\big) \subset \Omega$ for some $\varepsilon > 0$, a contradiction.

d) $\Longrightarrow$ e). The function $-\log d(z, \complement\Omega)$ is continuous on Ω and satisfies the mean value inequality because

$$-\log d(z, \complement\Omega) = \sup_{\xi \in \overline{B}} \big(-\log \delta_\Omega(z,\xi) \big).$$

e) $\Longrightarrow$ a). It is clear that

$$u(z) = |z|^2 + \max\{\log d(z, \complement\Omega)^{-1}, 0\}$$

is a continuous strictly plurisubharmonic exhaustion function. Richberg's theorem 5.21 implies that there exists $\psi \in C^\infty(\Omega)$ strictly plurisubharmonic such that $u \le \psi \le u+1$. Then ψ is the required exhaustion function. □

(7.3) Proposition.

a) *Let $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^p$ be pseudoconvex. Then $\Omega \times \Omega'$ is pseudoconvex. For every holomorphic map $F : \Omega \to \mathbb{C}^p$ the inverse image $F^{-1}(\Omega')$ is pseudoconvex.*

b) *If $(\Omega_\alpha)_{\alpha \in I}$ is a family of pseudoconvex open subsets of $\mathbb{C}^n$, the interior of the intersection $\Omega = \big(\bigcap_{\alpha\in I} \Omega_\alpha\big)^\circ$ is pseudoconvex.*

c) *If $(\Omega_j)_{j\in\mathbb{N}}$ is a non decreasing sequence of pseudoconvex open subsets of $\mathbb{C}^n$, then $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$ is pseudoconvex.*

Proof. a) Let φ, ψ be smooth plurisubharmonic exhaustions of Ω, Ω'. Then $(z,w) \longmapsto \varphi(z) + \psi(w)$ is an exhaustion of $\Omega \times \Omega'$ and $z \longmapsto \varphi(z) + \psi(F(z))$ is an exhaustion of $F^{-1}(\Omega')$.

b) We have $-\log d(z, \complement\Omega) = \sup_{\alpha\in I} -\log d(z, \complement\Omega_\alpha)$, so this function is plurisubharmonic.

c) The limit $-\log d(z, \complement\Omega) = \lim\!\downarrow_{j\to+\infty} -\log d(z, \complement\Omega_j)$ is plurisubharmonic, hence Ω is pseudoconvex. This result cannot be generalized to strongly pseudoconvex manifolds: J.E. Fornaess in (Fornaess 1977) has constructed an increasing sequence of 2-dimensional Stein (even affine algebraic) manifolds X_ν whose union is not Stein; see Exercise 8.16. □

(7.4) Examples.

a) An *analytic polyhedron* in $\mathbb{C}^n$ is an open subset of the form

$$P = \{z \in \mathbb{C}^n \,;\, |f_j(z)| < 1,\ 1 \le j \le N\}$$

where $(f_j)_{1\le j\le N}$ is a family of analytic functions on $\mathbb{C}^n$. By 7.3 a), every analytic polyhedron is pseudoconvex.

b) Let $\omega \subset \mathbb{C}^{n-1}$ be pseudoconvex and let $u : \omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. Then the *Hartogs domain*

$$\Omega = \big\{(z_1, z') \in \mathbb{C} \times \omega\,;\ \log|z_1| + u(z') < 0\big\}$$

is pseudoconvex if and and only if u is plurisubharmonic. To see that the plurisubharmonicity of u is necessary, observe that

$$u(z') = -\log \delta_\Omega\big((0, z'), (1, 0)\big).$$

Conversely, assume that u is plurisubharmonic and continuous. If ψ is a plurisubharmonic exhaustion of ω, then

$$\psi(z') + \big|\log|z_1| + u(z')\big|^{-1}$$

is an exhaustion of Ω. This is no longer true if u is not continuous, but in this case we may apply Property 7.3 c) to conclude that

$$\Omega_\varepsilon = \big\{(z_1, z')\,;\ d(z', \complement\omega) > \varepsilon,\ \log|z_1| + u \star \rho_\varepsilon(z') < 0\big\}, \qquad \Omega = \bigcup \Omega_\varepsilon$$

are pseudoconvex.

c) An open set $\Omega \subset \mathbb{C}^n$ is called a *tube* of base ω if $\Omega = \omega + \mathrm{i}\mathbb{R}^n$ for some open subset $\omega \subset \mathbb{R}^n$. Then of course $-\log d(z, \complement\Omega) = -\log(x, \complement\omega)$ depends only on the real part $x = \mathrm{Re}\, z$. By Th. 5.13, this function is plurisubharmonic if and only if it is locally convex in x. Therefore Ω if pseudoconvex if and only if every connected component of ω is convex.

d) An open set $\Omega \subset \mathbb{C}^n$ is called a *Reinhardt domain* if $(e^{\mathrm{i}\theta_1} z_1, \ldots, e^{\mathrm{i}\theta_n} z_n)$ is in Ω for every $z = (z_1, \ldots, z_n) \in \Omega$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}^n$. For such a domain, we consider the *logarithmic indicatrix*

$$\omega^\star = \Omega^\star \cap \mathbb{R}^n \quad \text{with} \quad \Omega^\star = \{\zeta \in \mathbb{C}^n\,;\ (e^{\zeta_1}, \ldots, e^{\zeta_n}) \in \Omega\}.$$

It is clear that $\Omega^\star$ is a tube of base $\omega^\star$. Therefore every connected component of $\omega^\star$ must be convex if Ω is pseudoconvex. The converse is not true: $\Omega = \mathbb{C}^n \smallsetminus \{0\}$ is not pseudoconvex for $n \geq 2$ although $\omega^\star = \mathbb{R}^n$ is convex. However, the Reinhardt open set

$$\Omega^\bullet = \big\{(z_1, \ldots, z_n) \in (\mathbb{C} \smallsetminus \{0\})^n\,;\ (\log|z_1|, \ldots, \log|z_n|) \in \omega^\star\big\} \subset \Omega$$

is easily seen to be pseudoconvex if $\omega^\star$ is convex: if χ is a convex exhaustion of $\omega^\star$, then $\psi(z) = \chi(\log|z_1|, \ldots, \log|z_n|)$ is a plurisubharmonic exhaustion of $\Omega^\bullet$. Similarly, if $\omega^\star$ is convex and such that $x \in \omega^\star \Longrightarrow y \in \omega^\star$ for $y_j \leq x_j$, we can take χ increasing in all variables and tending to $+\infty$ on $\partial\omega^\star$, hence the set

$$\widetilde{\Omega} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n\,;\ |z_j| \leq e^{x_j} \ \text{ for some } \ x \in \omega^\star\big\}$$

is a pseudoconvex Reinhardt open set containing 0. □

§7.B. Kiselman's Minimum Principle

We already know that a maximum of plurisubharmonic functions is plurisubharmonic. However, if v is a plurisubharmonic function on $X \times \mathbb{C}^n$, the partial minimum function on X defined by $u(\zeta) = \inf_{z\in\Omega} v(\zeta, z)$ need not be plurisubharmonic. A simple counterexample in $\mathbb{C} \times \mathbb{C}$ is given by

$$v(\zeta, z) = |z|^2 + 2\,\mathrm{Re}(z\zeta) = |z + \overline{\zeta}|^2 - |\zeta|^2, \qquad u(\zeta) = -|\zeta|^2.$$

It follows that the image $F(\Omega)$ of a pseudoconvex open set Ω by a holomorphic map F need not be pseudoconvex. In fact, if

$$\Omega = \{(t, \zeta, z) \in \mathbb{C}^3 \,;\, \log|t| + v(\zeta, z) < 0\}$$

and if $\Omega' \subset \mathbb{C}^2$ is the image of Ω by the projection map $(t, \zeta, z) \longmapsto (t, \zeta)$, then $\Omega' = \{(t, \zeta) \in \mathbb{C}^2 \,;\, \log|t| + u(\zeta) < 0\}$ is not pseudoconvex. However, the minimum property holds true when $v(\zeta, z)$ depends only on $\mathrm{Re}\, z$:

(7.5) Theorem (Kiselman 1978). *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a pseudoconvex open set such that each slice*

$$\Omega_\zeta = \{z \in \mathbb{C}^n \,;\, (\zeta, z) \in \Omega\}, \qquad \zeta \in \mathbb{C}^p,$$

is a convex tube $\omega_\zeta + \mathrm{i}\mathbb{R}^n$, $\omega_\zeta \subset \mathbb{R}^n$. For every plurisubharmonic function $v(\zeta, z)$ on Ω that does not depend on $\mathrm{Im}\, z$*, the function*

$$u(\zeta) = \inf_{z\in\Omega_\zeta} v(\zeta, z)$$

is plurisubharmonic or locally $\equiv -\infty$ on $\Omega' = \mathrm{pr}_{\mathbb{C}^n}(\Omega)$.

Proof. The hypothesis implies that $v(\zeta, z)$ is convex in $x = \mathrm{Re}\, z$. In addition, we first assume that v is smooth, plurisubharmonic in (ζ, z), strictly convex in x and $\lim_{x\to\{\infty\}\cup\partial\omega_\zeta} v(\zeta, x) = +\infty$ for every $\zeta \in \Omega'$. Then $x \longmapsto v(\zeta, x)$ has a unique minimum point $x = g(\zeta)$, solution of the equations $\partial v/\partial x_j(x, \zeta) = 0$. As the matrix $(\partial^2 v/\partial x_j \partial x_k)$ is positive definite, the implicit function theorem shows that g is smooth. Now, if $\mathbb{C} \ni w \longmapsto \zeta_0 + wa$, $a \in \mathbb{C}^n$, $|w| \le 1$ is a complex disk Δ contained in Ω, there exists a holomorphic function f on the unit disk, smooth up to the boundary, whose real part solves the Dirichlet problem

$$\mathrm{Re}\, f(e^{\mathrm{i}\theta}) = g(\zeta_0 + e^{\mathrm{i}\theta} a).$$

Since $v(\zeta_0 + wa, f(w))$ is subharmonic in w, we get the mean value inequality

$$v(\zeta_0, f(0)) \le \frac{1}{2\pi}\int_0^{2\pi} v\big(\zeta_0 + e^{\mathrm{i}\theta} a, f(e^{\mathrm{i}\theta})\big)d\theta = \frac{1}{2\pi}\int_{\partial\Delta} v(\zeta, g(\zeta))d\theta.$$

The last equality holds because $\operatorname{Re} f = g$ on $\partial\Delta$ and $v(\zeta, z) = v(\zeta, \operatorname{Re} z)$ by hypothesis. As $u(\zeta_0) \le v(\zeta_0, f(0))$ and $u(\zeta) = v(\zeta, g(\zeta))$, we see that u satisfies the mean value inequality, thus u is plurisubharmonic.

Now, this result can be extended to arbitrary functions v as follows: let $\psi(\zeta, z) \ge 0$ be a continuous plurisubharmonic function on Ω which is independent of $\operatorname{Im} z$ and is an exhaustion of $\Omega \cap (\mathbb{C}^p \times \mathbb{R}^n)$, e.g.

$$\psi(\zeta, z) = \max\{|\zeta|^2 + |\operatorname{Re} z|^2, -\log \delta_\Omega(\zeta, z)\}.$$

There is slowly increasing sequence $C_j \to +\infty$ such that each function $\psi_j = (C_j - \psi \star \rho_{1/j})^{-1}$ is an "exhaustion" of a pseudoconvex open set $\Omega_j \subset\subset \Omega$ whose slices are convex tubes and such that $d(\Omega_j, \complement\Omega) > 2/j$. Then

$$v_j(\zeta, z) = v \star \rho_{1/j}(\zeta, z) + \frac{1}{j}|\operatorname{Re} z|^2 + \psi_j(\zeta, z)$$

is a decreasing sequence of plurisubharmonic functions on Ω_j satisfying our previous conditions. As $v = \lim v_j$, we see that $u = \lim u_j$ is plurisubharmonic. □

(7.6) Corollary. *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a pseudoconvex open set such that all slices Ω_ζ, $\zeta \in \mathbb{C}^p$, are convex tubes in $\mathbb{C}^n$. Then the projection Ω' of Ω on $\mathbb{C}^p$ is pseudoconvex.*

Proof. Take $v \in \operatorname{Psh}(\Omega)$ equal to the function ψ defined in the proof of Th. 7.5. Then u is a plurisubharmonic exhaustion of Ω'. □

§7.C. Levi Form of the Boundary

For an arbitrary domain in $\mathbb{C}^n$, we first show that pseudoconvexity is a local property of the boundary.

(7.7) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset such that every point $z_0 \in \partial\Omega$ has a neighborhood V such that $\Omega \cap V$ is pseudoconvex. Then Ω is pseudoconvex.*

Proof. As $d(z, \complement\Omega)$ coincides with $d\big(z, \complement(\Omega \cap V)\big)$ in a neighborhood of z_0, we see that there exists a neighborhood U of $\partial\Omega$ such that $-\log d(z, \complement\Omega)$ is plurisubharmonic on $\Omega \cap U$. Choose a convex increasing function χ such that

$$\chi(r) > \sup_{(\Omega \smallsetminus U) \cap \overline{B}(0,r)} -\log d(z, \complement\Omega), \qquad \forall r \ge 0.$$

Then the function

$$\psi(z) = \max\big\{\chi(|z|), -\log d(z, \complement\Omega)\big\}$$

coincides with $\chi(|z|)$ in a neighborhood of $\Omega \smallsetminus U$. Therefore $\psi \in \mathrm{Psh}(\Omega)$, and ψ is clearly an exhaustion. $\square$

Now, we give a geometric characterization of the pseudoconvexity property when $\partial\Omega$ is of class C^2. Let $\rho \in C^2(\overline{\Omega})$ be a *defining function* of Ω, i.e. a function such that

$$(7.9)\quad \rho < 0 \ \text{ on } \ \Omega, \qquad \rho = 0 \ \text{ and } \ d\rho \neq 0 \ \text{ on } \ \partial\Omega.$$

The *holomorphic tangent space* to $\partial\Omega$ is by definition the largest complex subspace which is contained in the tangent space $T_{\partial\Omega}$ to the boundary:

$$(7.9)\quad {}^hT_{\partial\Omega} = T_{\partial\Omega} \cap JT_{\partial\Omega}.$$

It is easy to see that ${}^hT_{\partial\Omega,z}$ is the complex hyperplane of vectors $\xi \in \mathbb{C}^n$ such that

$$d'\rho(z)\cdot\xi = \sum_{1\le j\le n} \frac{\partial\rho}{\partial z_j}\,\xi_j = 0.$$

The *Levi form* on ${}^hT_{\partial\Omega}$ is defined at every point $z \in \partial\Omega$ by

$$(7.10)\quad L_{\partial\Omega,z}(\xi) = \frac{1}{|\nabla\rho(z)|}\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\xi}_k, \qquad \xi \in {}^hT_{\partial\Omega,z}.$$

The Levi form does not depend on the particular choice of ρ, as can be seen from the following intrinsic computation of $L_{\partial\Omega}$ (we still denote by $L_{\partial\Omega}$ the associated sesquilinear form).

(7.11) Lemma. *Let ξ,η be C^1 vector fields on $\partial\Omega$ with values in ${}^hT_{\partial\Omega}$. Then*

$$\langle[\xi,\eta], J\nu\rangle = 4\,\mathrm{Im}\,L_{\partial\Omega}(\xi,\eta)$$

where ν is the outward normal unit vector to $\partial\Omega$, $[\ ,\ ]$ the Lie bracket of vector fields and $\langle\ ,\ \rangle$ the hermitian inner product.

Proof. Extend first ξ,η as vector fields in a neighborhood of $\partial\Omega$ and set

$$\xi' = \sum \xi_j\frac{\partial}{\partial z_j} = \frac{1}{2}(\xi - \mathrm{i}J\xi), \qquad \eta'' = \sum \overline{\eta}_k\frac{\partial}{\partial\overline{z}_k} = \frac{1}{2}(\eta + \mathrm{i}J\eta).$$

As $\xi, J\xi, \eta, J\eta$ are tangent to $\partial\Omega$, we get on $\partial\Omega$:

$$0 = \xi'.(\eta''.\rho) + \eta''.(\xi'.\rho) = \sum_{1\le j,k\le n} 2\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\eta}_k + \xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k} + \overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial\rho}{\partial z_j}.$$

Since $[\xi,\eta]$ is also tangent to $\partial\Omega$, we have $\mathrm{Re}\langle[\xi,\eta],\nu\rangle = 0$, hence $\langle J[\xi,\eta],\nu\rangle$ is real and

$$\langle[\xi,\eta],J\nu\rangle=-\langle J[\xi,\eta],\nu\rangle=-\frac{1}{|\nabla\rho|}\big(J[\xi,\eta].\rho\big)=-\frac{2}{|\nabla\rho|}\operatorname{Re}\big(J[\xi',\eta''].\rho\big)$$

because $J[\xi',\eta']=i[\xi',\eta']$ and its conjugate $J[\xi'',\eta'']$ are tangent to $\partial\Omega$. We find now

$$J[\xi',\eta'']=-\mathrm{i}\sum\xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial}{\partial\overline{z}_k}+\overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial}{\partial\overline{z}_j},$$

$$\operatorname{Re}\big(J[\xi',\eta''].\rho\big)=\operatorname{Im}\sum\xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}+\overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial\rho}{\partial z_j}=-2\operatorname{Im}\sum\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\eta}_k,$$

$$\langle[\xi,\eta],J\nu\rangle=\frac{4}{|\nabla\rho|}\operatorname{Im}\sum\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\eta}_k=4\operatorname{Im}L_{\partial\Omega}(\xi,\eta).\qquad\square$$

(7.12) Theorem. *An open subset $\Omega\subset\mathbb{C}^n$ with C^2 boundary is pseudoconvex if and only if the Levi form $L_{\partial\Omega}$ is semipositive at every point of $\partial\Omega$.*

Proof. Set $\delta(z)=d(z,\complement\Omega)$, $z\in\overline{\Omega}$. Then $\rho=-\delta$ is C^2 near $\partial\Omega$ and satisfies (7.9). If Ω is pseudoconvex, the plurisubharmonicity of $-\log(-\rho)$ means that for all $z\in\Omega$ near $\partial\Omega$ and all $\xi\in\mathbb{C}^n$ one has

$$\sum_{1\le j,k\le n}\Big(\frac{1}{|\rho|}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}+\frac{1}{\rho^2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}\Big)\xi_j\overline{\xi}_k\ge 0.$$

Hence $\sum(\partial^2\rho/\partial z_j\partial\overline{z}_k)\xi_j\overline{\xi}_k\ge 0$ if $\sum(\partial\rho/\partial z_j)\xi_j=0$, and an easy argument shows that this is also true at the limit on $\partial\Omega$.

Conversely, if Ω is not pseudoconvex, Th. 7.2 and 7.7 show that $-\log\delta$ is not plurisubharmonic in any neighborhood of $\partial\Omega$. Hence there exists $\xi\in\mathbb{C}^n$ such that

$$c=\Big(\frac{\partial^2}{\partial t\partial\overline{t}}\log\delta(z+t\xi)\Big)_{|t=0}>0$$

for some z in the neighborhood of $\partial\Omega$ where $\delta\in C^2$. By Taylor's formula, we have

$$\log\delta(z+t\xi)=\log\delta(z)+\operatorname{Re}(at+bt^2)+c|t|^2+\mathrm{o}(|t|^2)$$

with $a,b\in\mathbb{C}$. Now, choose $z_0\in\partial\Omega$ such that $\delta(z)=|z-z_0|$ and set

$$h(t)=z+t\xi+e^{at+bt^2}(z_0-z),\qquad t\in\mathbb{C}.$$

Then we get $h(0)=z_0$ and

$$\begin{aligned}\delta(h(t))&\ge\delta(z+t\xi)-\delta(z)\left|e^{at+bt^2}\right|\\&\ge\delta(z)\left|e^{at+bt^2}\right|\big(e^{c|t|^2/2}-1\big)\ge\delta(z)\,c|t|^2/3\end{aligned}$$

when $|t|$ is sufficiently small. Since $\delta(h(0))=\delta(z_0)=0$, we obtain at $t=0$:

$$\frac{\partial}{\partial t}\,\delta(h(t)) = \sum \frac{\partial \delta}{\partial z_j}(z_0)\,h_j'(0) = 0,$$

$$\frac{\partial^2}{\partial t \partial \overline{t}}\,\delta(h(t)) = \sum \frac{\partial^2 \delta}{\partial z_j \partial \overline{z}_k}(z_0)\,h_j'(0)\overline{h_k'(0)} > 0,$$

hence $h'(0) \in {}^hT_{\partial\Omega,z_0}$ and $L_{\partial\Omega,z_0}(h'(0)) < 0$. □

(7.13) Definition. *The boundary $\partial\Omega$ is said to be weakly (resp. strongly) pseudoconvex if $L_{\partial\Omega}$ is semipositive (resp. positive definite) on $\partial\Omega$. The boundary is said to be Levi flat if $L_{\partial\Omega} \equiv 0$.*

(7.14) Remark. Lemma 7.11 shows that $\partial\Omega$ is Levi flat if and only if the subbundle ${}^hT_{\partial\Omega} \subset T_{\partial\Omega}$ is integrable (i.e. stable under the Lie bracket). Assume that $\partial\Omega$ is of class C^k, $k \geq 2$. Then ${}^hT_{\partial\Omega}$ is of class C^{k-1}. By Frobenius' theorem, the integrability condition implies that ${}^hT_{\partial\Omega}$ is the tangent bundle to a C^k foliation of $\partial\Omega$ whose leaves have real dimension $2n-2$. But the leaves themselves must be complex analytic since ${}^hT_{\partial\Omega}$ is a complex vector space (cf. Lemma 7.15 below). Therefore $\partial\Omega$ is Levi flat if and only if it is foliated by complex analytic hypersurfaces.

(7.15) Lemma. *Let Y be a C^1-submanifold of a complex analytic manifold X. If the tangent space $T_{Y,x}$ is a complex subspace of $T_{X,x}$ at every point $x \in Y$, then Y is complex analytic.*

Proof. Let $x_0 \in Y$. Select holomorphic coordinates $(z_1, \ldots, z_n)$ on X centered at x_0 such that T_{Y,x_0} is spanned by $\partial/\partial z_1, \ldots, \partial/\partial z_p$. Then there exists a neighborhood $U = U' \times U''$ of x_0 such that $Y \cap U$ is a graph

$$z'' = h(z'), \qquad z' = (z_1, \ldots, z_p) \in U', \quad z'' = (z_{p+1}, \ldots, z_n)$$

with $h \in C^1(U')$ and $dh(0) = 0$. The differential of h at z' is the composite of the projection of $\mathbb{C}^p \times \{0\}$ on $T_{Y,(z',h(z'))}$ along $\{0\} \times \mathbb{C}^{n-p}$ and of the second projection $\mathbb{C}^n \to \mathbb{C}^{n-p}$. Hence $dh(z')$ is $\mathbb{C}$-linear at every point and h is holomorphic. □

§8. Exercises

8.1. Let $\Omega \subset \mathbb{C}^n$ be an open set such that

$$z \in \Omega, \quad \lambda \in \mathbb{C}, \quad |\lambda| \leq 1 \Longrightarrow \lambda z \in \Omega.$$

Show that Ω is a union of polydisks of center 0 (with arbitrary linear changes of coordinates) and infer that the space of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ is dense in $\mathcal{O}(\Omega)$ for the topology of uniform convergence on compact subsets and in $\mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$

for the topology of uniform convergence on $\overline{\Omega}$.
Hint: consider the Taylor expansion of a function $f \in \mathcal{O}(\Omega)$ at the origin, writing it as a series of homogeneous polynomials. To deal with the case of $\overline{\Omega}$, first apply a dilation to f.

8.2. Let $B \subset \mathbb{C}^n$ be the unit euclidean ball, $S = \partial B$ and $f \in \mathcal{O}(B) \cap C^0(\overline{B})$. Our goal is to check the following Cauchy formula:

$$f(w) = \frac{1}{\sigma_{2n-1}} \int_S \frac{f(z)}{(1 - \langle w, z\rangle)^n}\, d\sigma(z).$$

a) By means of a unitary transformation and Exercise 8.1, reduce the question to the case when $w = (w_1, 0, \ldots, 0)$ and $f(z)$ is a monomial z^α.

b) Show that the integral $\int_B z^\alpha \overline{z}_1^k\, d\lambda(z)$ vanishes unless $\alpha = (k, 0, \ldots, 0)$. Compute the value of the remaining integral by the Fubini theorem, as well as the integrals $\int_S z^\alpha \overline{z}_1^k\, d\sigma(z)$.

c) Prove the formula by a suitable power series expansion.

8.3. A current $T \in \mathcal{D}'_p(M)$ is said to be *normal* if both T and dT are of order zero, i.e. have measure coefficients.

a) If T is normal and has support contained in a C^1 submanifold $Y \subset M$, show that there exists a normal current Θ on Y such that $T = j_\star \Theta$, where $j : Y \longrightarrow M$ is the inclusion.
Hint: if $x_1 = \ldots = x_q = 0$ are equations of Y in a coordinate system $(x_1, \ldots, x_n)$, observe that $x_j T = x_j dT = 0$ for $1 \le j \le q$ and infer that $dx_1 \wedge \ldots \wedge dx_q$ can be factorized in all terms of T.

b) What happens if $p > \dim Y$?

c) Are a) and b) valid when the normality assumption is dropped ?

8.4. Let $T = \sum_{1 \le j \le n} T_j d\overline{z}_j$ be a closed current of bidegree $(0,1)$ with compact support in $\mathbb{C}^n$ such that $d''T = 0$.

a) Show that the partial convolution $S = (1/\pi z_1) \star_1 T_1$ is a solution of the equation $d''S = T$.

b) Let $K = \operatorname{Supp} T$. If $n \ge 2$, show that S has support in the compact set $\widetilde{K}$ equal to the union of K and of all bounded components of $\mathbb{C}^n \smallsetminus K$.
Hint: observe that S is holomorphic on $\mathbb{C}^n \smallsetminus K$ and that S vanishes for $|z_2| + \ldots + |z_n|$ large.

8.5. Alternative proof of the Dolbeault-Grothendieck lemma. Let $v = \sum_{|J|=q} v_J d\overline{z}_J$, $q \ge 1$, be a smooth form of bidegree $(0,q)$ on a polydisk $\Omega = D(0,R) \subset \mathbb{C}^n$, such that $d''v = 0$, and let $\omega = D(0,r) \subset\subset \omega$. Let k be the smallest integer such that the monomials $d\overline{z}_J$ appearing in v only involve $d\overline{z}_1, \ldots, d\overline{z}_k$. Prove by induction on k that the equation $d''u = v$ can be solved on ω.
Hint: set $v = f \wedge d\overline{z}_k + g$ where f, g only involve $d\overline{z}_1, \ldots, d\overline{z}_{k-1}$. Then consider $v - d''F$ where

$$F = \sum_{|J|=q-1} F_J d\overline{z}_J, \qquad F_J(z) = (\psi(z_k) f_J) \star_k \left(\frac{1}{\pi z_k}\right),$$

where $\star_k$ denotes the partial convolution with respect to z_k, $\psi(z_k)$ is a cut-off function equal to 1 on $D(0, r_k + \varepsilon)$ and $f = \sum_{|J|=q-1} f_J d\overline{z}_J$.

8.6. Construct locally bounded non continuous subharmonic functions on $\mathbb{C}$.
Hint: consider e^u where $u(z) = \sum_{j\geq 1} 2^{-j} \log |z - 1/j|$.

8.7. Let ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, and u a subharmonic function which is not locally $-\infty$.

a) For every open set $\omega \subset\subset \Omega$, show that there is a positive measure μ with support in $\overline{\omega}$ and a harmonic function h on ω such that $u = N \star \mu + h$ on ω.

b) Use this representation to prove the following properties: $u \in L^p_{\mathrm{loc}}$ for all $p < n/(n-2)$ and $\partial u/\partial x_j \in L^p_{\mathrm{loc}}$ for all $p < n/(n-1)$.

8.8. Show that a connected open set $\Omega \subset \mathbb{R}^n$ is convex if and only if Ω has a locally convex exhaustion function φ.
Hint: to show the sufficiency, take a path $\gamma : [0,1] \to \Omega$ joining two arbitrary points $a, b \in \Omega$ and consider the restriction of φ to $[a, \gamma(t_0)] \cap \Omega$ where t_0 is the supremum of all t such that $[a, \gamma(u)] \subset \Omega$ for $u \in [0, t]$.

8.9. Let $r_1, r_2 \in \,]1, +\infty[$. Consider the compact set

$$K = \{|z_1| \leq r_1\,,\ |z_2| \leq 1\} \cup \{|z_1| \leq 1\,,\ |z_2| \leq r_2\} \subset \mathbb{C}^2.$$

Show that the holomorphic hull of K in $\mathbb{C}^2$ is

$$\widehat{K} = \{|z_1| \leq r_1\,,\ |z_2| \leq r_2\,,\ |z_1|^{1/\log r_1} |z_2|^{1/\log r_2} \leq e\}.$$

Hint: to show that $\widehat{K}$ is contained in this set, consider all holomorphic monomials $f(z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}$. To show the converse inclusion, apply the maximum principle to the domain $|z_1| \leq r_1$, $|z_2| \leq r_2$ on suitably chosen Riemann surfaces $z_1^{\alpha_1} z_2^{\alpha_2} = \lambda$.

8.10. Compute the rank of the Levi form of the ellipsoid $|z_1|^2 + |z_3|^4 + |z_3|^6 < 1$ at every point of the boundary.

8.11. Let X be a complex manifold and let $u(z) = \sum_{j\in\mathbb{N}} |f_j|^2$, $f_j \in \mathcal{O}(X)$, be a series converging uniformly on every compact subset of X. Prove that the limit is real analytic and that the series remains uniformly convergent by taking derivatives term by term.
Hint: since the problem is local, take $X = B(0, r)$, a ball in $\mathbb{C}^n$. Let $g_j(z) = \overline{g_j(\overline{z})}$ be the conjugate function of f_j and let $U(z, w) = \sum_{j\in\mathbb{N}} f_j(z) g_j(w)$ on $X \times X$. Using the Cauchy-Schwarz inequality, show that this series of holomorphic functions is uniformly convergent on every compact subset of $X \times X$.

8.12. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary.

a) Let $a \in \partial\Omega$ be a given point. Let e_n be the outward normal vector to $T_{\partial\Omega, a}$, $(e_1, \ldots, e_{n-1})$ an orthonormal basis of ${}^hT_a(\partial\Omega)$ in which the Levi form is diagonal and $(z_1, \ldots, z_n)$ the associated linear coordinates centered at a. Show that there is a neighborhood V of a such that $\partial\Omega \cap V$ is the graph $\mathrm{Re}\, z_n = -\varphi(z_1, \ldots, z_{n-1}, \mathrm{Im}\, z_n)$ of a function φ such that $\varphi(z) = O(|z|^2)$ and the matrix $\partial^2\varphi/\partial z_j \partial\overline{z}_k(0)$, $1 \leq j, k \leq n-1$ is diagonal.

b) Show that there exist local analytic coordinates $w_1 = z_1, \ldots, w_{n-1} = z_{n-1}$, $w_n = z_n + \sum c_{jk} z_j z_k$ on a neighborhood V' of $a = 0$ such that

$$\Omega \cap V' = V' \cap \{\mathrm{Re}\, w_n + \sum_{1\leq j\leq n} \lambda_j |w_j|^2 + \mathrm{o}(|w|^2) < 0\}, \qquad \lambda_j \in \mathbb{R}$$

and that λ_n can be assigned to any given value by a suitable choice of the coordinates.
Hint: Consider the Taylor expansion of order 2 of the defining function $\rho(z) = (\mathrm{Re}\, z_n + \varphi(z))(1 + \mathrm{Re} \sum c_j z_j)$ where $c_j \in \mathbb{C}$ are chosen in a suitable way.

c) Prove that $\partial\Omega$ is strongly pseudoconvex at a if and only if there is a neighborhood U of a and a biholomorphism Φ of U onto some open set of $\mathbb{C}^n$ such that $\Phi(\Omega \cap U)$ is strongly convex.

d) Assume that the Levi form of $\partial\Omega$ is not semipositive. Show that all holomorphic functions $f \in \mathcal{O}(\Omega)$ extend to some (fixed) neighborhood of a.
Hint: assume for example $\lambda_1 < 0$. For $\varepsilon > 0$ small, show that Ω contains the Hartogs figure

$$\{\varepsilon/2 < |w_1| < \varepsilon\} \times \{|w_j| < \varepsilon^2\}_{1<j<n} \times \{|w_n| < \varepsilon^{3/2}\,,\ \mathrm{Re}\, w_n < \varepsilon^3\} \ \cup$$
$$\{|w_1| < \varepsilon\} \times \{|w_j| < \varepsilon^2\}_{1<j<n} \times \{|w_n| < \varepsilon^{3/2}\,,\ \mathrm{Re}\, w_n < -\varepsilon^2\}.$$

8.13. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary and $\rho \in C^2(\Omega, \mathbb{R})$ such that $\rho < 0$ on Ω, $\rho = 0$ and $d\rho \neq 0$ on $\partial\Omega$. Let $f \in C^1(\partial\Omega, \mathbb{C})$ be a function satisfying the *tangential Cauchy-Riemann equations*

$$\xi'' \cdot f = 0, \qquad \forall \xi \in {}^hT_{\partial\Omega}, \qquad \xi'' = \frac{1}{2}(\xi + \mathrm{i}J\xi).$$

a) Let f_0 be a C^1 extension of f to $\overline{\Omega}$. Show that $d''f_0 \wedge d''\rho = 0$ on $\partial\Omega$ and infer that $v = \mathbb{1}_\Omega d''f_0$ is a d''-closed current on $\mathbb{C}^n$.

b) Show that the solution u of $d''u = v$ provided by Cor. 3.27 is continuous and that f admits an extension $\widetilde{f} \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$ if $\partial\Omega$ is connected.

8.14. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2 boundary and let $\delta(z) = d(z, \complement\Omega)$ be the euclidean distance to the boundary.

a) Use the plurisubharmonicity of $-\log\delta$ to prove the following fact: for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\frac{-H\delta_z(\xi)}{\delta(z)} + \varepsilon\frac{|d'\delta_z.\xi|^2}{|\delta(z)|^2} + C_\varepsilon|\xi|^2 \geq 0$$

for $\xi \in \mathbb{C}^n$ and z near $\partial\Omega$.

b) Set $\psi(z) = -\log\delta(z) + K|z|^2$. Show that for K large and α small the function

$$\rho(z) = -\exp\Big(-\alpha\psi(z)\Big) = -\Big(e^{-K|z|^2}\delta(z)\Big)^\alpha$$

is plurisubharmonic.

c) Prove the existence of a plurisubharmonic exhaustion function $u : \Omega \to [-1, 0[$ of class C^2 such that $|u(z)|$ has the same order of magnitude as $\delta(z)^\alpha$ when z tends to $\partial\Omega$.
Hint: consult (Diederich-Fornaess 1976).

8.15. Let $\Omega = \omega + \mathrm{i}\mathbb{R}^n$ be a connected tube in $\mathbb{C}^n$ of base ω.

a) Assume first that $n = 2$. Let $T \subset \mathbb{R}^2$ be the triangle $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq 1$, and assume that the two edges $[0,1] \times \{0\}$ and $\{0\} \times [0,1]$ are contained in ω. Show that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends to a neighborhood of $T + \mathrm{i}\mathbb{R}^2$.
Hint: let $\pi : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$ be the projection on the real part and M_ε the intersection of $\pi^{-1}((1+\varepsilon)T)$ with the Riemann surface $z_1 + z_2 - \frac{\varepsilon}{2}(z_1^2 + z_2^2) = 1$ (a non degenerate affine conic). Show that M_ε is compact and that

$$\pi(\partial M_\varepsilon) \subset ([0, 1+\varepsilon] \times \{0\}) \cup (\{0\} \times [0, 1+\varepsilon]) \subset \omega,$$
$$\pi([0,1] \cdot M_\varepsilon) \supset T$$

for ε small. Use the Cauchy formula along ∂M_ε (in some parametrization of the conic) to obtain an extension of f to $[0,1] \cdot M_\varepsilon + \mathrm{i}\mathbb{R}^n$.

b) In general, show that every $f \in \mathcal{O}(\Omega)$ extends to the convex hull $\widehat{\Omega}$.
Hint: given $a, b \in \omega$, consider a polygonal line joining a and b and apply a) inductively to obtain an extension along $[a,b] + \mathrm{i}\mathbb{R}^n$.

8.16. For each integer $\nu \geq 1$, consider the algebraic variety

$$X_\nu = \Big\{(z, w, t) \in \mathbb{C}^3 \,;\, wt = p_\nu(z)\Big\}, \qquad p_\nu(z) = \prod_{1 \leq k \leq \nu} (z - 1/k),$$

and the map $j_\nu : X_\nu \to X_{\nu+1}$ such that

$$j_\nu(z, w, t) = \bigg(z,\ w,\ t\Big(z - \frac{1}{\nu+1}\Big)\bigg).$$

a) Show that X_ν is a Stein manifold, and that j_ν is an embedding of X_ν onto an open subset of $X_{\nu+1}$.

b) Define $X = \lim(X_\nu, j_\nu)$, and let $\pi_\nu : X_\nu \to \mathbb{C}^2$ be the projection to the first two coordinates. Since $\pi_{\nu+1} \circ j_\nu = \pi_\nu$, there exists a holomorphic map $\pi : X \to \mathbb{C}^2$, $\pi = \lim \pi_\nu$. Show that

$$\mathbb{C}^2 \smallsetminus \pi(X) = \Big\{(z, 0) \in \mathbb{C}^2 \,;\, z \neq 1/\nu,\ \forall \nu \in \mathbb{N},\ \nu \geq 1\Big\},$$

and especially, that $(0,0) \notin \pi(X)$.

c) Consider the compact set

$$K = \pi^{-1}\Big(\{(z, w) \in \mathbb{C}^2 \,;\, |z| \leq 1,\ |w| = 1\}\Big).$$

By looking at points of the forms $(1/\nu, w, 0)$, $|w| = 1$, show that $\pi^{-1}(1/\nu, 1/\nu) \in \widehat{K}_{\mathcal{O}(X)}$. Conclude from this that X is not holomorphically convex (this example is due to Fornaess 1977).

8.17. Let X be a complex manifold, and let $\pi : \widetilde{X} \to X$ be a holomorphic unramified covering of X (X and $\widetilde{X}$ are assumed to be connected).

a) Let g be a complete riemannian metric on X, and let $\widetilde{d}$ be the geodesic distance on $\widetilde{X}$ associated to $\widetilde{g} = \pi^\star g$ (see VIII-2.3 for definitions). Show that $\widetilde{g}$ is complete and that $\delta_0(x) := \widetilde{d}(x, x_0)$ is a continuous exhaustion function on $\widetilde{X}$, for any given point $x_0 \in \widetilde{X}$.

b) Let (U_α) be a locally finite covering of X by open balls contained in coordinate open sets, such that all intersections $U_\alpha \cap U_\beta$ are diffeomorphic to convex open sets (see Lemma IV-6.9). Let θ_α be a partition of unity subordinate to the covering (U_α), and let $\delta_{\varepsilon_\alpha}$ be the convolution of δ_0 with a regularizing kernel $\rho_{\varepsilon_\alpha}$ on each piece of $\pi^{-1}(U_\alpha)$ which is mapped biholomorphically onto U_α. Finally, set $\delta = \sum(\theta_\alpha \circ \pi)\delta_{\varepsilon_\alpha}$. Show that if (ε_α) is a collection of sufficiently small positive numbers, then δ is a smooth exhaustion function on $\widetilde{X}$.

c) Using the fact that δ_0 is 1-Lipschitz with respect to $\widetilde{d}$, show that derivatives $\partial^{|\nu|}\delta(x)/\partial x^\nu$ of a given order with respect to coordinates in U_α are uniformly bounded in all components of $\pi^{-1}(U_\alpha)$, at least when x lies in the compact subset $\operatorname{Supp}\theta_\alpha$. Conclude from this that there exists a positive hermitian form with continuous coefficients on X such that $H\delta \geq -\pi^\star\gamma$ on $\widetilde{X}$.

d) If X is strongly pseudoconvex, show that $\widetilde{X}$ is also strongly pseudoconvex. *Hint*: let ψ be a smooth strictly plurisubharmonic exhaustion function on X. Show that there exists a smooth convex increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\delta + \chi \circ \psi$ is strictly plurisubharmonic.