Chapter II.
Coherent Sheaves and Analytic Spaces

The chapter starts with rather general and abstract concepts concerning sheaves and ringed spaces. Introduced in the decade 1950-1960 by Leray, Cartan, Serre and Grothendieck, sheaves and ringed spaces have since been recognized as the adequate tools to handle algebraic varieties and analytic spaces in a unified framework. We then concentrate ourselves on the theory of complex analytic functions. The second section is devoted to a proof of the Weierstrass preparation theorem, which is nothing but a division algorithm for holomorphic functions. It is used to derive algebraic properties of the ring $\mathcal{O}_n$ of germs of holomorphic functions in $\mathbb{C}^n$. Coherent analytic sheaves are then introduced and the fundamental coherence theorem of Oka is proved. Basic properties of analytic sets are investigated in detail: local parametrization theorem, Hilbert's Nullstellensatz, coherence of the ideal sheaf of an analytic set, analyticity of the singular set. The formalism of complex spaces is then developed and gives a natural setting for the proof of more global properties (decomposition into global irreducible components, maximum principle). After a few definitions concerning cycles, divisors and meromorphic functions, we investigate the important notion of normal space and establish the Oka normalization theorem. Next, the Remmert-Stein extension theorem and the Remmert proper mapping theorem on images of analytic sets are proved by means of semi-continuity results on the rank of morphisms. As an application, we give a proof of Chow's theorem asserting that every analytic subset of $\mathbb{P}^n$ is algebraic. Finally, the concept of analytic scheme with nilpotent elements is introduced as a generalization of complex spaces, and we discuss the concepts of bimeromorphic maps, modifications and blowing-up.

§1. Presheaves and Sheaves

§1.A. Main Definitions

Sheaves have become a very important tool in analytic or algebraic geometry as well as in algebraic topology. They are especially useful when one wants to relate global properties of an object to its local properties (the latter being usually easier to establish). We first introduce the axioms of presheaves and sheaves in full generality and give some basic examples.

(1.1) Definition. *Let X be a topological space. A presheaf $\mathcal{A}$ on X consists of the following data:*

a) *a collection of non empty sets* $\mathcal{A}(U)$ *associated with every open set* $U \subset X$*,*

b) *a collection of maps* $\rho_{U,V} : \mathcal{A}(V) \longrightarrow \mathcal{A}(U)$ *defined whenever* $U \subset V$ *and satisfying the transitivity property*

c) $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$ *for* $U \subset V \subset W$, $\qquad \rho_{U,U} = \mathrm{Id}_U$ *for every* U*.*

The set $\mathcal{A}(U)$ *is called the set of sections of the presheaf* $\mathcal{A}$ *over* U*.*

Most often, the presheaf $\mathcal{A}$ is supposed to carry an additional algebraic structure. For instance:

(1.2) Definition. *A presheaf* $\mathcal{A}$ *is said to be a presheaf of abelian groups (resp. rings, R-modules, algebras) if all sets* $\mathcal{A}(U)$ *are abelian groups (resp. rings, R-modules, algebras) and if the maps* $\rho_{U,V}$ *are morphisms of these algebraic structures. In this case, we always assume that* $\mathcal{A}(\emptyset) = \{0\}$*.*

(1.3) Example. If we assign to each open set $U \subset X$ the set $\mathcal{C}(U)$ of all real valued continuous functions on U and let $\rho_{U,V}$ be the obvious restriction morphism $\mathcal{C}(V) \to \mathcal{C}(U)$, then $\mathcal{C}$ is a presheaf of rings on X. Similarly if X is a differentiable (resp. complex analytic) manifold, there are well defined presheaves of rings $\mathcal{C}^k$ of functions of class C^k (resp. $\mathcal{O}$) of holomorphic functions) on X. Because of these examples, the maps $\rho_{U,V}$ in Def. 1.1 are often viewed intuitively as "restriction homomorphisms", although the sets $\mathcal{A}(U)$ are not necessarily sets of functions defined over U. For the simplicity of notation we often just write $\rho_{U,V}(f) = f_{\restriction U}$ whenever $f \in \mathcal{A}(V)$, $V \supset U$. $\square$

For the above presheaves $\mathcal{C}$, $\mathcal{C}^k$, $\mathcal{O}$, the properties of functions under consideration are purely local. As a consequence, these presheaves satisfy the following additional *gluing axioms*, where (U_α) and $U = \bigcup U_\alpha$ are arbitrary open subsets of X :

(1.4′) If $F_\alpha \in \mathcal{A}(U_\alpha)$ are such that $\rho_{U_\alpha \cap U_\beta, U_\alpha}(F_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(F_\beta)$ for all α, β, there exists $F \in \mathcal{A}(U)$ such that $\rho_{U_\alpha, U}(F) = F_\alpha$;

(1.4″) If $F, G \in \mathcal{A}(U)$ and $\rho_{U_\alpha, U}(F) = \rho_{U_\alpha, U}(G)$ for all α, then $F = G$;

in other words, local sections over the sets U_α can be glued together if they coincide in the intersections and the resulting section on U is uniquely defined. Not all presheaves satisfy (1.4′) and (1.4″):

(1.5) Example. Let E be an arbitrary set with a distinguished element 0 (e.g. an abelian group, a R-module, ...). The *constant presheaf* E_X on X is defined to be $E_X(U) = E$ for all $\emptyset \neq U \subset X$ and $E_X(\emptyset) = \{0\}$, with restriction maps $\rho_{U,V} = \mathrm{Id}_E$ if $\emptyset \neq U \subset V$ and $\rho_{U,V} = 0$ if $U = \emptyset$. Then axiom (1.4′) is not satisfied if U is the union of two disjoint open sets U_1, U_2 and E contains a non zero element.

(1.6) Definition. *A presheaf* $\mathcal{A}$ *is said to be a sheaf if it satisfies the gluing axioms* (1.4′) *and* (1.4″).

If $\mathcal{A}$, $\mathcal{B}$ are presheaves of abelian groups (or of some other algebraic structure) on the same space X, a presheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ commuting with the restriction morphisms, i.e. such that for each pair $U \subset V$ there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(V) & \xrightarrow{\varphi_V} & \mathcal{B}(V) \\ \rho^{\mathcal{B}}_{U,V}\downarrow & & \downarrow\rho^{\mathcal{A}}_{U,V} \\ \mathcal{A}(U) & \xrightarrow{\varphi_U} & \mathcal{B}(U). \end{array}$$

We say that $\mathcal{A}$ is a subpresheaf of $\mathcal{B}$ in the case where $\varphi_U : \mathcal{A}(U) \subset \mathcal{B}(U)$ is the inclusion morphism; the commutation property then means that $\rho^{\mathcal{B}}_{U,V}(\mathcal{A}(V)) \subset \mathcal{A}(U)$ for all U, V, and that $\rho^{\mathcal{A}}_{U,V}$ coincides with $\rho^{\mathcal{B}}_{U,V}$ on $\mathcal{A}(V)$. If $\mathcal{A}$ is a subpresheaf of a presheaf $\mathcal{B}$ of abelian groups, there is a presheaf quotient $\mathcal{C} = \mathcal{B}/\mathcal{A}$ defined by $\mathcal{C}(U) = \mathcal{B}(U)/\mathcal{A}(U)$. In a similar way, one defines the presheaf kernel (resp. presheaf image, presheaf cokernel) of a presheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ to be the presheaves

$$U \mapsto \operatorname{Ker}\varphi_U, \qquad U \mapsto \operatorname{Im}\varphi_U, \qquad U \mapsto \operatorname{Coker}\varphi_U.$$

The direct sum $\mathcal{A} \oplus \mathcal{B}$ of presheaves of abelian groups $\mathcal{A}$, $\mathcal{B}$ is the presheaf $U \mapsto \mathcal{A}(U) \oplus \mathcal{B}(U)$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ of presheaves of R-modules is $U \mapsto \mathcal{A}(U) \otimes_R \mathcal{B}(U)$, etc ...

(1.7) Remark. The reader should take care of the fact that the presheaf quotient of a sheaf by a subsheaf is not necessarily a sheaf. To give a specific example, let $X = S^1$ be the unit circle in $\mathbb{R}^2$, let $\mathcal{C}$ be the sheaf of continuous complex valued functions and $\mathcal{Z}$ the subsheaf of integral valued continuous functions (i.e. locally constant functions to $\mathbb{Z}$). The exponential map

$$\varphi = \exp(2\pi \mathrm{i}\bullet) : \mathcal{C} \longrightarrow \mathcal{C}^\star$$

is a morphism from $\mathcal{C}$ to the sheaf $\mathcal{C}^\star$ of invertible continuous functions, and the kernel of φ is precisely $\mathcal{Z}$. However φ_U is surjective for all $U \neq X$ but maps $\mathcal{C}(X)$ onto the multiplicative subgroup of continuous functions of $\mathcal{C}^\star(X)$ of degree 0. Therefore the quotient presheaf $\mathcal{C}/\mathcal{Z}$ is not isomorphic with $\mathcal{C}^\star$, although their groups of sections are the same for all $U \neq X$. Since $\mathcal{C}^\star$ is a sheaf, we see that $\mathcal{C}/\mathcal{Z}$ does not satisfy property (1.4′). $\square$

In order to overcome the difficulty appearing in Example 1.7, it is necessary to introduce a suitable process by which we can produce a sheaf from a presheaf. For this, it is convenient to introduce a slightly modified viewpoint for sheaves.

(1.8) Definition. *If $\mathcal{A}$ is a presheaf, we define the set $\widetilde{\mathcal{A}}_x$ of germs of $\mathcal{A}$ at a point $x \in X$ to be the abstract inductive limit*

$$\widetilde{\mathcal{A}}_x = \varinjlim_{U \ni x} \big(\mathcal{A}(U), \rho_{U,V}\big).$$

More explicitely, $\widetilde{\mathcal{A}}_x$ is the set of equivalence classes of elements in the disjoint union $\coprod_{U \ni x} \mathcal{A}(U)$ taken over all open neighborhoods U of x, with two elements $F_1 \in \mathcal{A}(U_1)$, $F_2 \in \mathcal{A}(U_2)$ being equivalent, $F_1 \sim F_2$, if and only if there is a neighborhood $V \subset U_1, U_2$ such that $F_{1\restriction V} = F_{2\restriction V}$, i.e., $\rho_{VU_1}(F_1) = \rho_{VU_2}(F_2)$. The germ of an element $F \in \mathcal{A}(U)$ at a point $x \in U$ will be denoted by F_x.

Let $\mathcal{A}$ be an arbitrary presheaf. The disjoint union $\widetilde{\mathcal{A}} = \coprod_{x \in X} \widetilde{\mathcal{A}}_x$ can be equipped with a natural topology as follows: for every $F \in \mathcal{A}(U)$, we set

$$\Omega_{F,U} = \big\{F_x \ ; \ x \in U\big\}$$

and choose the $\Omega_{F,U}$ to be a basis of the topology of $\widetilde{\mathcal{A}}$; note that this family is stable by intersection: $\Omega_{F,U} \cap \Omega_{G,V} = \Omega_{H,W}$ where W is the (open) set of points $x \in U \cap V$ at which $F_x = G_x$ and $H = \rho_{W,U}(F)$. The obvious projection map $\pi : \widetilde{\mathcal{A}} \to X$ which sends $\widetilde{\mathcal{A}}_x$ to $\{x\}$ is then a local homeomorphism (it is actually a homeomorphism from $\Omega_{F,U}$ onto U). This leads in a natural way to the following definition:

(1.9) Definition. *Let X and $\mathcal{S}$ be topological spaces (not necessarily Hausdorff), and let $\pi : \mathcal{S} \longrightarrow X$ be a mapping such that*

a) *π maps $\mathcal{S}$ onto X ;*

b) *π is a local homeomorphism, that is, every point in $\mathcal{S}$ has an open neighborhood which is mapped homeomorphically by π onto an open subset of X.*

Then $\mathcal{S}$ is called a sheaf-space on X and π is called the projection of $\mathcal{S}$ on X. If $x \in X$, then $\mathcal{S}_x = \pi^{-1}(x)$ is called the stalk of $\mathcal{S}$ at x.

If Y is a subset of X, we denote by $\Gamma(Y, \mathcal{S})$ the set of sections of $\mathcal{S}$ on Y, i.e. the set of continuous functions $F : Y \to \mathcal{S}$ such that $\pi \circ F = \mathrm{Id}_Y$. It is clear that the presheaf defined by the collection of sets $\mathcal{S}'(U) := \Gamma(U, \mathcal{S})$ for all open sets $U \subset X$ together with the restriction maps $\rho_{U,V}$ satisfies axioms (1.4′) and (1.4″), hence $\mathcal{S}'$ is a sheaf. The set of germs of $\mathcal{S}'$ at x is in one-to-one correspondence with the stalk $\mathcal{S}_x = \pi^{-1}(x)$, thanks to the local homeomorphism assumption 1.9 b). This shows that one can associate in a natural way a sheaf $\mathcal{S}'$ to every sheaf-space $\mathcal{S}$, and that the sheaf-space $(\mathcal{S}')^\sim$ can be considered to be identical to the original sheaf-space $\mathcal{S}$. Since the assignment $\mathcal{S} \mapsto \mathcal{S}'$ from sheaf-spaces to sheaves is an equivalence of categories, we will usually omit the prime sign in the notation of $\mathcal{S}'$ and thus use the same symbols for a sheaf-space and its associated sheaf of sections; in a corresponding way, we write $\Gamma(U, \mathcal{S}) = \mathcal{S}(U)$ when U is an open set.

Conversely, given a presheaf $\mathcal{A}$ on X, we have an associated sheaf-space $\widetilde{\mathcal{A}}$ and an obvious presheaf morphism

$$(1.10)\quad \mathcal{A}(U) \longrightarrow \widetilde{\mathcal{A}}'(U) = \Gamma(U, \widetilde{\mathcal{A}}), \qquad F \longmapsto \widetilde{F} = (U \ni x \mapsto F_x).$$

This morphism is clearly injective if and only if $\mathcal{A}$ satisfies axiom $(1.4'')$, and it is not difficult to see that $(1.4')$ and $(1.4'')$ together imply surjectivity. Therefore $\mathcal{A} \to \widetilde{\mathcal{A}}'$ is an isomorphism if and only if $\mathcal{A}$ is a sheaf. According to the equivalence of categories between sheaves and sheaf-spaces mentioned above, we will use from now on the same symbol $\widetilde{\mathcal{A}}$ for the sheaf-space and its associated sheaf $\widetilde{\mathcal{A}}'$; one says that $\widetilde{\mathcal{A}}$ is the *sheaf associated with the presheaf* $\mathcal{A}$. If $\mathcal{A}$ itself is a sheaf, we will again identify $\widetilde{\mathcal{A}}$ and $\mathcal{A}$, but we will of course keep the notational difference for a presheaf $\mathcal{A}$ which is not a sheaf.

(1.11) Example. The sheaf associated to the constant presheaf of stalk E over X is the sheaf of locally constant functions $X \to E$. This sheaf will be denoted merely by E_X or E if there is no risk of confusion with the corresponding presheaf. In Example 1.7, we have $\mathcal{Z} = \mathbb{Z}_X$ and the sheaf $(\mathcal{C}/\mathbb{Z}_X)^{\sim}$ associated with the quotient presheaf $\mathcal{C}/\mathbb{Z}_X$ is isomorphic to $\mathcal{C}^\star$ via the exponential map. $\square$

In the sequel, we usually work in the category of sheaves rather than in the category of presheaves themselves. For instance, the quotient $\mathcal{B}/\mathcal{A}$ of a sheaf $\mathcal{B}$ by a subsheaf $\mathcal{A}$ generally refers to the sheaf associated with the quotient presheaf: its stalks are equal to $\mathcal{B}_x/\mathcal{A}_x$, but a section G of $\mathcal{B}/\mathcal{A}$ over an open set U need not necessarily come from a global section of $\mathcal{B}(U)$; what can be only said is that there is a covering (U_α) of U and local sections $F_\alpha \in \mathcal{B}(U_\alpha)$ representing $G_{\restriction U_\alpha}$ such that $(F_\beta - F_\alpha)_{\restriction U_\alpha \cap U_\beta}$ belongs to $\mathcal{A}(U_\alpha \cap U_\beta)$. A sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be injective (resp. surjective) if the germ morphism $\varphi_x : \mathcal{A}_x \to \mathcal{B}_x$ is injective (resp. surjective) for every $x \in X$. Let us note again that a surjective sheaf morphism φ does not necessarily give rise to surjective morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$.

§1.B. Direct and Inverse Images of Sheaves

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If $\mathcal{A}$ is a presheaf on X, the *direct image* $f_\star\mathcal{A}$ is the presheaf on Y defined by

$$(1.12)\quad f_\star\mathcal{A}(U) = \mathcal{A}\big(f^{-1}(U)\big)$$

for all open sets $U \subset Y$. When $\mathcal{A}$ is a sheaf, it is clear that $f_\star\mathcal{A}$ also satisfies axioms $(1.4')$ and $(1.4'')$, thus $f_\star\mathcal{A}$ is a sheaf. Its stalks are given by

$$(1.13)\quad (f_\star\mathcal{A})_y = \varinjlim_{V \ni y} \mathcal{A}\big(f^{-1}(V)\big)$$

where V runs over all open neighborhoods of $y \in Y$.

Now, let $\mathcal{B}$ be a sheaf on Y, viewed as a sheaf-space with projection map $\pi : \mathcal{B} \to Y$. We define the *inverse image* $f^{-1}\mathcal{B}$ by

$$(1.14) \quad f^{-1}\mathcal{B} = \mathcal{B} \times_Y X = \big\{(s,x) \in \mathcal{B} \times X\ ;\ \pi(s) = f(x)\big\}$$

with the topology induced by the product topology on $\mathcal{B} \times X$. It is then easy to see that the projection $\pi' = \mathrm{pr}_2 : f^{-1}\mathcal{B} \to X$ is a local homeomorphism, therefore $f^{-1}\mathcal{B}$ is a sheaf on X. By construction, the stalks of $f^{-1}\mathcal{B}$ are

$$(1.15) \quad (f^{-1}\mathcal{B})_x = \mathcal{B}_{f(x)},$$

and the sections $\sigma \in f^{-1}\mathcal{B}(U)$ can be considered as continuous mappings $s : U \to \mathcal{B}$ such that $\pi \circ \sigma = f$. In particular, any section $s \in \mathcal{B}(V)$ on an open set $V \subset Y$ has a *pull-back*

$$(1.16) \quad f^\star s = s \circ f \quad \in \quad f^{-1}\mathcal{B}\big(f^{-1}(V)\big).$$

There are always natural sheaf morphisms

$$(1.17) \quad f^{-1}f_\star\mathcal{A} \longrightarrow \mathcal{A}, \qquad \mathcal{B} \longrightarrow f_\star f^{-1}\mathcal{B}$$

defined as follows. A germ in $(f^{-1}f_\star\mathcal{A})_x = (f_\star\mathcal{A})_{f(x)}$ is defined by a local section $s \in (f_\star\mathcal{A})(V) = \mathcal{A}(f^{-1}(V))$ for some neighborhood V of $f(x)$; this section can be mapped to the germ $s_x \in \mathcal{A}_x$. In the opposite direction, the pull-back $f^\star s$ of a section $s \in \mathcal{B}(V)$ can be seen by (1.16) as a section of $f_\star f^{-1}\mathcal{B}(V)$. It is not difficult to see that these natural morphisms are not isomorphisms in general. For instance, if f is a finite covering map with q sheets and if we take $\mathcal{A} = E_X$, $\mathcal{B} = E_Y$ to be constant sheaves, then $f_\star E_X \simeq E_Y^q$ and $f^{-1}E_Y = E_X$, thus $f^{-1}f_\star E_X \simeq E_X^q$ and $f_\star f^{-1}E_Y \simeq E_Y^q$.

§1.C. Ringed Spaces

Many natural geometric structures considered in analytic or algebraic geometry can be described in a convenient way as topological spaces equipped with a suitable "structure sheaf" which, most often, is a sheaf of commutative rings. For instance, a lot of properties of C^k differentiable (resp. real analytic, complex analytic) manifolds can be described in terms of their sheaf of rings $\mathcal{C}_X^k$ of differentiable functions (resp. $\mathcal{C}_X^\omega$ of real analytic functions, $\mathcal{O}_X$ of holomorphic functions). We first recall a few standard definitions concerning rings, referring to textbooks on algebra for more details (see e.g. Lang 1965).

(1.18) Some definitions and conventions about rings. *All our rings R are supposed implicitly to have a unit element 1_R (if $R = \{0\}$, we agree that $1_R = 0_R$), and a ring morphism $R \to R'$ is supposed to map 1_R to $1_{R'}$. In the subsequent definitions, we assume that all rings under consideration are commutative.*

a) *An ideal $I \subset R$ is said to be prime if $xy \in I$ implies $x \in I$ or $y \in I$, i.e., if the quotient ring R/I is entire.*

b) *An ideal $I \subset R$ is said to be maximal if $I \neq R$ and there are no ideals J such that $I \subsetneq J \subsetneq R$ (equivalently, if the quotient ring R/I is a field).*

c) *The ring R is said to be a local ring if R has a unique maximal ideal $\mathfrak{m}$ (equivalently, if R has an ideal $\mathfrak{m}$ such that all elements of $R \smallsetminus \mathfrak{m}$ are invertible). Its residual field is defined to be the quotient field $R/\mathfrak{m}$.*

d) *The ring R is said to be Noetherian if every ideal $I \subset R$ is finitely generated (equivalently, if every increasing sequence of ideals $I_1 \subset I_2 \subset \ldots$ is stationary).*

e) *The radical $\sqrt{I}$ of an ideal I is the set of all elements $x \in R$ such that some power x^m, $m \in \mathbb{N}^\star$, lies in in I. Then $\sqrt{I}$ is again an ideal of R.*

f) *The nilradical $N(R) = \sqrt{\{0\}}$ is the ideal of nilpotent elements of R. The ring R is said to be reduced if $N(R) = \{0\}$. Otherwise, its reduction is defined to be the reduced ring $R/N(R)$.*

We now introduce the general notion of a ringed space.

(1.19) Definition. *A ringed space is a pair $(X, \mathcal{R}_X)$ consisting of a topological space X and of a sheaf of rings $\mathcal{R}_X$ on X, called the structure sheaf. A morphism*

$$F : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$$

of ringed spaces is a pair $(f, F^\star)$ where $f : X \to Y$ is a continuous map and

$$F^\star \ : \ f^{-1}\mathcal{R}_Y \to \mathcal{R}_X, \qquad F^\star_x \ : \ \mathcal{R}_{Y,f(x)} \to \mathcal{R}_{X,x}$$

a homomorphism of sheaves of rings on X, called the comorphism of F.

If $F : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ and $G : (Y, \mathcal{R}_Y) \to (Z, \mathcal{R}_Z)$ are morphisms of ringed spaces, the composite $G \circ F$ is the pair consisting of the map $g \circ f : X \to Z$ and of the comorphism $(G \circ F)^\star = F^\star \circ f^{-1}G^\star$:

$$(1.20) \quad \begin{array}{lcl} F^\star \circ f^{-1}G^\star & : & f^{-1}g^{-1}\mathcal{R}_Z \xrightarrow{f^{-1}G^\star} f^{-1}\mathcal{R}_Y \xrightarrow{F^\star} \mathcal{R}_X, \\ F^\star_x \circ G^\star_{f(x)} & : & \mathcal{R}_{Z,g\circ f(x)} \longrightarrow \mathcal{R}_{Y,f(x)} \longrightarrow \mathcal{R}_{X,x}. \end{array}$$

We say of course that F is an isomorphism of ringed spaces if there exists G such that $G \circ F = \mathrm{Id}_X$ and $F \circ G = \mathrm{Id}_Y$.

If $(X, \mathcal{R}_X)$ is a ringed space, the nilradical of $\mathcal{R}_X$ defines an ideal subsheaf $\mathcal{N}_X$ of $\mathcal{R}_X$, and the identity map $\mathrm{Id}_X : X \to X$ together with the ring homomorphism $\mathcal{R}_X \to \mathcal{R}_X/\mathcal{N}_X$ defines a ringed space morphism

$$(1.21) \quad (X, \mathcal{R}_X/\mathcal{N}_X) \to (X, \mathcal{R}_X)$$

called the *reduction morphism.* Quite often, the letter X by itself is used to denote the ringed space $(X, \mathcal{R}_X)$; we then denote by $X_{\mathrm{red}} = (X, \mathcal{R}_X/\mathcal{N}_X)$ its reduction. The ringed space X is said to be *reduced* if $\mathcal{N}_X = 0$, in which case the reduction morphism $X_{\mathrm{red}} \to X$ is an isomorphism. In all examples considered later on in this book, the structure sheaf $\mathcal{R}_X$ will be a sheaf of *local rings* over some field k. The relevant definition is as follows.

(1.22) Definition.

a) *A locally ringed space is a ringed space $(X, \mathcal{R}_X)$ such that all stalks $\mathcal{R}_{X,x}$ are local rings. The maximal ideal of $\mathcal{R}_{X,x}$ will be denoted by $\mathfrak{m}_{X,x}$. A morphism $F = (f, F^\star) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that $F_x^\star(\mathfrak{m}_{Y,f(x)}) \subset \mathfrak{m}_{X,x}$ at any point $x \in X$ (i.e., $F_x^\star$ is a "local" homomorphism of rings).*

b) *A locally ringed space over a field k is a locally ringed space $(X, \mathcal{R}_X)$ such that all rings $\mathcal{R}_{X,x}$ are local k-algebras with residual field $\mathcal{R}_{X,x}/\mathfrak{m}_{X,x} \simeq k$. A morphism F between such spaces is supposed to have its comorphism defined by local k-homomorphisms $F_x^\star : \mathcal{R}_{Y,f(x)} \to \mathcal{R}_{X,x}$.*

If $(X, \mathcal{R}_X)$ is a locally ringed space over k, we can associate to each section $s \in \mathcal{R}_X(U)$ a function

$$\overline{s} : U \to k, \qquad x \mapsto \overline{s}(x) \in k = \mathcal{R}_{X,x}/\mathfrak{m}_{X,x},$$

and we get a sheaf morphism $\mathcal{R}_X \to \overline{\mathcal{R}}_X$ onto a subsheaf of rings $\overline{\mathcal{R}}_X$ of the sheaf of functions from X to k. We clearly have a factorization

$$\mathcal{R}_X \to \mathcal{R}_X/\mathcal{N}_X \to \overline{\mathcal{R}}_X,$$

and thus a corresponding factorization of ringed space morphisms (with Id_X as the underlying set theoretic map)

$$X_{\text{st-red}} \to X_{\mathrm{red}} \to X$$

where $X_{\text{st-red}} = (X, \overline{R}_X)$ is called the *strong reduction* of $(X, \mathcal{R}_X)$. It is easy to see that $X_{\text{st-red}}$ is actually a reduced locally ringed space over k. We say that X is strongly reduced if $\mathcal{R}_X \to \overline{\mathcal{R}}_X$ is an isomorphism, that is, if $\mathcal{R}_X$ can be identified with a subsheaf of the sheaf of functions $X \to k$ (in our applications to the theory of algebraic or analytic schemes, the concepts of reduction and strong reduction will actually be the same ; in general, these notions differ, see Exercise ??.??). It is important to observe that reduction (resp. strong reduction) is a fonctorial process:
if $F = (f, F^\star) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is a morphism of ringed spaces (resp. of locally ringed spaces over k), there are natural reductions

$$F_{\mathrm{red}} = (f, F^\star_{\mathrm{red}}) : X_{\mathrm{red}} \to Y_{\mathrm{red}}, \quad F^\star_{\mathrm{red}} : \mathcal{R}_{Y,f(x)}/\mathcal{N}_{Y,f(x)} \to \mathcal{R}_{X,x}/\mathcal{N}_{X,x},$$

$$F_{\text{st-red}} = (f, f^\star) : X_{\text{st-red}} \to Y_{\text{st-red}}, \quad f^\star : \overline{\mathcal{R}}_{Y,f(x)} \to \overline{\mathcal{R}}_{X,x}, \quad \overline{s} \mapsto \overline{s} \circ f$$

where $f^\star$ is the usual pull-back comorphism associated with f. Therefore, if $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ are strongly reduced, the morphism F is completely determined by the underlying set-theoretic map f. Our first basic examples of (strongly reduced) ringed spaces are the various types of manifolds already defined in Chapter I. The language of ringed spaces provides an equivalent but more elegant and more intrinsic definition.

(1.23) Definition. *Let X be a Hausdorff separable topological space. One can define the category of C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, differentiable manifolds (resp. complex analytic manifolds) to be the category of reduced locally ringed spaces $(X, \mathcal{R}_X)$ over $\mathbb{R}$ (resp. over $\mathbb{C}$), such that every point $x \in X$ has a neighborhood U on which the restriction $(U, \mathcal{R}_{X\restriction U})$ is isomorphic to a ringed space $(\Omega, \mathcal{C}^k_\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set and $\mathcal{C}^k_\Omega$ is the sheaf of C^k differentiable functions (resp. $(\Omega, \mathcal{O}_\Omega)$, where $\Omega \subset \mathbb{C}^n$ is an open subset, and $\mathcal{O}_\Omega$ is the sheaf of holomorphic functions on Ω).*

We say that the ringed spaces $(\Omega, \mathcal{C}^k_\Omega)$ and $(\Omega, \mathcal{O}_\Omega)$ are the *models* of the category of differentiable (resp. complex analytic) manifolds, and that a general object $(X, \mathcal{R}_X)$ in the category is *locally isomorphic* to one of the given model spaces. It is easy to see that the corresponding ringed spaces morphisms are nothing but the usual concepts of differentiable and holomorphic maps.

§1.D. Algebraic Varieties over a Field

As a second illustration of the notion of ringed space, we present here a brief introduction to the formalism of algebraic varieties, referring to (Hartshorne 1977) or (EGA 1967) for a much more detailed exposition. Our hope is that the reader who already has some background of analytic or algebraic geometry will find some hints of the strong interconnections between both theories. Beginners are invited to skip this section and proceed directly to the theory of complex analytic sheaves in §,2. All rings or algebras occurring in this section are supposed to be commutative rings with unit.

§1.D.1. Affine Algebraic Sets. Let k be an algebraically closed field of any characteristic. An *affine algebraic set* is a subset $X \subset k^N$ of the affine space k^N defined by an arbitrary collection $S \subset k[T_1, \ldots, T_N]$ of polynomials, that is,

$$X = V(S) = \big\{(z_1, \ldots, z_N) \in k^N \,;\, P(z_1, \ldots, z_N) = 0,\ \forall P \in S\big\}.$$

Of course, if $J \subset k[T_1, \ldots, T_N]$ is the ideal generated by S, then $V(S) = V(J)$. As $k[T_1, \ldots, T_N]$ is Noetherian, J is generated by finitely many elements $(P_1, \ldots, P_m)$, thus $X = V(\{P_1, \ldots, P_m\})$ is always defined by finitely many equations. Conversely, for any subset $Y \subset k^N$, we consider the ideal $I(Y)$ of $k[T_1, \ldots, T_N]$, defined by

$$I(Y) = \{P \in k[T_1, \ldots, T_N]\,;\; P(z) = 0,\ \forall z \in Y\}.$$

Of course, if $Y \subset k^N$ is an algebraic set, we have $V(I(Y)) = Y$. In the opposite direction, we have the following fundamental result.

(1.24) Hilbert's Nullstellensatz (see Lang 1965). *If $J \subset k[T_1, \ldots, T_N]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

If $X = V(J) \subset k^N$ is an affine algebraic set, we define the (reduced) ring $\mathcal{O}(X)$ of algebraic functions on X to be the set of all functions $X \to k$ which are restrictions of polynomials, i.e.,

$$(1.25) \quad \mathcal{O}(X) = k[T_1, \ldots, T_N]/I(X) = k[T_1, \ldots, T_N]/\sqrt{J}.$$

This is clearly a reduced k-algebra. An (algebraic) morphism of affine algebraic sets $X = V(J) \subset k^N$, $Y = V(J') \subset k^{N'}$ is a map $f : Y \to X$ which is the restriction of a polynomial map $k^{N'} tok^N$. We then get a k-algebra homomomorphism

$$f^\star : \mathcal{O}(X) \to \mathcal{O}(Y), \qquad s \mapsto s \circ f,$$

called the *comorphism* of f. In this way, we have defined a contravariant fonctor

$$(1.26) \quad X \mapsto \mathcal{O}(X), \qquad f \mapsto f^\star$$

from the category of affine algebraic sets to the category of finitely generated reduced k-algebras.

We are going to show the existence of a natural fonctor going in the opposite direction. In fact, let us start with an arbitrary finitely generated algebra A (not necessarily reduced at this moment). For any choice of generators $(g_1, \ldots, g_N)$ of A we get a surjective morphism of the polynomial ring $k[T_1, \ldots, T_N]$ onto A,

$$k[T_1, \ldots, T_N] \to A, \qquad T_j \mapsto g_j,$$

and thus $A \simeq k[T_1, \ldots, T_N]/J$ with the ideal J being the kernel of this morphism. It is well-known that every maximal ideal $\mathfrak{m}$ of A has codimension 1 in A (see Lang 1965), so that $\mathfrak{m}$ gives rise to a k-algebra homomorphism $A \to A/\mathfrak{m} = k$. We thus get a bijection

$$\mathrm{Hom}_{\mathrm{alg}}(A, k) \to \mathrm{Spm}(A), \qquad u \mapsto \mathrm{Ker}\, u$$

between the set of k-algebra homomorphisms and the set $\mathrm{Spm}(A)$ of maximal ideals of A. In fact, if $A = k[T_1, \ldots, T_N]/J$, an element $\varphi \in \mathrm{Hom}_{\mathrm{alg}}(A, k)$ is completely determined by the values $z_j = \varphi(T_j \bmod J)$, and the corresponding algebra homomorphism $k[T_1, \ldots, T_N] \to k$, $P \mapsto P(z_1, \ldots, z_N)$ can be factorized mod J if and only if $z = (z_1, \ldots, z_N) \in k^N$ satisfies the equations

$$P(z_1,\ldots,z_N)=0,\qquad \forall P\in J.$$

We infer from this that

$$\mathrm{Spm}(A)\simeq V(J)=\big\{(z_1,\ldots,z_N)\in k^N\,;\,P(z_1,\ldots,z_N)=0,\,\forall P\in J\big\}$$

can be identified with the *affine algebraic set* $V(J)\subset k^N$. If we are given an algebra homomorphism $\Phi:A\to B$ of finitely generated k-algebras we get a corresponding map $\mathrm{Spm}(\Phi):\mathrm{Spm}(B)\to\mathrm{Spm}(A)$ described either as

$$\begin{aligned}&\mathrm{Spm}(B)\to\mathrm{Spm}(A),\quad \mathfrak{m}\mapsto\Phi^{-1}(\mathfrak{m})\quad\text{or}\\ &\mathrm{Hom}_{\mathrm{alg}}(B,k)\to\mathrm{Hom}_{\mathrm{alg}}(A,k),\quad v\mapsto v\circ\Phi.\end{aligned}$$

If $B=k[T'_1,\ldots,T'_{N'}]/J'$ and $\mathrm{Spm}(B)=V(J')\subset k^{N'}$, it is easy to see that $\mathrm{Spm}(\Phi):\mathrm{Spm}(B)\to\mathrm{Spm}(A)$ is the restriction of the polynomial map

$$f:k^{N'}\to k^N,\qquad w\mapsto f(w)=(P_1(w),\ldots,P_N(w)),$$

where $P_j\in k[T'_1,\ldots,T'_{N'}]$ are polynomials such that $P_j=\Phi(T_j)\bmod J'$ in B. We have in this way defined a contravariant fonctor

$$(1.27)\quad A\mapsto\mathrm{Spm}(A),\qquad \Phi\mapsto\mathrm{Spm}(\Phi)$$

from the category of finitely generated k-algebras to the category of affine algebraic sets.

Since $A=k[T_1,\ldots,T_N]/J$ and its reduction $A/N(A)=k[T_1,\ldots,T_N]/\sqrt{J}$ give rise to the same algebraic set

$$V(J)=\mathrm{Spm}(A)=\mathrm{Spm}(A/N(A))=V(\sqrt{J}),$$

we see that the category of affine algebraic sets is actually equivalent to the subcategory of *reduced* finitely generated k-algebras.

(1.28) Example. The simplest example of an affine algebraic set is the affine space

$$k^N=\mathrm{Spm}(k[T_1,\ldots,T_N]),$$

in particular $\mathrm{Spm}(k)=k^0$ is just one point. We agree that $\mathrm{Spm}(\{0\})=\emptyset$ (observe that $V(J)=\emptyset$ when J is the unit ideal in $k[T_1,\ldots,T_N]$).

§1.D.2. Zariski Topology and Affine Algebraic Schemes. Let A be a finitely generated algebra and $X=\mathrm{Spm}(A)$. To each ideal $\mathfrak{a}\subset A$ we associate the zero variety $V(\mathfrak{a})\subset X$ which consists of all elements $\mathfrak{m}\in X=\mathrm{Spm}(A)$ such that $\mathfrak{m}\supset\mathfrak{a}$; if

$$A\simeq k[T_1,\ldots,T_N]/J\quad\text{and}\quad X\simeq V(J)\subset k^N,$$

then $V(\mathfrak{a})$ can be identified with the zero variety $V(J_{\mathfrak{a}})\subset X$ of the inverse image $J_{\mathfrak{a}}$ of $\mathfrak{a}$ in $k[T_1,\ldots,T_N]$. For any family $(\mathfrak{a}_\alpha)$ of ideals in A we have

$$V(\sum \mathfrak{a}_\alpha) = \bigcap V(\mathfrak{a}_\alpha), \qquad V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2),$$

hence there exists a unique topology on X such that the closed sets consist precisely of all algebraic subsets $(V(\mathfrak{a}))_{\mathfrak{a}\subset A}$ of X. This topology is called the Zariski topology. The Zariski topology is almost never Hausdorff (for example, if $X = k$ is the affine line, the open sets are $\emptyset$ and complements of finite sets, thus any two nonempty open sets have nonempty intersection). However, X is a *Noetherian space*, that is, a topological space in which every decreasing sequence of closed sets is stationary; an equivalent definition is to require that every open set is quasi-compact (from any open covering of an open set, one can extract a finite covering).

We now come to the concept of affine open subsets. For $s \in A$, the open set $D(s) = X \smallsetminus V(s)$ can be given the structure of an affine algebraic variety. In fact, if $A = k[T_1, \ldots, T_N]/J$ and s is represented by a polynomial in $k[T_1, \ldots, T_N]$, the localized ring $A[1/s]$ can be written as $A[1/s] = k[T_1, \ldots, T_N, T_{N+1}]/J_s$ where $J_s = J[T_{N+1}] + (sT_{N+1} - 1)$, thus

$$V(J_s) = \{(z, w) \in V(J) \times k\,;\ s(z)\,w = 1\} \simeq V(I) \smallsetminus s^{-1}(0)$$

and $D(s)$ can be identified with $\mathrm{Spm}(A[1/s])$. We have $D(s_1) \cap D(s_2) = D(s_1 s_2)$, and the sets $(D(s))_{s\in A}$ are easily seen to be a basis of the Zariski topology on X. The open sets $D(s)$ are called *affine open sets*. Since the open sets $D(s)$ containing a given point $x \in X$ form a basis of neighborhoods, one can define a sheaf space $\mathcal{O}_X$ such that the ring of germs $\mathcal{O}_{X,x}$ is the inductive limit

$$\mathcal{O}_{X,x} = \varinjlim_{D(s)\ni x} A[1/s] = \{\text{fractions } p/q\,;\ p, q \in A,\ q(x) \neq 0\}.$$

This is a local ring with maximal ideal

$$\mathfrak{m}_{X,x} = \{p/q\,;\ p, q \in A,\ p(x) = 0,\ q(x) \neq 0\},$$

and residual field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k$. In this way, we get a ringed space $(X, \mathcal{O}_X)$ over k. It is easy to see that $\Gamma(X, \mathcal{O}_X)$ coincides with the finitely generated k-algebra A. In fact, from the definition of $\mathcal{O}_X$, a global section is obtained by gluing together local sections p_j/s_j on affine open sets $D(s_j)$ with $\bigcup D(s_j) = X$, $1 \le j \le m$. This means that the ideal $\mathfrak{a} = (s_1, \ldots, s_m) \subset A$ has an empty zero variety $V(\mathfrak{a})$, thus $\mathfrak{a} = A$ and there are elements $u_j \in A$ with $\sum u_j s_j = 1$. The compatibility condition $p_j/s_j = p_k/s_k$ implies that these elements are induced by

$$\sum u_j p_j / \sum u_j s_j = \sum u_j p_j \in A,$$

as desired. More generally, since the open sets $D(s)$ are affine, we get

$$\Gamma(D(s), \mathcal{O}_X) = A[1/s].$$

It is easy to see that the ringed space $(X, \mathcal{O}_X)$ is reduced if and only if A itself is reduced; in this case, X is even strongly reduced as Hilbert's Nullstellensatz shows. Otherwise, the reduction X_{red} can obtained from the reduced algebra $A_{\mathrm{red}} = A/N(A)$.

Ringed spaces $(X, \mathcal{O}_X)$ as above are called *affine algebraic schemes* over k (although substantially different from the usual definition, our definition can be shown to be equivalent in this special situation; compare with (Hartshorne 1977); see also Exercise ??.??). The category of affine algebraic schemes is equivalent to the category of finitely generated k-algebras (with the arrows reversed).

1.D.3. Algebraic Schemes. Algebraic schemes over k are defined to be ringed spaces over k which are locally isomorphic to affine algebraic schemes, modulo an ad hoc separation condition.

(1.29) Definition. *An algebraic scheme over k is a locally ringed space $(X, \mathcal{O}_X)$ over k such that*

a) *X has a finite covering by open sets U_α such that $(U_\alpha, \mathcal{O}_{X \restriction U_\alpha})$ is isomorphic as a ringed space to an affine algebraic scheme* $(\mathrm{Spm}(A_\alpha), \mathcal{O}_{\mathrm{Spm}(A_\alpha)})$.

b) *X satisfies the algebraic separation axiom, namely the diagonal Δ_X of $X \times X$ is closed for the Zariski topology.*

A morphism of algebraic schemes is just a morphism of the underlying locally ringed spaces. An (*abstract*) *algebraic variety is the same as a reduced algebraic scheme.*

In the above definition, some words of explanation are needed for b), since the product $X \times Y$ of algebraic schemes over k *is not* the ringed space theoretic product, i.e., the product topological space equipped with the structure sheaf $\mathrm{pr}_1^\star \mathcal{O}_X \otimes_k \mathrm{pr}_2^\star \mathcal{O}_Y$. Instead, we define the product of two affine algebraic schemes $X = \mathrm{Spm}(A)$ and $Y = \mathrm{Spm}(B)$ to be $X \times Y = \mathrm{Spm}(A \otimes_k B)$, equipped with the Zariski topology and the structural sheaf associated with $A \otimes_k B$. Notice that the Zariski topology on $X \times Y$ *is not the product topology* of the Zariski topologies on X, Y, as the example $k^2 = k \times k$ shows; also, the rational function $1/(1 - z_1 - z_2) \in \mathcal{O}_{k^2,(0,0)}$ is not in $\mathcal{O}_{k,0} \otimes_k \mathcal{O}_{k,0}$. In general, if X, Y are written as $X = \bigcup U_\alpha$ and $Y = \bigcup V_\beta$ with affine open sets U_α, V_β, we define $X \times Y$ to be the union of all open affine charts $U_\alpha \times V_\beta$ with their associated structure sheaves of affine algebraic varieties, the open sets of $X \times Y$ being all unions of open sets in the various charts $U_\alpha \times V_\beta$. The separation axiom b) is introduced for the sake of excluding pathological examples such as an affine line $k \amalg \{0'\}$ with the origin changed into a double point.

1.D.4. Subschemes. If $(X, \mathcal{O}_X)$ is an affine algebraic scheme and $A = \Gamma(X, \mathcal{O}_X)$ is the associated algebra, we say that $(Y, \mathcal{O}_Y)$ is a *subscheme* of $(X, \mathcal{O}_X)$ if there is an ideal $\mathfrak{a}$ of A such that $Y \hookrightarrow X$ is the morphism defined by the algebra morphism $A \to A/\mathfrak{a}$ as its comorphism. As $\mathrm{Spm}(A/\mathfrak{a}) \to \mathrm{Spm}(A)$ has for image the set $V(\mathfrak{a})$ of maximal ideals $\mathfrak{m}$ of A containing $\mathfrak{a}$, we see that $Y = V(\mathfrak{a})$ as a set; let us introduce the ideal subsheaf $\mathcal{J} = \mathfrak{a}\mathcal{O}_X \subset \mathcal{O}_X$. Since the structural sheaf $\mathcal{O}_Y$ is obtained by taken localizations $A/\mathfrak{a}[1/s]$, it is easy to see that $\mathcal{O}_Y$ coincides with the quotient sheaf $\mathcal{O}_X/\mathcal{J}$ restricted to Y. Since $\mathfrak{a}$ has finitely many generators, the ideal sheaf $\mathcal{J}$ is locally finitely generated (see § 2 below). This leads to the following definition.

(1.30) Definition. *If $(X, \mathcal{O}_X)$ is an algebraic scheme, a (closed) subscheme is an algebraic scheme $(Y, \mathcal{O}_Y)$ such that Y is a Zariski closed subset of X, and there is a locally finitely generated ideal subsheaf $\mathcal{J} \subset \mathcal{O}_X$ such that $Y = V(\mathcal{J})$ and $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})_{\restriction Y}$.*

If $(Y, \mathcal{O}_Y)$, $(Z, \mathcal{O}_Z)$ are subschemes of $(X, \mathcal{O}_X)$ defined by ideal subsheaves $\mathcal{J}$, $\mathcal{J}' \subset \mathcal{O}_X$, there are corresponding subschemes $Y \cap Z$ and $Y \cup Z$ defined as ringed spaces

$$(Y \cap Z, \mathcal{O}_X/(\mathcal{J} + \mathcal{J}')), \qquad (Y \cup Z, \mathcal{O}_X/\mathcal{J}\mathcal{J}').$$

§1.D.5. Projective Algebraic Varieties. A very important subcategory of the category of algebraic varieties is provided by *projective algebraic varieties*. Let $\mathbb{P}^N_k$ be the projective N-space, that is, the set $k^{N+1} \smallsetminus \{0\}/k^\star$ of equivalence classes of $(N+1)$-tuples $(z_0, \ldots, z_N) \in k^{N+1} \smallsetminus \{0\}$ under the equivalence relation given by $(z_0, \ldots, z_N) \sim \lambda(z_0, \ldots, z_N)$, $\lambda \in k^\star$. The corresponding element of $\mathbb{P}^N_k$ will be denoted $[z_0 : z_1 : \ldots : z_N]$. It is clear that $\mathbb{P}^k_N$ can be covered by the $(N+1)$ affine charts U_α, $0 \le \alpha \le N$, such that

$$U_\alpha = \big\{[z_0 : z_1 : \ldots : z_N] \in \mathbb{P}^N_k \quad z_\alpha \neq 0\big\}.$$

The set U_α can be identified with the affine N-space k^N by the map

$$U_\alpha \to k^N, \qquad [z_0 : z_1 : \ldots : z_N] \mapsto \Big(\frac{z_0}{z_\alpha}, \frac{z_1}{z_\alpha}, \ldots, \frac{z_{\alpha-1}}{z_\alpha}, \frac{z_{\alpha+1}}{z_\alpha}, \ldots, \frac{z_N}{z_\alpha}\Big).$$

With this identification, $\mathcal{O}(U_\alpha)$ is the algebra of homogeneous rational functions of degree 0 in $z_0, \ldots, z_N$ which have just a power of z_α in their denominator. It is easy to see that the structure sheaves $\mathcal{O}_{U_\alpha}$ and $\mathcal{O}_{U_\beta}$ coincide in the intersections $U_\alpha \cap U_\beta$; they can be glued together to define an algebraic variety structure $(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N})$, such that $\mathcal{O}_{\mathbb{P}^N,[z]}$ consists of all homogeneous rational functions p/q of degree 0 (i.e., $\deg p = \deg q$), such that $q(z) \neq 0$.

(1.30) Definition. *An algebraic scheme or variety $(X, \mathcal{O}_X)$ is said to be projective if it is isomorphic to a closed subscheme of some projective space $(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N})$.*

We now indicate a standard way of constructing projective schemes. Let S be a collection of homogeneous polynomials $P_j \in k[z_0, \ldots, z_N]$, of degree $d_j \in \mathbb{N}$. We define an associated *projective algebraic set*

$$\widetilde{V}(S) = \big\{[z_0 : \ldots : z_N] \in \mathbb{P}^N_k \,;\, P(z) = 0,\ \forall P \in S\big\}.$$

Let J be the *homogeneous ideal* of $k[z_0, \ldots, z_N]$ generated by S (recall that an ideal J is said to be homogeneous if $J = \bigoplus J_m$ is the direct sum of its homogeneous components, or equivalently, if J is generated by homogeneous elements). We have an associated graded algebra

$$B = k[z_0, \ldots, z_N]/J = \bigoplus B_m, \qquad B_m = k[z_0, \ldots, z_N]_m/J_m$$

such that B is generated by B_1 and B_m is a finite dimensional vector space over k for each k. This is enough to construct the desired scheme structure on $\widetilde{V}(J) := \bigcap \widetilde{V}(J_m)$, as we see in the next subsection.

1.D.6. Projective Scheme Associated with a Graded Algebra. Let us start with a reduced graded k-algebra

$$B = \bigoplus_{m \in \mathbb{N}} B_m$$

such that B is generated by B_0 and B_1 as an algebra, and B_0, B_1 are finite dimensional vector spaces over k (it then follows that B is finitely generated and that all B_m are finite dimensional vector spaces). Given $s \in B_m$, $m > 0$, we define a k-algebra A_s to be the ring of all fractions of homogeneous degree 0 with a power of s as their denominator, i.e.,

$$(1.31) \quad A_s = \big\{p/s^d \,;\, p \in B_{dm},\ d \in \mathbb{N}\big\}.$$

Since A_s is generated by $\frac{1}{s}B_1^m$ over B_0, A_s is a finitely generated algebra. We define $U_s = \mathrm{Spm}(A_s)$ to be the associated affine algebraic variety. For $s \in B_m$ and $s' \in B_{m'}$, we clearly have algebra homomorphisms

$$A_s \to A_{ss'}, \qquad A_{s'} \to A_{ss'},$$

since $A_{ss'}$ is the algebra of all 0-homogeneous fractions with powers of s and s' in the denominator. As $A_{ss'}$ is the same as the localized ring $A_s[s^{m'}/s'^m]$, we see that $U_{ss'}$ can be identified with an affine open set in U_s, and we thus get canonical injections

$$U_{ss'} \hookrightarrow U_s, \qquad U_{ss'} \hookrightarrow U_{s'}.$$

(1.32) Definition. *If $B = \bigoplus_{m \in \mathbb{N}} B_m$ is a reduced graded algebra generated by its finite dimensional vector subspaces B_0 and B_1, we associate an algebraic scheme $(X, \mathcal{O}_X) = \mathrm{Proj}(B)$ as follows. To each finitely generated algebra $A_s = \big\{p/s^d \,;\, p \in B_{dm},\ d \in \mathbb{N}\big\}$ we associate an affine algebraic variety*

$U_s = \mathrm{Spm}(A_s)$. *We let* X *be the union of all open charts* U_s *with the identifications* $U_s \cap U_{s'} = U_{ss'}$ *; then the collection* (U_s) *is a basis of the topology of* X*, and* $\mathcal{O}_X$ *is the unique sheaf of local* k*-algebras such that* $\Gamma(U_s, \mathcal{O}_X) = A_s$ *for each* U_s.

The following proposition shows that only finitely many open charts are actually needed to describe X (as required in Def. 1.29 a)).

(1.33) Lemma. *If* $s_0, \ldots, s_N$ *is a basis of* B_1*, then* $\mathrm{Proj}(B) = \bigcup_{0 \le j \le N} U_{s_j}$.

Proof. In fact, if $x \in X$ is contained in a chart U_s for some $s \in B_m$, then $U_s = \mathrm{Spm}(A_s) \neq \emptyset$, and therefore $A_s \neq \{0\}$. As A_s is generated by $\frac{1}{s} B_1^m$, we can find a fraction $f = s_{j_1} \ldots s_{j_m}/s$ representing an element $f \in \mathcal{O}(U_s)$ such that $f(x) \neq 0$. Then $x \in U_s \setminus f^{-1}(0)$, and $U_s \setminus f^{-1}(0) = \mathrm{Spm}(A_s[1/f]) = U_s \cap U_{s_{j_1}} \cap \ldots \cap U_{s_{j_m}}$. In particular $x \in U_{s_{j_1}}$. □

(1.34) Example. One can consider the *projective space* $\mathbb{P}_k^N$ to be the algebraic scheme

$$\mathbb{P}_k^N = \mathrm{Proj}(k[T_0, \ldots, T_N]).$$

The Proj construction is fonctorial in the following sense: if we have a graded homomorphism $\Phi : B \to B'$ (i.e. an algebra homomorphism such that $\Phi(B_m) \subset B'_m$, then there are corresponding morphisms $A_s \to A'_{\Phi(s)}$, $U'_{\Phi(s)} \to U_s$, and we thus find a scheme morphism

$$F : \mathrm{Proj}(B') \to \mathrm{Proj}(B).$$

Also, since $p/s^d = ps^l/s^{d+l}$, the algebras A_s depend only on components B_m of large degree, and we have $A_s = A_{s^l}$. It follows easily that there is a canonical isomorphism

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}\Big(\bigoplus_m B_{lm}\Big).$$

Similarly, we may if we wish change a finite number of components B_m without affecting $\mathrm{Proj}(B)$. In particular, we may alway assume that $B_0 = k\,1_B$. By selecting finitely many generators $g_0, \ldots, g_N$ in B_1, we then find a surjective graded homomorphism $k[T_0, \ldots, T_N] \to B$, thus $B \simeq k[T_0, \ldots, T_N]/J$ for some graded ideal $J \subset B$. The algebra homomorphism $k[T_0, \ldots, T_N] \to B$ therefore yields a scheme embedding $\mathrm{Proj}(B) \to \mathbb{P}^N$ onto $V(J)$.

We will not pursue further the study of algebraic varieties from this point of view; in fact we are mostly interested in the case $k = \mathbb{C}$, and algebraic varieties over $\mathbb{C}$ are a special case of the more general concept of complex analytic space.

§2. The Local Ring of Germs of Analytic Functions

§2.A. The Weierstrass Preparation Theorem

Our first goal is to establish a basic factorization and division theorem for analytic functions of several variables, which is essentially due to Weierstrass. We follow here a simple proof given by C.L. Siegel, based on a clever use of the Cauchy formula. Let g be a holomorphic function defined on a neighborhood of 0 in $\mathbb{C}^n$, $g \not\equiv 0$. There exists a dense set of vectors $v \in \mathbb{C}^n \smallsetminus \{0\}$ such that the function $\mathbb{C} \ni t \longmapsto g(tv)$ is not identically zero. In fact the Taylor series of g at the origin can be written

$$g(tv) = \sum_{k=0}^{+\infty} \frac{1}{k!}\, t^k\, g^{(k)}(v)$$

where $g^{(k)}$ is a homogeneous polynomial of degree k on $\mathbb{C}^n$ and $g^{(k_0)} \not\equiv 0$ for some index k_0. Thus it suffices to select v such that $g^{(k_0)}(v) \neq 0$. After a change of coordinates, we may assume that $v = (0, \ldots, 0, 1)$. Let s be the vanishing order of $z_n \longmapsto g(0, \ldots, 0, z_n)$ at $z_n = 0$. There exists $r_n > 0$ such that $g(0, \ldots, 0, z_n) \neq 0$ when $0 < |z_n| \leq r_n$. By continuity of g and compactness of the circle $|z_n| = r_n$, there exists $r' > 0$ and $\varepsilon > 0$ such that

$$g(z', z_n) \neq 0 \quad \text{for } z' \in \mathbb{C}^{n-1}, \quad |z'| \leq r', \quad r_n - \varepsilon \leq |z_n| \leq r_n + \varepsilon.$$

For every integer $k \in \mathbb{N}$, let us consider the integral

$$S_k(z') = \frac{1}{2\pi \mathrm{i}} \int_{|z_n| = r_n} \frac{1}{g(z', z_n)} \frac{\partial g}{\partial z_n}(z', z_n)\, z_n^k\, dz_n.$$

Then S_k is holomorphic in a neighborhood of $|z'| \leq r'$. Rouché's theorem shows that $S_0(z')$ is the number of roots z_n of $g(z', z_n) = 0$ in the disk $|z_n| < r_n$, thus by continuity $S_0(z')$ must be a constant s. Let us denote by $w_1(z'), \ldots, w_s(z')$ these roots, counted with multiplicity. By definition of r_n, we have $w_1(0) = \ldots = w_s(0) = 0$, and by the choice of r', ε we have $|w_j(z')| < r_n - \varepsilon$ for $|z'| \leq r'$. The Cauchy residue formula yields

$$S_k(z') = \sum_{j=1}^{s} w_j(z')^k.$$

Newton's formula shows that the elementary symmetric function $c_k(z')$ of degree k in $w_1(z'), \ldots, w_s(z')$ is a polynomial in $S_1(z'), \ldots, S_k(z')$. Hence $c_k(z')$ is holomorphic in a neighborhood of $|z'| \leq r'$. Let us set

$$P(z', z_n) = z_n^s - c_1(z') z_n^{s-1} + \cdots + (-1)^s c_s(z') = \prod_{j=1}^{s} \big(z_n - w_j(z')\big).$$

For $|z'| \le r'$, the quotient $f = g/P$ (resp. $f = P/g$) is holomorphic in z_n on the disk $|z_n| < r_n + \varepsilon$, because g and P have the same zeros with the same multiplicities, and $f(z', z_n)$ is holomorphic in z' for $r_n - \varepsilon \le |z_n| \le r_n + \varepsilon$. Therefore

$$f(z', z_n) = \frac{1}{2\pi \mathrm{i}} \int_{|w_n| = r_n + \varepsilon} \frac{f(z', w_n)\, dw_n}{w_n - z_n}$$

is holomorphic in z on a neighborhood of the closed polydisk $\overline{\Delta}(r', r_n) = \{|z'| \le r'\} \times \{|z_n| \le r_n\}$. Thus g/P is invertible and we obtain:

(2.1) Weierstrass preparation theorem. *Let g be holomorphic on a neighborhood of 0 in $\mathbb{C}^n$, such that $g(0, z_n)/z_n^s$ has a not zero finite limit at $z_n = 0$. With the above choice of r' and r_n, one can write $g(z) = u(z)P(z', z_n)$ where u is an invertible holomorphic function in a neighborhood of the polydisk $\overline{\Delta}(r', r_n)$, and P is a Weierstrass polynomial in z_n, that is, a polynomial of the form*

$$P(z', z_n) = z_n^s + a_1(z') z_n^{s-1} + \cdots + a_s(z'), \qquad a_k(0) = 0,$$

with holomorphic coefficients $a_k(z')$ on a neighborhood of $|z'| \le r'$ in $\mathbb{C}^{n-1}$.

(2.2) Remark. If g vanishes at order m at 0 and $v \in \mathbb{C}^n \smallsetminus \{0\}$ is selected such that $g^{(m)}(v) \ne 0$, then $s = m$ and P must also vanish at order m at 0. In that case, the coefficients $a_k(z')$ are such that $a_k(z') = O(|z'|^k)$, $1 \le k \le s$.

(2.3) Weierstrass division theorem. *Every bounded holomorphic function f on $\Delta = \Delta(r', r_n)$ can be represented in the form*

$$f(z) = g(z) q(z) + R(z', z_n), \tag{2.4}$$

where q and R are analytic in Δ, $R(z', z_n)$ is a polynomial of degree $\le s - 1$ in z_n, and

$$\sup_{\Delta} |q| \le C \sup_{\Delta} |f|, \qquad \sup_{\Delta} |R| \le C \sup_{\Delta} |f| \tag{2.5}$$

for some constant $C \ge 0$ independent of f. The representation (2.4) *is unique.*

Proof (Siegel) It is sufficient to prove the result when $g(z) = P(z', z_n)$ is a Weierstrass polynomial.

Let us first prove the uniqueness. If $f = Pq_1 + R_1 = Pq_2 + R_2$, then

$$P(q_2 - q_1) + (R_2 - R_1) = 0.$$

It follows that the s roots z_n of $P(z', \bullet) = 0$ are zeros of $R_2 - R_1$. Since $\deg_{z_n}(R_2 - R_1) \le s - 1$, we must have $R_2 - R_1 \equiv 0$, thus $q_2 - q_1 \equiv 0$.

In order to prove the existence of (q, R), we set

$$q(z', z_n) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi \mathrm{i}} \int_{|w_n| = r_n - \varepsilon} \frac{f(z', w_n)}{P(z', w_n)(w_n - z_n)} \, dw_n, \qquad z \in \Delta \ ;$$

observe that the integral does not depend on ε when $\varepsilon < r_n - |z_n|$ is small enough. Then q is holomorphic on Δ. The function $R = f - Pq$ is also holomorphic on Δ and

$$R(z) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi \mathrm{i}} \int_{|w_n| = r_n - \varepsilon} \frac{f(z', w_n)}{P(z', w_n)} \Big[\frac{P(z', w_n) - P(z', z_n)}{(w_n - z_n)} \Big] \, dw_n.$$

The expression in brackets has the form

$$\big[(w_n^s - z_n^s) + \sum_{j=1}^{s} a_j(z')(w_n^{s-j} - z_n^{s-j})\big] / (w_n - z_n)$$

hence is a polynomial in z_n of degree $\le s - 1$ with coefficients that are holomorphic functions of z'. Thus we have the asserted decomposition $f = Pq + R$ and

$$\sup_{\Delta} |R| \le C_1 \sup_{\Delta} |f|$$

where C_1 depends on bounds for the $a_j(z')$ and on $\mu = \min |P(z', z_n)|$ on the compact set $\{|z'| \le r'\} \times \{|z_n| = r_n\}$. By the maximum principle applied to $q = (f - R)/P$ on each disk $\{z'\} \times \{|z_n| < r_n - \varepsilon\}$, we easily get

$$\sup_{\Delta} |q| \le \mu^{-1}(1 + C_1) \sup_{\Delta} |f|. \qquad \square$$

§2.B. Algebraic Properties of the Ring $\mathcal{O}_n$

We give here important applications of the Weierstrass preparation theorem to the study of the ring of germs of holomorphic functions in $\mathbb{C}^n$.

(2.6) Notation. *We let $\mathcal{O}_n$ be the ring of germs of holomorphic functions on $\mathbb{C}^n$ at 0. Alternatively, $\mathcal{O}_n$ can be identified with the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series in $z_1, \ldots, z_n$.*

(2.7) Theorem. *The ring $\mathcal{O}_n$ is Noetherian, i.e. every ideal $\mathcal{I}$ of $\mathcal{O}_n$ is finitely generated.*

Proof. By induction on n. For $n = 1$, $\mathcal{O}_n$ is principal: every ideal $\mathcal{I} \ne \{0\}$ is generated by z^s, where s is the minimum of the vanishing orders at 0 of the non zero elements of $\mathcal{I}$. Let $n \ge 2$ and $\mathcal{I} \subset \mathcal{O}_n$, $\mathcal{I} \ne \{0\}$. After a change of variables, we may assume that $\mathcal{I}$ contains a Weierstrass polynomial $P(z', z_n)$. For every $f \in \mathcal{I}$, the Weierstrass division theorem yields

$$f(z) = P(z', z_n)q(z) + R(z', z_n), \qquad R(z', z_n) = \sum_{k=0}^{s-1} c_k(z')\, z_n^k,$$

and we have $R \in \mathcal{I}$. Let us consider the set $\mathcal{M}$ of coefficients $(c_0, \ldots, c_{s-1})$ in $\mathcal{O}_{n-1}^{\oplus s}$ corresponding to the polynomials $R(z', z_n)$ which belong to $\mathcal{I}$. Then $\mathcal{M}$ is a $\mathcal{O}_{n-1}$-submodule of $\mathcal{O}_{n-1}^{\oplus s}$. By the induction hypothesis $\mathcal{O}_{n-1}$ is Noetherian; furthermore, every submodule of a finitely generated module over a Noetherian ring is finitely generated (Lang 1965, Chapter VI). Therefore $\mathcal{M}$ is finitely generated, and $\mathcal{I}$ is generated by P and by polynomials $R_1, \ldots, R_N$ associated with a finite set of generators of $\mathcal{M}$. □

Before going further, we need two lemmas which relate the algebraic properties of $\mathcal{O}_n$ to those of the polynomial ring $\mathcal{O}_{n-1}[z_n]$.

(2.8) Lemma. *Let $P, F \in \mathcal{O}_{n-1}[z_n]$ where P is a Weierstrass polynomial. If P divides F in $\mathcal{O}_n$, then P divides F in $\mathcal{O}_{n-1}[z_n]$.*

Proof. Assume that $F(z', z_n) = P(z', z_n)h(z)$, $h \in \mathcal{O}_n$. The standard division algorithm of F by P in $\mathcal{O}_{n-1}[z_n]$ yields

$$F = PQ + R, \qquad Q, R \in \mathcal{O}_{n-1}[z_n], \quad \deg R < \deg P.$$

The uniqueness part of Th. 2.3 implies $h(z) = Q(z', z_n)$ and $R \equiv 0$. □

(2.9) Lemma. *Let $P(z', z_n)$ be a Weierstrass polynomial.*

a) *If $P = P_1 \ldots P_N$ with $P_j \in \mathcal{O}_{n-1}[z_n]$, then, up to invertible elements of $\mathcal{O}_{n-1}$, all P_j are Weierstrass polynomials.*

b) *$P(z', z_n)$ is irreducible in $\mathcal{O}_n$ if and only if it is irreducible in $\mathcal{O}_{n-1}[z_n]$.*

Proof. a) Assume that $P = P_1 \ldots P_N$ with polynomials $P_j \in \mathcal{O}_{n-1}[z_n]$ of respective degrees s_j, $\sum_{1 \le j \le N} s_j = s$. The product of the leading coefficients of $P_1, \ldots, P_N$ in $\mathcal{O}_{n-1}$ is equal to 1; after normalizing these polynomials, we may assume that $P_1, \ldots, P_N$ are unitary and $s_j > 0$ for all j. Then

$$P(0, z_n) = z_n^s = P_1(0, z_n) \ldots P_N(0, z_n),$$

hence $P_j(0, z_n) = z_n^{s_j}$ and therefore P_j is a Weierstrass polynomial.

b) Set $s = \deg P$ and $P(0, z_n) = z_n^s$. Assume that P is reducible in $\mathcal{O}_n$, with $P(z', z_n) = g_1(z)g_2(z)$ for non invertible elements $g_1, g_2 \in \mathcal{O}_n$. Then $g_1(0, z_n)$ and $g_2(0, z_n)$ have vanishing orders $s_1, s_2 > 0$ with $s_1 + s_2 = s$, and

$$g_j = u_j P_j, \qquad \deg P_j = s_j, \quad j = 1, 2,$$

where P_j is a Weierstrass polynomial and $u_j \in \mathcal{O}_n$ is invertible. Therefore $P_1 P_2 = uP$ for an invertible germ $u \in \mathcal{O}_n$. Lemma 2.8 shows that P divides $P_1 P_2$ in $\mathcal{O}_{n-1}[z_n]$; since P_1, P_2 are unitary and $s = s_1 + s_2$, we get $P = P_1 P_2$, hence P is reducible in $\mathcal{O}_{n-1}[z_n]$. The converse implication is obvious from a). $\square$

(2.10) Theorem. *$\mathcal{O}_n$ is a factorial ring, i.e. $\mathcal{O}_n$ is entire and:*

a) *every non zero germ $f \in \mathcal{O}_n$ admits a factorization $f = f_1 \ldots f_N$ in irreducible elements;*

b) *the factorization is unique up to invertible elements.*

Proof. The existence part a) follows from Lemma 2.9 if we take f to be a Weierstrass polynomial and $f = f_1 \ldots f_N$ be a decomposition of maximal length N into polynomials of positive degree. In order to prove the uniqueness, it is sufficient to verify the following statement:

b$'$) *If g is an irreducible element that divides a product $f_1 f_2$, then g divides either f_1 or f_2.*

By Th. 2.1, we may assume that f_1, f_2, g are Weierstrass polynomials in z_n. Then g is irreducible and divides $f_1 f_2$ in $\mathcal{O}_{n-1}[z_n]$ thanks to Lemmas 2.8 and 2.9 b). By induction on n, we may assume that $\mathcal{O}_{n-1}$ is factorial. The standard Gauss lemma (Lang 1965, Chapter V) says that the polynomial ring $A[T]$ is factorial if the ring A is factorial. Hence $\mathcal{O}_{n-1}[z_n]$ is factorial by induction and thus g must divide f_1 or f_2 in $\mathcal{O}_{n-1}[z_n]$. $\square$

(2.11) Theorem. *If $f, g \in \mathcal{O}_n$ are relatively prime, then the germs f_z, g_z at every point $z \in \mathbb{C}^n$ near 0 are again relatively prime.*

Proof. One may assume that $f = P$, $g = Q$ are Weierstrass polynomials. Let us recall that unitary polynomials $P, Q \in \mathcal{A}[X]$ ($\mathcal{A}$ = a factorial ring) are relatively prime if and only if their resultant $R \in \mathcal{A}$ is non zero. Then the resultant $R(z') \in \mathcal{O}_{n-1}$ of $P(z', z_n)$ and $Q(z', z_n)$ has a non zero germ at 0. Therefore the germ $R_{z'}$ at points $z' \in \mathbb{C}^{n-1}$ near 0 is also non zero. $\square$

§3. Coherent Sheaves

§3.1. Locally Free Sheaves and Vector Bundles

Section 9 will greatly develope this philosophy. Before introducing the more general notion of a coherent sheaf, we discuss the notion of locally free sheaves over a sheaf a ring. All rings occurring in the sequel are supposed to be commutative with unit (the non commutative case is also of considerable interest, e.g. in view of the theory of $\mathcal{D}$-modules, but this subject is beyond the scope of the present book).

(3.1) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally* *free of rank r over $\mathcal{A}$, if $\mathcal{S}$ is locally isomorphic to $\mathcal{A}^{\oplus r}$ on a neighborhood of every point, i.e. for every $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_r \in \mathcal{S}(\Omega)$ such that the sheaf homomorphism*

$$F : \mathcal{A}^{\oplus r}_{\restriction\Omega} \longrightarrow \mathcal{S}_{\restriction\Omega}, \qquad \mathcal{A}^{\oplus r}_x \ni (w_1, \ldots, w_r) \longmapsto \sum_{1\le j\le r} w_j F_{j,x} \in \mathcal{S}_x$$

is an isomorphism.

By definition, if $\mathcal{S}$ is locally free, there is a covering $(U_\alpha)_{\alpha\in I}$ by open sets on which $\mathcal{S}$ admits free generators $F^1_\alpha, \ldots, F^r_\alpha \in \mathcal{S}(U_\alpha)$. Because the generators can be uniquely expressed in terms of any other system of independent generators, there is for each pair (α, β) a $r \times r$ matrix

$$G_{\alpha\beta} = (G^{jk}_{\alpha\beta})_{1\le j,k\le r}, \qquad G^{jk}_{\alpha\beta} \in \mathcal{A}(U_\alpha \cap U_\beta),$$

such that

$$F^k_\beta = \sum_{1\le j\le r} F^j_\alpha G^{jk}_{\alpha\beta} \qquad \text{on} \quad U_\alpha \cap U_\beta.$$

In other words, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{A}^{\oplus r}_{\restriction U_\alpha\cap U_\beta} & \xrightarrow{F_\alpha} & \mathcal{S}_{\restriction U_\alpha\cap U_\beta} \\
G_{\alpha\beta}\Big\uparrow & & \Big\| \\
\mathcal{A}^{\oplus r}_{\restriction U_\alpha\cap U_\beta} & \xrightarrow[F_\beta]{} & \mathcal{S}_{\restriction U_\alpha\cap U_\beta}
\end{array}$$

It follows easily from the equality $G_{\alpha\beta} = F_\alpha^{-1} \circ F_\beta$ that the *transition matrices* $G_{\alpha\beta}$ are invertible matrices satisfying the transition relation

$$(3.2) \quad G_{\alpha\gamma} = G_{\alpha\beta} G_{\beta\gamma} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma$$

for all indices $\alpha, \beta, \gamma \in I$. In particular $G_{\alpha\alpha} = \mathrm{Id}$ on U_α and $G^{-1}_{\alpha\beta} = G_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.

Conversely, if we are given a system of invertible $r \times r$ matrices $G_{\alpha\beta}$ with coefficients in $\mathcal{A}(U_\alpha \cap U_\beta)$ satisfying the transition relation (3.2), we can define a locally free sheaf $\mathcal{S}$ of rank r over $\mathcal{A}$ by taking $\mathcal{S} \simeq \mathcal{A}^{\oplus r}$ over each U_α, the identification over $U_\alpha \cap U_\beta$ being given by the isomorphism $G_{\alpha\beta}$. A section H of $\mathcal{S}$ over an open set $\Omega \subset X$ can just be seen as a collection of sections $H_\alpha = (H^1_\alpha, \ldots, H^r_\alpha)$ of $\mathcal{A}^{\oplus r}(\Omega \cap U_\alpha)$ satisfying the transition relations $H_\alpha = G_{\alpha\beta} H_\beta$ over $\Omega \cap U_\alpha \cap U_\beta$.

The notion of locally free sheaf is closely related to another essential notion of differential geometry, namely the notion of vector bundle (resp. topological, differentiable, holomorphic ..., vector bundle). To describe the relation between these notions, we assume that the sheaf of rings $\mathcal{A}$ is a subsheaf of the sheaf $\mathcal{C}_{\mathbb{K}}$ of continous functions on X with values in the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, containing the sheaf of locally constant functions $X \to \mathbb{K}$. Then, for each $x \in X$, there is an evaluation map

$$\mathcal{A}_x \to \mathbb{K}, \qquad w \mapsto w(x)$$

whose kernel is a maximal ideal $\mathfrak{m}_x$ of $\mathcal{A}_x$, and $\mathcal{A}_x/\mathfrak{m}_x = \mathbb{K}$. Let $\mathcal{S}$ be a locally free sheaf of rank r over $\mathcal{A}$. To each $x \in X$, we can associate a $\mathbb{K}$-vector space $E_x = \mathcal{S}_x/\mathfrak{m}_x\mathcal{S}_x$: since $\mathcal{S}_x \simeq \mathcal{A}_x^{\oplus r}$, we have $E_x \simeq (\mathcal{A}_x/\mathfrak{m}_x)^{\oplus r} = \mathbb{K}^r$. The set $E = \coprod_{x \in X} E_x$ is equipped with a natural projection

$$\pi : E \to X, \qquad \xi \in E_x \mapsto \pi(\xi) := x,$$

and the fibers $E_x = \pi^{-1}(x)$ have a structure of r-dimensional $\mathbb{K}$-vector space: such a structure E is called a $\mathbb{K}$-*vector bundle of rank* r over X. Every section $s \in \mathcal{S}(U)$ gives rise to a *section* of E over U by setting $s(x) = s_x \bmod \mathfrak{m}_x$. We obtain a function (still denoted by the same symbol) $s : U \to E$ such that $s(x) \in E_x$ for every $x \in U$, i.e. $\pi \circ s = \mathrm{Id}_U$. It is clear that $\mathcal{S}(U)$ can be considered as a $\mathcal{A}(U)$-submodule of the $\mathbb{K}$-vector space of functions $U \to E$ mapping a point $x \in U$ to an element in the fiber E_x. Thus we get a subsheaf of the sheaf of E-valued sections, which is in a natural way a $\mathcal{A}$-module isomorphic to $\mathcal{S}$. This subsheaf will be denoted by $\mathcal{A}(E)$ and will be called the *sheaf of* $\mathcal{A}$*-sections* of E. If we are given a $\mathbb{K}$-vector bundle E over X and a subsheaf $\mathcal{S} = \mathcal{A}(E)$ of the sheaf of all sections of E which is in a natural way a locally free $\mathcal{A}$-module of rank r, we say that E (or more precisely the pair $(E, \mathcal{A}(E))$) is a $\mathcal{A}$-vector bundle of rank r over X.

(3.3) Example. In case $\mathcal{A} = \mathcal{C}_{X,\mathbb{K}}$ is the sheaf of all $\mathbb{K}$-valued continuous functions on X, we say that E is a *topological* vector bundle over X. When X is a manifold and $\mathcal{A} = \mathcal{C}^p_{X,\mathbb{K}}$, we say that E is a C^p*-differentiable* vector bundle; finally, when X is complex analytic and $\mathcal{A} = \mathcal{O}_X$, we say that E is a *holomorphic* vector bundle.

Let us introduce still a little more notation. Since $\mathcal{A}(E)$ is a locally free sheaf of rank r over any open set U_α in a suitable covering of X, a choice of generators $(F^1_\alpha, \ldots, F^r_\alpha)$ for $\mathcal{A}(E)_{\restriction U_\alpha}$ yields corresponding generators $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of the fibers E_x over $\mathbb{K}$. Such a system of generators is called a $\mathcal{A}$*-admissible frame* of E over U_α. There is a corresponding isomorphism

$$\text{(3.4)} \quad \theta_\alpha : E_{\restriction U_\alpha} := \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^r$$

which to each $\xi \in E_x$ associates the pair $(x, (\xi^1_\alpha, \ldots, \xi^r_\alpha)) \in U_\alpha \times \mathbb{K}^r$ composed of x and of the components $(\xi^j_\alpha)_{1 \le j \le r}$ of ξ in the basis $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of E_x. The bundle E is said to be *trivial* if it is of the form $X \times \mathbb{K}^r$, which is the same as saying that $\mathcal{A}(E) = \mathcal{A}^{\oplus r}$. For this reason, the isomorphisms

θ_α are called *trivializations* of E over U_α. The corresponding *transition automorphisms* are

$$\theta_{\alpha\beta} := \theta_\alpha \circ \theta_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^r \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{K}^r,$$
$$\theta_{\alpha\beta}(x,\xi) = (x, g_{\alpha\beta}(x)\cdot\xi), \qquad (x,\xi) \in (U_\alpha \cap U_\beta) \times \mathbb{K}^r,$$

where $(g_{\alpha\beta}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ are the transition matrices already described (except that they are just seen as matrices with coefficients in $\mathbb{K}$ rather than with coefficients in a sheaf). Conversely, if we are given a collection of matrices $g_{\alpha\beta} = (g_{\alpha\beta}^{jk}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ satisfying the transition relation

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma,$$

we can define a $\mathcal{A}$-vector bundle

$$E = \Big(\coprod_{\alpha\in I} U_\alpha \times \mathbb{K}^r\Big)/\sim$$

by gluing the charts $U_\alpha \times \mathbb{K}^r$ via the identification $(x_\alpha,\xi_\alpha) \sim (x_\beta,\xi_\beta)$ if and only if $x_\alpha = x_\beta = x \in U_\alpha \cap U_\beta$ and $\xi_\alpha = g_{\alpha\beta}(x)\cdot\xi_\beta$.

(3.5) Example. When X is a real differentiable manifold, an interesting example of real vector bundle is the *tangent bundle* T_X ; if $\tau_\alpha : U_\alpha \to \mathbb{R}^n$ is a collection of coordinate charts on X, then $\theta_\alpha = \pi \times d\tau_\alpha : T_{X\restriction U_\alpha} \to U_\alpha \times \mathbb{R}^m$ define trivializations of T_X and the transition matrices are given by $g_{\alpha\beta}(x) = d\tau_{\alpha\beta}(x^\beta)$ where $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ and $x^\beta = \tau_\beta(x)$. The dual $T_X^\star$ of T_X is called the *cotangent bundle* of X. If X is complex analytic, then T_X has the structure of a holomorphic vector bundle.

We now briefly discuss the concept of sheaf and bundle morphisms. If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of $\mathcal{A}$-modules over a topological space X, then by a morphism $\varphi : \mathcal{S} \to \mathcal{S}'$ we just mean a $\mathcal{A}$-linear sheaf morphism. If $\mathcal{S} = \mathcal{A}(E)$ and $\mathcal{S}' = \mathcal{A}(E')$ are locally free sheaves, this is the same as a $\mathcal{A}$-linear bundle morphism, that is, a fiber preserving $\mathbb{K}$-linear morphism $\varphi(x) : E_x \to E'_x$ such that the matrix representing φ in any local $\mathcal{A}$-admissible frames of E and E' has coefficients in $\mathcal{A}$.

(3.6) Proposition. *Suppose that $\mathcal{A}$ is a sheaf of local rings, i.e. that a section of $\mathcal{A}$ is invertible in $\mathcal{A}$ if and only if it never takes the zero value in $\mathbb{K}$. Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a $\mathcal{A}$-morphism of locally free $\mathcal{A}$-modules of rank r, r'. If the rank of the $r'\times r$ matrix $\varphi(x) \in M_{r'r}(\mathbb{K})$ is constant for all $x \in X$, then* $\mathrm{Ker}\,\varphi$ *and* $\mathrm{Im}\,\varphi$ *are locally free subsheaves of $\mathcal{S}$, $\mathcal{S}'$ respectively, and* $\mathrm{Coker}\,\varphi = \mathcal{S}'/\mathrm{Im}\,\varphi$ *is locally free.*

Proof. This is just a consequence of elementary linear algebra, once we know that non zero determinants with coefficients in $\mathcal{A}$ can be inverted. □

Note that all three sheaves $\mathcal{C}_{X,\mathbb{K}}$, $\mathcal{C}^p_{X,\mathbb{K}}$, $\mathcal{O}_X$ are sheaves of local rings, so Prop. 3.6 applies to these cases. However, even if we work in the holomorphic category ($\mathcal{A} = \mathcal{O}_X$), a difficulty immediately appears that the kernel or cokernel of an arbitrary morphism of locally free sheaves is in general not locally free.

(3.7) Examples.

a) Take $X = \mathbb{C}$, let $\mathcal{S} = \mathcal{S}' = \mathcal{O}$ be the trivial sheaf, and let $\varphi : \mathcal{O} \to \mathcal{O}$ be the morphism $u(z) \mapsto z\,u(z)$. It is immediately seen that φ is injective as a sheaf morphism ($\mathcal{O}$ being an entire ring), and that $\operatorname{Coker}\varphi$ is the *skyscraper sheaf* $\mathbb{C}_0$ of stalk $\mathbb{C}$ at $z = 0$, having zero stalks at all other points $z \neq 0$. Thus $\operatorname{Coker}\varphi$ is not a locally free sheaf, although φ is everywhere injective (note however that the corresponding morphism $\varphi : E \to E'$, $(z,\xi) \mapsto (z, z\xi)$ of trivial rank 1 vector bundles $E = E' = \mathbb{C} \times \mathbb{C}$ is *not injective* on the zero fiber E_0).

b) Take $X = \mathbb{C}^3$, $\mathcal{S} = \mathcal{O}^{\oplus 3}$, $\mathcal{S}' = \mathcal{O}$ and

$$\varphi : \mathcal{O}^{\oplus 3} \to \mathcal{O}, \qquad (u_1, u_2, u_3) \mapsto \sum_{1 \le j \le 3} z_j u_j(z_1, z_2, z_3).$$

Since φ yields a surjective bundle morphism on $\mathbb{C}^3 \smallsetminus \{0\}$, one easily sees that $\operatorname{Ker}\varphi$ is locally free of rank 2 over $\mathbb{C}^3 \smallsetminus \{0\}$. However, by looking at the Taylor expansion of the u_j's at 0, it is not difficult to check that $\operatorname{Ker}\varphi$ is the $\mathcal{O}$-submodule of $\mathcal{O}^{\oplus 3}$ generated by the three sections $(-z_2, z_1, 0)$, $(-z_3, 0, z_1)$ and $(0, z_3, -z_2)$, and that any two of these three sections cannot generate the 0-stalk $(\operatorname{Ker}\varphi)_0$. Hence $\operatorname{Ker}\varphi$ is not locally free.

Since the category of locally free $\mathcal{O}$-modules is not stable by taking kernels or cokernels, one is led to introduce a more general category which will be stable under these operations. This leads to the notion of *coherent sheaves.*

§3.2. Notion of Coherence

The notion of coherence again deals with sheaves of modules over a sheaf of rings. It is a semi-local property which says roughly that the sheaf of modules locally has a finite presentation in terms of generators and relations. We describe here some general properties of this notion, before concentrating ourselves on the case of coherent $\mathcal{O}_X$-modules.

(3.8) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally finitely generated if for every point $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_q \in \mathcal{S}(\Omega)$ such that for every $x \in \Omega$ the stalk $\mathcal{S}_x$ is generated by the germs $F_{1,x}, \ldots, F_{q,x}$ as an $\mathcal{A}_x$-module.*

(3.9) Lemma. *Let $\mathcal{S}$ be a locally finitely generated sheaf of $\mathcal{A}$-modules on X and $G_1, \ldots, G_N$ sections in $\mathcal{S}(U)$ such that $G_{1,x_0}, \ldots, G_{N,x_0}$ generate $\mathcal{S}_{x_0}$ at $x_0 \in U$. Then $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for x near x_0.*

Proof. Take $F_1, \ldots, F_q$ as in Def. 3.8. As $G_1, \ldots, G_N$ generate $\mathcal{S}_{x_0}$, one can find a neighborhood $\Omega' \subset \Omega$ of x_0 and $H_{jk} \in \mathcal{A}(\Omega')$ such that $F_j = \sum H_{jk} G_k$ on Ω'. Thus $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for all $x \in \Omega'$. □

§**3.2.1.** Definition of Coherent Sheaves. If U is an open subset of X, we denote by $\mathcal{S}_{\restriction U}$ the restriction of $\mathcal{S}$ to U, i.e. the union of all stalks $\mathcal{S}_x$ for $x \in U$. If $F_1, \ldots, F_q \in \mathcal{S}(U)$, the kernel of the sheaf homomorphism $F : \mathcal{A}^{\oplus q}_{\restriction U} \longrightarrow \mathcal{S}_{\restriction U}$ defined by

$$(3.10) \quad \mathcal{A}^{\oplus q}_x \ni (g^1, \ldots, g^q) \longmapsto \sum_{1 \le j \le q} g^j F_{j,x} \in \mathcal{S}_x, \qquad x \in U$$

is a subsheaf $\mathcal{R}(F_1, \ldots, F_q)$ of $\mathcal{A}^{\oplus q}_{\restriction U}$, called the *sheaf of relations* between $F_1, \ldots, F_q$.

(3.11) Definition. *A sheaf $\mathcal{S}$ of $\mathcal{A}$-modules on X is said to be coherent if:*

a) $\mathcal{S}$ *is locally finitely generated* ;

b) *for any open subset U of X and any $F_1, \ldots, F_q \in \mathcal{S}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is locally finitely generated.*

Assumption a) means that every point $x \in X$ has a neighborhood Ω such that there is a surjective sheaf morphism $F : \mathcal{A}^{\oplus q}_{\restriction \Omega} \longrightarrow \mathcal{S}_{\restriction \Omega}$, and assumption b) implies that the kernel of F is locally finitely generated. Thus, after shrinking Ω, we see that $\mathcal{S}$ admits over Ω a finite presentation under the form of an exact sequence

$$(3.12) \quad \mathcal{A}^{\oplus p}_{\restriction \Omega} \xrightarrow{G} \mathcal{A}^{\oplus q}_{\restriction \Omega} \xrightarrow{F} \mathcal{S}_{\restriction \Omega} \longrightarrow 0,$$

where G is given by a $q \times p$ matrix (G_{jk}) of sections of $\mathcal{A}(\Omega)$ whose columns $(G_{j1}), \ldots, (G_{jp})$ are generators of $\mathcal{R}(F_1, \ldots, F_q)$.

It is clear that every locally finitely generated subsheaf of a coherent sheaf is coherent. From this we easily infer:

(3.13) Theorem. *Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a $\mathcal{A}$-morphism of coherent sheaves. Then* $\operatorname{Im} \varphi$ *and* $\ker \varphi$ *are coherent.*

Proof. Clearly $\operatorname{Im} \varphi$ is a locally finitely generated subsheaf of $\mathcal{G}$, so it is coherent. Let $x_0 \in X$, let $F_1, \ldots, F_q \in \mathcal{F}(\Omega)$ be generators of $\mathcal{F}$ on a neighborhood Ω of x_0, and $G_1, \ldots, G_r \in \mathcal{A}(\Omega')^{\oplus q}$ be generators of $\mathcal{R}\big(\varphi(F_1), \ldots, \varphi(F_q)\big)$ on a neighborhood $\Omega' \subset \Omega$ of x_0. Then $\ker \varphi$ is generated over Ω' by the sections

$$H_j = \sum_{k=1}^{q} G_j^k F_k \in \mathcal{F}(\Omega'), \qquad 1 \le j \le r. \qquad \square$$

(3.14) Theorem. *Let* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow 0$ *be an exact sequence of* $\mathcal{A}$*-modules. If two of the sheaves* $\mathcal{F}, \mathcal{S}, \mathcal{G}$ *are coherent, then all three are coherent.*

Proof. If $\mathcal{S}$ and $\mathcal{G}$ are coherent, then $\mathcal{F} = \ker(\mathcal{S} \to \mathcal{G})$ is coherent by Th. 3.13. If $\mathcal{S}$ and $\mathcal{F}$ are coherent, then $\mathcal{G}$ is locally finitely generated; to prove the coherence, let $G_1, \ldots, G_q \in \mathcal{G}(U)$ and $x_0 \in U$. Then there is a neighborhood Ω of x_0 and sections $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega)$ which are mapped to $G_1, \ldots, G_q$ on Ω. After shrinking Ω, we may assume also that $\mathcal{F}_{\restriction\Omega}$ is generated by sections $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$. Then $\mathcal{R}(G_1, \ldots, G_q)$ is the projection on the last q-components of $\mathcal{R}(F_1, \ldots, F_p, \tilde{G}_1, \ldots, \tilde{G}_q) \subset \mathcal{A}^{p+q}$, which is finitely generated near x_0 by the coherence of $\mathcal{S}$. Hence $\mathcal{R}(G_1, \ldots, G_q)$ is finitely generated near x_0 and $\mathcal{G}$ is coherent.

Finally, assume that $\mathcal{F}$ and $\mathcal{G}$ are coherent. Let $x_0 \in X$ be any point, let $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$ and $G_1, \ldots, G_q \in \mathcal{G}(\Omega)$ be generators of $\mathcal{F}$, $\mathcal{G}$ on a neighborhood Ω of x_0. There is a neighborhood Ω' of x_0 such that $G_1, \ldots, G_q$ admit liftings $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega')$. Then $(F_1, \ldots, F_q, \tilde{G}_1, \ldots, \tilde{G}_q)$ generate $\mathcal{S}_{\restriction\Omega'}$, so $\mathcal{S}$ is locally finitely generated. Now, let $S_1, \ldots, S_q$ be arbitrary sections in $\mathcal{S}(U)$ and $\overline{S}_1, \ldots, \overline{S}_q$ their images in $\mathcal{G}(U)$. For any $x_0 \in U$, the sheaf of relations $\mathcal{R}(\overline{S}_1, \ldots, \overline{S}_q)$ is generated by sections $P_1, \ldots, P_s \in \mathcal{A}(\Omega)^{\oplus q}$ on a small neighborhood Ω of x_0. Set $P_j = (P_j^k)_{1 \le k \le q}$. Then $H_j = P_j^1 S_1 + \ldots + P_j^q S_q$, $1 \le j \le s$, are mapped to 0 in $\mathcal{G}$ so they can be seen as sections of $\mathcal{F}$. The coherence of $\mathcal{F}$ shows that $\mathcal{R}(H_1, \ldots, H_s)$ has generators $Q_1, \ldots, Q_t \in \mathcal{A}(\Omega')^s$ on a small neighborhood $\Omega' \subset \Omega$ of x_0. Then $\mathcal{R}(S_1, \ldots, S_q)$ is generated over Ω' by $R_j = \sum Q_j^k P_k \in \mathcal{A}(\Omega')$, $1 \le j \le t$, and $\mathcal{S}$ is coherent. $\square$

(3.15) Corollary. *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are coherent subsheaves of a coherent analytic sheaf* $\mathcal{S}$*, the intersection* $\mathcal{F} \cap \mathcal{G}$ *is a coherent sheaf.*

Proof. Indeed, the intersection sheaf $\mathcal{F} \cap \mathcal{G}$ is the kernel of the composite morphism $\mathcal{F} \hookrightarrow \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{G}$, and $\mathcal{S}/\mathcal{G}$ is coherent. $\square$

§**3.2.2. Coherent Sheaf of Rings.** A sheaf of rings $\mathcal{A}$ is said to be coherent if it is coherent as a module over itself. By Def. 3.11, this means that for any open set $U \subset X$ and any sections $F_j \in \mathcal{A}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is finitely generated. The above results then imply that all free modules $\mathcal{A}^{\oplus p}$ are coherent. As a consequence:

(3.16) Theorem. *If* $\mathcal{A}$ *is a coherent sheaf of rings, any locally finitely generated subsheaf of* $\mathcal{A}^{\oplus p}$ *is coherent. In particular, if* $\mathcal{S}$ *is a coherent* $\mathcal{A}$*-module* *and* $F_1, \ldots, F_q \in \mathcal{S}(U)$*, the sheaf of relations* $\mathcal{R}(F_1, \ldots, F_q) \subset \mathcal{A}^{\oplus q}$ *is also coherent.*

Let $\mathcal{S}$ be a coherent sheaf of modules over a coherent sheaf of ring $\mathcal{A}$. By an iteration of construction (3.12), we see that for every integer $m \geq 0$ and every point $x \in X$ there is a neighborhood Ω of x on which there is an exact sequence of sheaves

$$(3.17) \quad \mathcal{A}^{\oplus p_m}_{\restriction\Omega} \xrightarrow{F_m} \mathcal{A}^{\oplus p_{m-1}}_{\restriction\Omega} \longrightarrow \cdots \longrightarrow \mathcal{A}^{\oplus p_1}_{\restriction\Omega} \xrightarrow{F_1} \mathcal{A}^{\oplus p_0}_{\restriction\Omega} \xrightarrow{F_0} \mathcal{S}_{\restriction\Omega} \longrightarrow 0,$$

where F_j is given by a $p_{j-1} \times p_j$ matrix of sections in $\mathcal{A}(\Omega)$.

§3.3. Analytic Sheaves and the Oka Theorem

Many properties of holomorphic functions which will be considered in this book can be expressed in terms of sheaves. Among them, analytic sheaves play a central role. The Oka theorem (Oka 1950) asserting the coherence of the sheaf of holomorphic functions can be seen as a far-reaching deepening of the noetherian property seen in Sect. 1. The theory of analytic sheaves could not be presented without it.

(3.18) Definition. *Let* M *be a* n*-dimensional complex analytic manifold and let* $\mathcal{O}_M$ *be the sheaf of germs of analytic functions on* M*. An analytic sheaf over* M *is by definition a sheaf* $\mathcal{S}$ *of modules over* $\mathcal{O}_M$*.*

(3.19) Coherence theorem of Oka. *The sheaf of rings* $\mathcal{O}_M$ *is coherent for any complex manifold* M*.*

Let $F_1, \ldots, F_q \in \mathcal{O}(U)$. Since $\mathcal{O}_{M,x}$ is Noetherian, we already know that every stalk $\mathcal{R}(F_1, \ldots, F_q)_x \subset \mathcal{O}^{\oplus q}_{M,x}$ is finitely generated, but the important new fact expressed by the theorem is that the sheaf of relations is locally finitely generated, namely that the "same" generators can be chosen to generate each stalk in a neighborhood of a given point.

Proof. By induction on $n = \dim_{\mathbb{C}} M$. For $n = 0$, the stalks $\mathcal{O}_{M,x}$ are equal to $\mathbb{C}$ and the result is trivial. Assume now that $n \geq 1$ and that the result has already been proved in dimension $n - 1$. Let U be an open set of M and $F_1, \ldots, F_q \in \mathcal{O}_M(U)$. To show that $\mathcal{R}(F_1, \ldots, F_q)$ is locally finitely generated, we may assume that $U = \Delta = \Delta' \times \Delta_n$ is a polydisk in $\mathbb{C}^n$ centered at $x_0 = 0$; after a change of coordinates and multiplication of $F_1, \ldots, F_q$ by invertible functions, we may also suppose that $F_1, \ldots, F_q$ are Weierstrass polynomials in z_n with coefficients in $\mathcal{O}(\Delta')$. We need a lemma.

(3.20) Lemma. *If* $x = (x', x_n) \in \Delta$*, the* $\mathcal{O}_{\Delta,x}$*-module* $\mathcal{R}(F_1, \ldots, F_q)_x$ *is generated by those of its elements whose components are germs of analytic* *polynomials in $\mathcal{O}_{\Delta',x'}[z_n]$ with a degree in z_n at most equal to μ, the maximum of the degrees of $F_1,\ldots,F_q$.*

Proof. Assume for example that F_q is of the maximum degree μ. By the Weierstrass preparation Th. 1.1 and Lemma 1.9 applied at x, we can write $F_{q,x} = f'f''$ where $f', f'' \in \mathcal{O}_{\Delta',x'}[z_n]$, f' is a Weierstrass polynomial in $z_n - x_n$ and $f''(x) \neq 0$. Let μ' and μ'' denote the degrees of f' and f'' with respect to z_n, so $\mu' + \mu'' = \mu$. Now, take $(g^1,\ldots,g^q) \in \mathcal{R}(F_1,\ldots,F_q)_x$. The Weierstrass division theorem gives

$$g^j = F_{q,x}t^j + r^j, \qquad j = 1,\ldots,q-1,$$

where $t^j \in \mathcal{O}_{\Delta,x}$ and $r^j \in \mathcal{O}_{\Delta',x'}[z_n]$ is a polynomial of degree $< \mu'$. For $j = q$, define $r^q = g^q + \sum_{1\le j\le q-1} t^j F_{j,x}$. We can write

$$(3.21) \quad (g^1,\ldots,g^q) = \sum_{1\le j\le q} t^j(0,\ldots,F_q,\ldots,0,-F_j)_x + (r^1,\ldots,r^q)$$

where F_q is in the j-th position in the q-tuples of the summation. Since these q-tuples are in $\mathcal{R}(F_1,\ldots,F_q)_x$, we have $(r^1,\ldots,r^q) \in \mathcal{R}(F_1,\ldots,F_q)_x$, thus

$$\sum_{1\le j\le q-1} F_{j,x}r^j + f'f''r^q = 0.$$

As the sum is a polynomial in z_n of degree $< \mu+\mu'$, it follows from Lemma 1.9 that $f''r^q$ is a polynomial in z_n of degree $< \mu$. Now we have

$$(r^1,\ldots,r^q) = 1/f''(f''r^1,\ldots,f''r^q)$$

where $f''r^j$ is of degree $< \mu'+\mu'' = \mu$. In combination with (3.21) this proves the lemma. □

Proof of Theorem 3.19 (end) If $g = (g^1,\ldots,g^q)$ is one of the polynomials of $\mathcal{R}(F_1,\ldots,F_q)_x$ described in Lemma 3.20, we can write

$$g^j = \sum_{0\le k\le\mu} u^{jk}z_n^k, \qquad u^{jk} \in \mathcal{O}_{\Delta',x'}.$$

The condition for $(g^1,\ldots,g^q)$ to belong to $\mathcal{R}(F_1,\ldots,F_q)_x$ therefore consists of $2\mu+1$ linear conditions for the germ $u = (u^{jk})$ with coefficients in $\mathcal{O}(\Delta')$. By the induction hypothesis, $\mathcal{O}_{\Delta'}$ is coherent and Th. 3.16 shows that the corresponding modules of relations are generated over $\mathcal{O}_{\Delta',x'}$, for x' in a neighborhood Ω' of 0, by finitely many $(q\times\mu)$-tuples $U_1,\ldots,U_N \in \mathcal{O}(\Omega')^{q\mu}$. By Lemma 3.20, $\mathcal{R}(F_1,\ldots,F_q)_x$ is generated at every point $x \in \Omega = \Omega'\times\Delta_n$ by the germs of the corresponding polynomials

$$G_l(z) = \Big(\sum_{0\le k\le\mu} U_l^{jk}(z')z_n^k\Big)_{1\le j\le q}, \qquad z\in\Omega, \quad 1\le l\le N. \qquad \square$$

(3.22) Strong Noetherian property. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold M and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an increasing sequence of coherent subsheaves of $\mathcal{F}$. Then the sequence $(\mathcal{F}_k)$ is stationary on every compact subset of M.*

Proof. Since $\mathcal{F}$ is locally a quotient of a free module $\mathcal{O}_M^{\oplus q}$, we can pull back the sequence to $\mathcal{O}_M^{\oplus q}$ and thus suppose $\mathcal{F} = \mathcal{O}_M$ (by easy reductions similar to those in the proof of Th. 3.14). Suppose M connected and $\mathcal{F}_{k_0} \neq \{0\}$ for some index k_0 (otherwise, there is nothing to prove). By the analytic continuation theorem, we easily see that $\mathcal{F}_{k_0,x} \neq \{0\}$ for every $x \in M$. We can thus find a non zero Weierstrass polynomial $P \in \mathcal{F}_{k_0}(V)$, $\deg_{z_n} P(z', z_n) = \mu$, in a coordinate neighborhood $V = \Delta' \times \Delta_n$ of any point $x \in M$. A division by P shows that for $k \geq k_0$ and $x \in V$, all stalks $\mathcal{F}_{k,x}$ are generated by P_x and by polynomials of degree $< \mu$ in z_n with coefficients in $\mathcal{O}_{\Delta',x'}$. Therefore, we can apply induction on n to the coherent $\mathcal{O}_{\Delta'}$-module

$$\mathcal{F}' = \mathcal{F} \cap \big\{Q \in \mathcal{O}_{\Delta'}[z_n]\,;\, \deg Q \leq \mu\big\}$$

and its increasing sequence of coherent subsheaves $\mathcal{F}'_k = \mathcal{F}_k \cap \mathcal{F}'$. □

§4. Complex Analytic Sets. Local Properties

§4.1. Definition. Irreducible Components

A complex analytic set is a set which can be defined locally by finitely many holomorphic equations; such a set has in general singular points, because no assumption is made on the differentials of the equations. We are interested both in the description of the singularities and in the study of algebraic properties of holomorphic functions on analytic sets. For a more detailed study than ours, we refer to H. Cartan's seminar (Cartan 1950), to the books of (Gunning-Rossi 1965), (Narasimhan 1966) or the recent book by (Grauert-Remmert 1984).

(4.1) Definition. *Let M be a complex analytic manifold. A subset $A \subset M$ is said to be an analytic subset of M if A is closed and if for every point $x_0 \in A$ there exist a neighborhood U of x_0 and holomorphic functions $g_1, \ldots, g_n$ in $\mathcal{O}(U)$ such that*

$$A \cap U = \{z \in U \;;\; g_1(z) = \ldots = g_N(z) = 0\}.$$

Then $g_1, \ldots, g_N$ are said to be (*local*) *equations of A in U.*

It is easy to see that a finite union or intersection of analytic sets is analytic: if (g'_j), (g''_k) are equations of A', A'' in the open set U, then the family of all products $(g'_j g''_k)$ and the family $(g'_j) \cup (g''_k)$ define equations of $A' \cup A''$ and $A' \cap A''$ respectively.

(4.2) Remark. Assume that M is connected. The analytic continuation theorem shows that either $A = M$ or A has no interior point. In the latter case, each piece $A \cap U = g^{-1}(0)$ is the set of points where the function $\log|g|^2 = \log(|g_1|^2 + \cdots + |g_N|^2) \in \mathrm{Psh}(U)$ takes the value $-\infty$, hence A is pluripolar. In particular $M \smallsetminus A$ is connected and every function $f \in \mathcal{O}(M \smallsetminus A)$ that is locally bounded near A can be extended to a function $\tilde{f} \in \mathcal{O}(M)$. □

We focus now our attention on local properties of analytic sets. By definition, a germ of set at a point $x \in M$ is an equivalence class of elements in the power set $\mathcal{P}(M)$, with $A \sim B$ if there is an open neighborhood V of x such that $A \cap V = B \cap V$. The germ of a subset $A \subset M$ at x will be denoted by (A, x). We most often consider the case when $A \subset M$ is a analytic set in a neighborhood U of x, and in this case we denote by $\mathcal{I}_{A,x}$ the ideal of germs $f \in \mathcal{O}_{M,x}$ which vanish on (A, x). Conversely, if $\mathcal{J} = (g_1, \ldots, g_N)$ is an ideal of $\mathcal{O}_{M,x}$, we denote by $\big(V(\mathcal{J}), x\big)$ the germ at x of the zero variety $V(\mathcal{J}) = \{z \in U \,;\, g_1(z) = \ldots = g_N(z) = 0\}$, where U is a neighborhood of x such that $g_j \in \mathcal{O}(U)$. It is easy to check that the germ $(V(cJ), x)$ does not depend on the choice of generators. Moreover, it is clear that

(4.3′) for every ideal $\mathcal{J}$ in the ring $\mathcal{O}_{M,x}$, $\qquad \mathcal{I}_{V(\mathcal{J}),x} \supset \mathcal{J}$,

(4.3″) for every germ of analytic set (A, x), $\qquad \big(V(\mathcal{I}_{A,x}), x\big) = (A, x)$.

(4.4) Definition. *A germ (A, x) is said to be irreducible if it has no decomposition $(A, x) = (A_1, x) \cup (A_2, x)$ with analytic sets $(A_j, x) \neq (A, x)$, $j = 1, 2$.*

(4.5) Proposition. *A germ (A, x) is irreducible if and only if $\mathcal{I}_{A,x}$ is a prime ideal of the ring $\mathcal{O}_{M,x}$.*

Proof. Let us recall that an ideal $\mathcal{J}$ is said to be *prime* if $fg \in \mathcal{J}$ implies $f \in \mathcal{J}$ or $g \in \mathcal{J}$. Assume that (A, x) is irreducible and that $fg \in \mathcal{I}_{A,x}$. As we can write $(A, x) = (A_1, x) \cup (A_2, x)$ with $A_1 = A \cap f^{-1}(0)$ and $A_2 = A \cap g^{-1}(0)$, we must have for example $(A_1, x) = (A, x)$; thus $f \in \mathcal{I}_{A,x}$ and $\mathcal{I}_{A,x}$ is prime. Conversely, if $(A, x) = (A_1, x) \cup (A_2, x)$ with $(A_j, x) \neq (A, x)$, there exist $f \in \mathcal{I}_{A_1,x}$, $g \in \mathcal{I}_{A_2,x}$ such that $f, g \notin \mathcal{I}_{A,x}$. However $fg \in \mathcal{I}_{A,x}$, thus $\mathcal{I}_{A,x}$ is not prime. □

(4.6) Theorem. *Every decreasing sequence of germs of analytic sets (A_k, x) is stationary.*

Proof. In fact, the corresponding sequence of ideals $\mathcal{J}_k = \mathcal{I}_{A_k,x}$ is increasing, thus $\mathcal{J}_k = \mathcal{J}_{k_0}$ for $k \geq k_0$ large enough by the Noetherian property of $\mathcal{O}_{M,x}$. Hence $(A_k, x) = \big(V(\mathcal{J}_k), x\big)$ is constant for $k \geq k_0$. This result has the following straightforward consequence: □

(4.7) Theorem. *Every analytic germ (A, x) has a finite decomposition*

$$(A, x) = \bigcup_{1 \leq k \leq N} (A_k, x)$$

where the germs (A_j, x) are irreducible and $(A_j, x) \not\subset (A_k, x)$ for $j \neq k$. The decomposition is unique apart from the ordering.

Proof. If (A, x) can be split in several components, we split repeatedly each component as long as one of them is reducible. The process must stop by Th. 4.6, whence the existence. For the uniqueness, assume that $(A, x) = \bigcup (A'_l, x)$, $1 \leq l \leq N'$, is another decomposition. Since $(A_k, x) = \bigcup_l (A_k \cap A'_l, x)$, we must have $(A_k, x) = (A_k \cap A'_l, x)$ for some $l = l(k)$, i.e. $(A_k, x) \subset (A'_{l(k)}, x)$, and likewise $(A'_{l(k)}, x) \subset (A_j, x)$ for some j. Hence $j = k$ and $(A'_{l(k)}, x) = (A_k, x)$. □

§4.2. Local Structure of a Germ of Analytic Set

We are going to describe the local structure of a germ, both from the holomorphic and topological points of view. By the above decomposition theorem, we may restrict ourselves to the case of irreducible germs Let $\mathcal{J}$ be a prime ideal of $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0}$ and let $A = V(\mathcal{J})$ be its zero variety. We set $\mathcal{J}_k = \mathcal{J} \cap \mathbb{C}\{z_1, \ldots, z_k\}$ for each $k = 0, 1, \ldots, n$.

(4.8) Proposition. *There exist an integer d, a basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$ and associated coordinates $(z_1, \ldots, z_n)$ with the following properties: $\mathcal{J}_d = \{0\}$ and for every integer $k = d+1, \ldots, n$ there is a Weierstrass polynomial $P_k \in \mathcal{J}_k$ of the form*

$$(4.9) \quad P_k(z', z_k) = z_k^{s_k} + \sum_{1 \leq j \leq s_k} a_{j,k}(z')\, z_k^{s_k - j}, \qquad a_{j,k}(z') \in \mathcal{O}_{k-1},$$

where $a_{j,k}(z') = O(|z'|^j)$. Moreover, the basis $(e_1, \ldots, e_n)$ can be chosen arbitrarily close to any preassigned basis $(e_1^0, \ldots, e_n^0)$.

Proof. By induction on n. If $\mathcal{J} = \mathcal{J}_n = \{0\}$, then $d = n$ and there is nothing to prove. Otherwise, select a non zero element $g_n \in \mathcal{J}$ and a vector e_n such that $\mathbb{C} \ni w \longmapsto g_n(we_n)$ has minimum vanishing order s_n. This choice excludes at most the algebraic set $g_n^{(s_n)}(v) = 0$, so e_n can be taken arbitrarily close to e_n^0. Let $(\tilde{z}_1, \ldots, \tilde{z}_{n-1}, z_n)$ be the coordinates associated to the basis $(e_1^0, \ldots, e_{n-1}^0, e_n)$. After multiplication by an invertible element, we may assume that g_n is a Weierstrass polynomial

$$P_n(\tilde z, z_n) = z_n^{s_n} + \sum_{1\le j\le s_n} a_{j,n}(\tilde z)\, z_n^{s_n-j}, \qquad a_{j,n}\in \mathcal{O}_{n-1},$$

and $a_{j,n}(\tilde z) = O(|\tilde z|^j)$ by Remark 2.2. If $\mathcal{J}_{n-1} = \mathcal{J}\cap\mathbb{C}\{\tilde z\} = \{0\}$ then $d = n-1$ and the construction is finished. Otherwise we apply the induction hypothesis to the ideal $\mathcal{J}_{n-1}\subset\mathcal{O}_{n-1}$ in order to find a new basis $(e_1,\ldots,e_{n-1})$ of $\mathrm{Vect}(e_1^0,\ldots,e_{n-1}^0)$, associated coordinates $(z_1,\ldots,z_{n-1})$ and Weierstrass polynomials $P_k\in\mathcal{J}_k$, $d+1\le k\le n-1$, as stated in the lemma. □

(4.10) Lemma. *If $w\in\mathbb{C}$ is a root of $w^d + a_1w^{d-1}+\cdots+a_d = 0$, $a_j\in\mathbb{C}$, then $|w|\le 2\max|a_j|^{1/j}$.*

Proof. Otherwise $|w| > 2|a_j|^{1/j}$ for all $j=1,\ldots,d$ and the given equation $-1 = a_1/w+\cdots+a_d/w^d$ implies $1\le 2^{-1}+\cdots+2^{-d}$, a contradiction. □

(4.11) Corollary. *Set $z'=(z_1,\ldots,z_d)$, $z''=(z_{d+1},\ldots,z_n)$, and let Δ' in $\mathbb{C}^d$, Δ'' in $\mathbb{C}^{n-d}$ be polydisks of center 0 and radii $r', r''>0$. Then the germ $(A,0)$ is contained in a cone $|z''|\le C|z'|$, $C=$ constant, and the restriction of the projection map $\mathbb{C}^n\longrightarrow\mathbb{C}^d$, $(z',z'')\longmapsto z'$:*

$$\pi : A\cap(\Delta'\times\Delta'')\longrightarrow\Delta'$$

is proper if r'' is small enough and $r'\le r''/C$.

Proof. The polynomials $P_k(z_1,\ldots,z_{k-1}\,;\,z_k)$ vanish on $(A,0)$. By Lemma 4.10 and (4.9), every point $z\in A$ sufficiently close to 0 satisfies

$$|z_k|\le C_k(|z_1|+\cdots+|z_{k-1}|),\qquad d+1\le k\le n,$$

thus $|z''|\le C|z'|$ and the Corollary follows. □

Since $\mathcal{J}_d=\{0\}$, we have an injective ring morphism

$$\mathcal{O}_d = \mathbb{C}\{z_1,\ldots,z_d\}\hookrightarrow\mathcal{O}_n/\mathcal{J}. \tag{4.12}$$

(4.13) Proposition. *$\mathcal{O}_n/\mathcal{J}$ is a finite integral extension of $\mathcal{O}_d$.*

Proof. Let $f\in\mathcal{O}_n$. A division by P_n yields $f=P_nq_n+R_n$ with a remainder $R_n\in\mathcal{O}_{n-1}[z_n]$, $\deg_{z_n}R_n<s_n$. Further divisions of the coefficients of R_n by P_{n-1}, P_{n-2} etc … yield

$$R_{k+1}=P_kq_k+R_k,\qquad R_k\in\mathcal{O}_k[z_{k+1},\ldots,z_n],$$

where $\deg_{z_j}R_k<s_j$ for $j>k$. Hence

$$f = R_d+\sum_{d+1\le k\le n}P_kq_k = R_d \mod (P_{d+1},\ldots,P_n)\subset\mathcal{J} \tag{4.14}$$

and $\mathcal{O}_n/\mathcal{J}$ is finitely generated as an $\mathcal{O}_d$-module by the family of monomials $z_{d+1}^{\alpha_{d+1}} \ldots z_n^{\alpha_n}$ with $\alpha_j < s_j$. □

As $\mathcal{J}$ is prime, $\mathcal{O}_n/\mathcal{J}$ is an entire ring. We denote by $\tilde{f}$ the class of any germ $f \in \mathcal{O}_n$ in $\mathcal{O}_n/\mathcal{J}$, by $\mathcal{M}_A$ and $\mathcal{M}_d$ the quotient fields of $\mathcal{O}_n/\mathcal{J}$ and $\mathcal{O}_d$ respectively. Then $\mathcal{M}_A = \mathcal{M}_d[\tilde{z}_{d+1}, \ldots, \tilde{z}_n]$ is a finite algebraic extension of $\mathcal{M}_d$. Let $q = [\mathcal{M}_A{:}\mathcal{M}_d]$ be its degree and let $\sigma_1, \ldots, \sigma_q$ be the embeddings of $\mathcal{M}_A$ over $\mathcal{M}_d$ in an algebraic closure $\overline{\mathcal{M}}_A$. Let us recall that a factorial ring is integrally closed in its quotient field (Lang 1965, Chapter IX). Hence every element of $\mathcal{M}_d$ which is integral over $\mathcal{O}_d$ lies in fact in $\mathcal{O}_d$. By the primitive element theorem, there exists a linear form $u(z'') = c_{d+1}z_{d+1} + \cdots + c_n z_n$, $c_k \in \mathbb{C}$, such that $\mathcal{M}_A = \mathcal{M}_d[\tilde{u}]$; in fact, u is of degree q if and only if $\sigma_1\tilde{u}, \ldots, \sigma_q\tilde{u}$ are all distinct, and this excludes at most a finite number of vector subspaces in the space $\mathbb{C}^{n-d}$ of coefficients $(c_{d+1}, \ldots, c_n)$. As $\tilde{u} \in \mathcal{O}_n/\mathcal{J}$ is integral over the integrally closed ring $\mathcal{O}_d$, the unitary irreducible polynomial W_u of $\tilde{u}$ over $\mathcal{M}_d$ has coefficients in $\mathcal{O}_d$:

$$W_u(z'\,;T) = T^q + \sum_{1\le j\le q} a_j(z_1, \ldots, z_d)\, T^{q-j}, \qquad a_j \in \mathcal{O}_d.$$

W_u must be a Weierstrass polynomial, otherwise there would exist a factorization $W_u = W'Q$ in $\mathcal{O}_d[T]$ with a Weierstrass polynomial W' of degree $\deg W' < q = \deg\, \tilde{u}$ and $Q(0) \neq 0$, hence $W'(\tilde{u}) = 0$, a contradiction. In the same way, we see that $\tilde{z}_{d+1}, \ldots, \tilde{z}_n$ have irreducible equations $W_k(z'\,;\tilde{z}_k) = 0$ where $W_k \in \mathcal{O}_d[T]$ is a Weierstrass polynomial of degree $= \deg\, \tilde{z}_k \le q$, $d+1 \le k \le n$.

(4.15) Lemma. *Let $\delta(z') \in \mathcal{O}_d$ be the discriminant of $W_u(z'\,;T)$. For every element g of $\mathcal{M}_A$ which is integral over $\mathcal{O}_d$ (or equivalently over $\mathcal{O}_n/\mathcal{J}$) we have $\delta g \in \mathcal{O}_d[\tilde{u}]$.*

Proof. We have $\delta(z') = \prod_{j<k}(\sigma_k\tilde{u} - \sigma_j\tilde{u})^2 \not\equiv 0$, and $g \in \mathcal{M}_A = \mathcal{M}_d[\tilde{u}]$ can be written

$$g = \sum_{0\le j\le q-1} b_j\, \tilde{u}^j, \qquad b_j \in \mathcal{M}_d,$$

where $b_0, \ldots, b_{d-1}$ are the solutions of the linear system $\sigma_k g = \sum b_j(\sigma_k\tilde{u})^j$; the determinant (of Van der Monde type) is $\delta^{1/2}$. It follows that $\delta b_j \in \mathcal{M}_d$ are polynomials in $\sigma_k g$ and $\sigma_k\tilde{u}$, thus δb_j is integral over $\mathcal{O}_d$. As $\mathcal{O}_d$ is integrally closed, we must have $\delta b_j \in \mathcal{O}_d$, hence $\delta g \in \mathcal{O}_d[\tilde{u}]$. □

In particular, there exist unique polynomials $B_{d+1}, \ldots, B_n \in \mathcal{O}_d[T]$ with $\deg B_k \le q-1$, such that

$$(4.16)\qquad \delta(z')z_k = B_k(z'\,;u(z'')) \pmod{\mathcal{J}}.$$

Then we have

$$(4.17)\quad \delta(z')^q W_k\big(z'\,;B_k(z'\,;T)/\delta(z')\big) \in \text{ideal}\ \ W_u(z'\,;T)\,\mathcal{O}_d[T]\ ;$$

indeed, the left-hand side is a polynomial in $\mathcal{O}_d[T]$ and admits $T=\tilde u$ as a root in $\mathcal{O}_n/\mathcal{J}$ since $B_k(z'\,;\tilde u)/\delta(z')=\tilde z_k$ and $W_k(z'\,;\tilde z_k)=0$.

(4.18) Lemma. *Consider the ideal*

$$\mathcal{G}=\big(W_u(z'\,;u(z''))\,,\,\delta(z')z_k-B_k(z'\,;u(z''))\big)\subset\mathcal{J}$$

and set $m=\max\{q,(n-d)(q-1)\}$*. For every germ* $f\in\mathcal{O}_n$*, there exists a unique polynomial* $R\in\mathcal{O}_d[T]$*,* $\deg_T R\le q-1$*, such that*

$$\delta(z')^m f(z)=R(z'\,;u(z''))\qquad(\operatorname{mod}\mathcal{G}).$$

Moreover $f\in\mathcal{J}$ *implies* $R=0$*, hence* $\delta^m\mathcal{J}\subset\mathcal{G}$*.*

Proof. By (4.17) and a substitution of z_k, we find $\delta(z')^qW_k(z'\,;z_k)\in\mathcal{G}$. The analogue of formula (4.14) with W_k in place of P_k yields

$$f=R_d+\sum_{d+1\le k\le n}W_kq_k,\qquad R_d\in\mathcal{O}_d[z_{d+1},\ldots,z_n],$$

with $\deg_{z_k}R_d<\deg W_k\le q$, thus $\delta^mf=\delta^mR_d \bmod \mathcal{G}$. We may therefore replace f by R_d and assume that $f\in\mathcal{O}_d[z_{d+1},\ldots,z_n]$ is a polynomial of total degree $\le(n-d)(q-1)\le m$. A substitution of z_k by $B_k(z'\,;u(z''))/\delta(z')$ yields $G\in\mathcal{O}_d[T]$ such that

$$\delta(z')^mf(z)=G(z'\,;u(z''))\quad \operatorname{mod}\ \big(\delta(z')z_k-B_k(z'\,;u(z''))\big).$$

Finally, a division $G=W_uQ+R$ gives the required polynomial $R\in\mathcal{O}_d[T]$. The last statement is clear: if $f\in\mathcal{J}$ satisfies $\delta^m(z')f(z)=R(z\,;u(z''))$ mod $\mathcal{G}$ for $\deg_T R<q$, then $R(z'\,;\tilde u)=0$, and as $\tilde u\in\mathcal{O}_n/\mathcal{J}$ is of degree q, we must have $R=0$. The uniqueness of R is proved similarly. □

(4.19) Local parametrization theorem. *Let* $\mathcal{J}$ *be a prime ideal of* $\mathcal{O}_n$ *and let* $A=V(\mathcal{J})$*. Assume that the coordinates*

$$(z'\,;z'')=(z_1,\ldots,z_d\,;z_{d+1},\ldots,z_n)$$

are chosen as above. Then the ring $\mathcal{O}_n/\mathcal{J}$ *is a finite integral extension of* $\mathcal{O}_d$*; let* q *be the degree of the extension and let* $\delta(z')\in\mathcal{O}_d$ *be the discriminant of the irreducible polynomial of a primitive element* $u(z'')=\sum_{k>d}c_kz_k$*. If* Δ',Δ'' *are polydisks of sufficiently small radii* r',r'' *and if* $r'\le r''/C$ *with* C *large, the projection map* $\pi:A\cap(\Delta'\times\Delta'')\longrightarrow\Delta'$ *is a ramified covering with* q *sheets, whose ramification locus is contained in* $S=\{z'\in\Delta';\delta(z')=0\}$*. This means that:*

a) *the open subset* $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$ *is a smooth* d*-dimensional manifold, dense in* $A \cap (\Delta' \times \Delta'')$;

b) $\pi : A_S \longrightarrow \Delta' \smallsetminus S$ *is a covering* ;

c) *the fibers* $\pi^{-1}(z')$ *have exactly* q *elements if* $z' \notin S$ *and at most* q *if* $z' \in S$.

Moreover, A_S *is a connected covering of* $\Delta' \smallsetminus S$*, and* $A \cap (\Delta' \times \Delta'')$ *is contained in a cone* $|z''| \le C|z'|$ *(see Fig. 1).*

Fig. 1 Ramified covering from A to $\Delta' \subset \mathbb{C}^p$.

Proof. After a linear change in the coordinates $z_{d+1}, \ldots, z_n$, we may assume $u(z'') = z_{d+1}$, so $W_u = W_{d+1}$ and $B_{d+1}(z' ; T) = \delta(z')T$. By Lemma 4.18, we have

$$\mathcal{G} = \big(W_{d+1}(z', z_{d+1})\,,\ \delta(z')z_k - B_k(z', z_{d+1})\big)_{k \ge d+2} \subset \mathcal{J}, \qquad \delta^m \mathcal{J} \subset \mathcal{G}.$$

We can thus find a polydisk $\Delta = \Delta' \times \Delta''$ of sufficiently small radii r', r'' such that $V(\mathcal{J}) \subset V(\mathcal{G}) \subset V(\delta^m \mathcal{J})$ in Δ. As $V(\mathcal{J}) = A$ and $V(\delta) \cap \Delta = S \times \Delta''$, this implies

$$A \cap \Delta \subset V(\mathcal{G}) \cap \Delta \subset (A \cap \Delta) \cup (S \times \Delta'').$$

In particular, the set $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$ lying above $\Delta' \smallsetminus S$ coincides with $V(\mathcal{G}) \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$, which is the set of points $z \in \Delta$ parametrized by the equations

$$\text{(4.20)} \qquad \begin{cases} \delta(z') \ne 0, \quad W_{d+1}(z', z_{d+1}) = 0, \\ z_k = B_k(z', z_{d+1})/\delta(z'), \quad d+2 \le k \le n. \end{cases}$$

As $\delta(z')$ is the resultant of W_{d+1} and $\partial W_{d+1}/\partial T$, we have

$$\partial W_{d+1}/\partial T(z', z_{d+1}) \ne 0 \quad \text{on } A_S.$$

The implicit function theorem shows that z_{d+1} is locally a holomorphic function of z' on A_S, and the same is true for $z_k = B_k(z', z_{d+1})/\delta(z')$, $k \geq d+2$. Hence A_S is a smooth manifold, and for $r' \leq r''/C$ small, the projection map $\pi : A_S \longrightarrow \Delta' \setminus S$ is a proper local diffeomorphism; by (4.20) the fibers $\pi^{-1}(z')$ have at most q points corresponding to some of the q roots w of $W_{d+1}(z'\,; w) = 0$. Since $\Delta' \setminus S$ is connected (Remark 4.2), either $A_S = \emptyset$ or the map π is a covering of constant sheet number $q' \leq q$. However, if w is a root of $W_{d+1}(z', w) = 0$ with $z' \in \Delta' \setminus S$ and if we set $z_{d+1} = w$, $z_k = B_k(z', w)/\delta(z')$, $k \geq d+2$, relation (4.17) shows that $W_k(z', z_k) = 0$, in particular $|z_k| = O(|z'|^{1/q})$ by Lemma 4.10. For z' small enough, the q points $z = (z', z'')$ defined in this way lie in Δ, thus $q' = q$. Property b) and the first parts of a) and c) follow. Now, we need the following lemma.

(4.21) Lemma. *If $\mathcal{J} \subset \mathcal{O}_n$ is prime and $A = V(\mathcal{J})$, then $\mathcal{I}_{A,0} = \mathcal{J}$.*

Proof I t is obvious that $\mathcal{I}_{A,0} \supset \mathcal{J}$. Now, for any $f \in \mathcal{I}_{A,0}$, Prop. 4.13 implies that $\tilde{f}$ satisfies in $\mathcal{O}_n/\mathcal{I}$ an irreducible equation

$$f^r + b_1(z')\, f^{r-1} + \cdots + b_r(z') = 0 \quad (\mathrm{mod}\ \mathcal{J}).$$

Then $b_r(z')$ vanishes on $(A, 0)$ and the first part of c) gives $b_r = 0$ on $\Delta' \setminus S$. Hence $\tilde{b}_r = 0$ and the irreducibility of the equation of $\tilde{f}$ implies $r = 1$, so $f \in \mathcal{J}$, as desired. □

Proof of Theorem 4.19 (end). It only remains to prove that A_S is connected and dense in $A \cap \Delta$ and that the fibers $\pi^{-1}(z')$, $z' \in S$, have at most q elements. Let $A_{S,1}, \ldots, A_{S,N}$ be the connected components of A_S. Then $\pi : A_{S,j} \longrightarrow \Delta' \setminus S$ is a covering with q_j sheets, $q_1 + \cdots + q_N = q$. For every point $\zeta' \in \Delta' \setminus S$, there exists a neighborhood U of ζ' such that $A_{S,j} \cap \pi^{-1}(U)$ is a disjoint union of graphs $z'' = g_{j,k}(z')$ of analytic functions $g_{j,k} \in \mathcal{O}(U)$, $1 \leq k \leq q_j$. If $\lambda(z'')$ is an arbitrary linear form in $z_{d+1}, \ldots, z_n$ and $z' \in \Delta' \setminus S$, we set

$$P_{\lambda,j}(z'\,; T) = \prod_{\{z''\,;\,(z',z'') \in A_{S,j}\}} \big(T - \lambda(z'')\big) = \prod_{1 \leq k \leq k_j} \big(T - \lambda \circ g_{j,k}(z')\big).$$

This defines a polynomial in T with bounded analytic coefficients on $\Delta' \setminus S$. These coefficients have analytic extensions to Δ' (Remark 4.2), thus $P_{\lambda,j} \in \mathcal{O}(\Delta')[T]$. By construction, $P_{\lambda,j}\big(z'\,; \lambda(z'')\big)$ vanishes identically on $A_{S,j}$. Set

$$P_\lambda = \prod_{1 \leq j \leq N} P_{\lambda,j}, \qquad f(z) = \delta(z')\, P_\lambda\big(z'\,; \lambda(z'')\big)\ ;$$

f vanishes on $A_{S,1} \cup \ldots \cup A_{S,N} \cup (S \times \Delta'') \supset A \cap \Delta$. Lemma 4.21 shows that $\mathcal{I}_{A,0}$ is prime; as $\delta \notin \mathcal{I}_{A,0}$, we get $P_{\lambda,j}\big(z'\,; \lambda(z'')\big) \in \mathcal{I}_{A,0}$ for some j. This is a contradiction if $N \geq 2$ and if λ is chosen in such a way that λ

separates the q points z''_ν in each fiber $\pi^{-1}(z'_\nu)$, for a sequence $z'_\nu \to 0$ in $\Delta' \smallsetminus S$. Hence $N = 1$, A_S is connected, and for every $\lambda \in (\mathbb{C}^{n-d})^\star$ we have $P_\lambda\big(z', \lambda(z'')\big) \in \mathcal{I}_{(A,0)}$. By construction $P_\lambda\big(z', \lambda(z'')\big)$ vanishes on A_S, so it vanishes on $\overline{A}_S$; hence, for every $z' \in S$, the fiber $\overline{A}_S \cap \pi^{-1}(z')$ has at most q elements, otherwise selecting λ which separates $q+1$ of these points would yield $q+1$ roots $\lambda(z'')$ of $P_\lambda(z' ; T)$, a contradiction. Assume now that A_S is not dense in $A \cap \Delta$ for arbitrarily small polydisks Δ. Then there exists a sequence $A \ni z_\nu = (z'_\nu, z''_\nu) \to 0$ such that $z'_\nu \in S$ and z''_ν is not in $F_\nu := \mathrm{pr}''\big(\overline{A}_S \cap \pi^{-1}(z'_\nu)\big)$. The continuity of the roots of the polynomial $P_\lambda(z' ; T)$ as $\Delta' \smallsetminus S \ni z' \to z'_\nu$ implies that the set of roots of $P_\lambda(z'_\nu ; T)$ is $\lambda(F_\nu)$. Select λ such that $\lambda(z''_\nu) \notin \lambda(F_\nu)$ for all ν. Then $P_\lambda\big(z'_\nu ; \lambda(z''_\nu)\big) \neq 0$ for every ν and $P_\lambda\big(z' ; \lambda(z'')\big) \notin \mathcal{I}_{A,0}$, a contradiction. □

At this point, it should be observed that many of the above statements completely fail in the case of real analytic sets. In $\mathbb{R}^2$, for example, the prime ideal $\mathcal{J} = (x^5 + y^4)$ defines an irreducible germ of curve $(A, 0)$ and there is an injective integral extension of rings $\mathbb{R}\{x\} \hookrightarrow \mathbb{R}\{x, y\}/\mathcal{J}$ of degree 4; however, the projection of $(A, 0)$ on the first factor, $(x, y) \mapsto x$, has not a constant sheet number near 0, and this number is not related to the degree of the extension. Also, the prime ideal $\mathcal{J} = (x^2 + y^2)$ has an associated variety $V(\mathcal{J})$ reduced to $\{0\}$, hence $\mathcal{I}_{A,0} = (x, y)$ is strictly larger than $\mathcal{J}$, in contrast with Lemma 4.21.

Let us return to the complex situation, which is much better behaved. The result obtained in Lemma 4.21 can then be extended to non prime ideals and we get the following important result:

(4.22) Hilbert's Nullstellensatz. *For every ideal $\mathcal{J} \subset \mathcal{O}_n$*

$$\mathcal{I}_{V(\mathcal{J}),0} = \sqrt{\mathcal{J}},$$

where $\sqrt{\mathcal{J}}$ is the radical of $\mathcal{J}$, i.e. the set of germs $f \in \mathcal{O}_n$ such that some power f^k lies in $\mathcal{J}$.

Proof. Set $B = V(\mathcal{J})$. If $f^k \in \mathcal{J}$, then f^k vanishes on $(B, 0)$ and $f \in \mathcal{I}_{B,0}$. Thus $\sqrt{\mathcal{J}} \subset \mathcal{I}_{B,0}$. Conversely, it is well known that $\sqrt{\mathcal{J}}$ is the intersection of all prime ideals $\mathcal{P} \supset \mathcal{J}$ (Lang 1965, Chapter VI). For such an ideal $(B, 0) = \big(V(\mathcal{J}), 0\big) \supset \big(V(\mathcal{P}), 0\big)$, thus $\mathcal{I}_{B,0} \subset \mathcal{I}_{V(\mathcal{P}),0} = \mathcal{P}$ in view of Lemma 4.21. Therefore $\mathcal{I}_{B,0} \subset \bigcap_{\mathcal{P} \supset \mathcal{J}} \mathcal{P} = \sqrt{\mathcal{J}}$ and the Theorem is proved. □

In other words, if a germ $(B, 0)$ is defined by an arbitrary ideal $\mathcal{J} \subset \mathcal{O}_n$ and if $f \in \mathcal{O}_n$ vanishes on $(B, 0)$, then some power f^k lies in $\mathcal{J}$.

§4.3. Regular and Singular Points. Dimension

The above powerful results enable us to investigate the structure of singularities of an analytic set. We first give a few definitions.

(4.23) Definition. *Let $A \subset M$ be an analytic set and $x \in A$. We say that $x \in A$ is a regular point of A if $A \cap \Omega$ is a $\mathbb{C}$-analytic submanifold of Ω for some neighborhood Ω of x. Otherwise x is said to be singular. The corresponding subsets of A will be denoted respectively A_{reg} and A_{sing}.*

It is clear from the definition that A_{reg} is an open subset of A (thus A_{sing} is closed), and that the connected components of A_{reg} are $\mathbb{C}$-analytic submanifolds of M (non necessarily closed).

(4.24) Proposition. *If (A, x) is irreducible, there exist arbitrarily small neighborhoods Ω of x such that $A_{\mathrm{reg}} \cap \Omega$ is dense and connected in $A \cap \Omega$.*

Proof. Take $\Omega = \Delta$ as in Th. 4.19. Then $A_S \subset A_{\mathrm{reg}} \cap \Omega \subset A \cap \Omega$, where A_S is connected and dense in $A \cap \Omega$; hence $A_{\mathrm{reg}} \cap \Omega$ has the same properties. $\square$

(4.25) Definition. *The dimension of an irreducible germ of analytic set (A, x) is defined by* $\dim(A, x) = \dim(A_{\mathrm{reg}}, x)$. *If (A, x) has several irreducible components (A_l, x), we set*

$$\dim(A, x) = \max\{\dim(A_l, x)\}, \qquad \operatorname{codim}(A, x) = n - \dim(A, x).$$

(4.26) Proposition. *Let $(B, x) \subset (A, x)$ be germs of analytic sets. If (A, x) is irreducible and $(B, x) \neq (A, x)$, then* $\dim(B, x) < \dim(A, x)$ *and $B \cap \Omega$ has empty interior in $A \cap \Omega$ for all sufficiently small neighborhoods Ω of x.*

Proof. We may assume $x = 0$, $(A, 0) \subset (\mathbb{C}^n, 0)$ and $(B, 0)$ irreducible. Then $\mathcal{J}_{A,0} \subset \mathcal{J}_{B,0}$ are prime ideals. When we choose suitable coordinates for the ramified coverings, we may at each step select vectors $e_n, e_{n-1}, \ldots$ that work simultaneously for A and B. If $\dim B = \dim A$, the process stops for both at the same time, i.e. we get ramified coverings

$$\pi : A \cap (\Delta' \times \Delta'') \longrightarrow \Delta', \qquad \pi : B \cap (\Delta' \times \Delta'') \longrightarrow \Delta'$$

with ramification loci S_A, S_B. Then $B \cap \big((\Delta' \smallsetminus (S_A \cup S_B)) \times \Delta''\big)$ is an open subset of the manifold $A_S = A \cap \big((\Delta' \smallsetminus S_A) \times \Delta''\big)$, therefore $B \cap A_S$ is an analytic subset of A_S with non empty interior. The same conclusion would hold if $B \cap \Delta$ had non empty interior in $A \cap \Delta$. As A_S is connected, we get $B \cap A_S = A_S$, and as $B \cap \Delta$ is closed in Δ we infer $B \cap \Delta \supset \overline{A}_S = A \cap \Delta$, hence $(B, 0) = (A, 0)$, in contradiction with the hypothesis. $\square$

(4.27) Example: parametrization of curves. Suppose that $(A,0)$ is an irreducible germ of curve ($\dim(A,0)=1$). If the disk $\Delta' \subset \mathbb{C}$ is chosen so small that $S=\{0\}$, then A_S is a connected covering of $\Delta' \smallsetminus \{0\}$ with q sheets. Hence, there exists a covering isomorphism between π and the standard covering

$$\mathbb{C} \supset \Delta(r) \smallsetminus \{0\} \longrightarrow \Delta(r^q) \smallsetminus \{0\}, \qquad t \longmapsto t^q, \qquad r^q = \text{radius of } \Delta',$$

i.e. a map $\gamma : \Delta(r) \smallsetminus \{0\} \longrightarrow A_S$ such that $\pi \circ \gamma(t) = t^q$. This map extends into a bijective holomorphic map $\gamma : \Delta(r) \longrightarrow A \cap \Delta$ with $\gamma(0)=0$. This means that every irreducible germ of curve can be parametrized by a bijective holomorphic map defined on a disk in $\mathbb{C}$ (see also Exercise 10.8).

§4.4. Coherence of Ideal Sheaves

Let A be an analytic set in a complex manifold M. The *sheaf of ideals* $\mathcal{I}_A$ is the subsheaf of $\mathcal{O}_M$ consisting of germs of holomorphic functions on M which vanish on A. Its stalks are the ideals $\mathcal{I}_{A,x}$ already considered; note that $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ if $x \notin A$. If $x \in A$, we let $\mathcal{O}_{A,x}$ be the ring of germs of functions on (A,x) which can be extended as germs of holomorphic functions on (M,x). By definition, there is a surjective morphism $\mathcal{O}_{M,x} \longrightarrow \mathcal{O}_{A,x}$ whose kernel is $\mathcal{I}_{A,x}$, thus

$$(4.28) \qquad \mathcal{O}_{A,x} = \mathcal{O}_{M,x}/\mathcal{I}_{A,x}, \qquad \forall x \in A,$$

i.e. $\mathcal{O}_A = (\mathcal{O}_M/\mathcal{I}_A)_{\restriction A}$. Since $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ for $x \notin A$, the quotient sheaf $\mathcal{O}_M/\mathcal{I}_A$ is zero on $M \smallsetminus A$.

(4.29) Theorem (Cartan 1950). *For any analytic set $A \subset M$, the sheaf of ideals $\mathcal{I}_A$ is a coherent analytic sheaf.*

Proof. It is sufficient to prove the result when A is an analytic subset in a neighborhood of 0 in $\mathbb{C}^n$. If $(A,0)$ is not irreducible, there exists a neighborhood Ω such that $A \cap \Omega = A_1 \cup \ldots \cup A_N$ where A_k are analytic sets in Ω and $(A_k,0)$ is irreducible. We have $\mathcal{I}_{A\cap\Omega} = \bigcap \mathcal{I}_{A_k}$, so by Cor. 3.15 we may assume that $(A,0)$ is irreducible. Then we can choose coordinates z', z'', polydisks Δ', Δ'' and a primitive element $u(z'') = c_{d+1}z_{d+1} + \cdots + c_n z_n$ such that Th. 4.19 is valid. Since $\delta(z') = \prod_{j<k}(\sigma_k\tilde{u} - \sigma_j\tilde{u})^2$, we see that $\delta(z')$ is in fact a polynomial in the c_j 's with coefficients in $\mathcal{O}_d$. The same is true for the coefficients of the polynomials $W_u(z';T)$ and $B_k(z';T)$ which can be expressed in terms of the elementary symmetric functions of the $\sigma_k\tilde{u}$'s. We suppose that Δ' is chosen small enough in order that all coefficients of these $\mathcal{O}_d[c_{d+1},\ldots,c_n]$ polynomials are in $\mathcal{O}(\Delta')$. Let $\delta_\alpha \in \mathcal{O}(\Delta')$ be some non zero coefficient appearing in $\delta^m = \sum \delta_\alpha c^\alpha$. Also, let $G_1,\ldots,G_N \in \mathcal{O}(\Delta')[z'']$ be

the coefficients of all monomials c^α appearing in the expansion of the functions $W_u(z'\,;u(z''))$ or $\delta(z')z_k - B_k(z'\,;u(z''))$. Clearly, $G_1,\ldots,G_N$ vanish on $A\cap\Delta$. We contend that

$$(4.30)\quad \mathcal{I}_{A,x} = \big\{f\in\mathcal{O}_{M,x}\ ;\ \delta_\alpha f\in(G_{1,x},\ldots,G_{N,x})\big\}.$$

This implies that the sheaf $\mathcal{I}_A$ is the projection on the first factor of the sheaf of relations $\mathcal{R}(\delta_\alpha,G_1,\ldots,G_N)\subset\mathcal{O}_\Delta^{N+1}$, which is coherent by the Oka theorem; Theorem 4.29 then follows.

We first prove that the inclusion $\mathcal{I}_{A,x}\supset\{\ldots\}$ holds in (4.30). In fact, if $\delta_\alpha f\in(G_{1,x},\ldots,G_{N,x})$, then f vanishes on $A\smallsetminus\{\delta_\alpha=0\}$ in some neighborhood of x. Since $(A\cap\Delta)\smallsetminus\{\delta_\alpha=0\}$ is dense in $A\cap\Delta$, we conclude that $f\in\mathcal{I}_{A,x}$.

To prove the other inclusion $\mathcal{I}_{A,x}\subset\{\ldots\}$, we repeat the proof of Lemma 4.18 with a few modifications. Let $x\in\Delta$ be a fixed point. At x, the irreducible polynomials $W_u(z'\,;T)$ and $W_k(z'\,;T)$ of $\tilde u$ and $\tilde z_k$ in $\mathcal{O}_{M,0}/\mathcal{I}_{A,0}$ split into

$$\begin{aligned}W_u(z'\,;T) &= W_{u,x}\big(z'\,;T-u(x'')\big)\,Q_{u,x}\big(z'\,;T-u(x'')\big),\\ W_k(z'\,;T) &= W_{k,x}(z'\,;T-x_k)\,Q_{k,x}(z'\,;T-x_k),\end{aligned}$$

where $W_{u,x}(z'\,;T)$ and $W_{k,x}(z'\,;T)$ are Weierstrass polynomials in T and $Q_{u,x}(x',0)\neq0$, $Q_{k,x}(x',0)\neq0$. For all $z'\in\Delta'$, the roots of $W_u(z'\,;T)$ are the values $u(z'')$ at all points $z\in A\cap\pi^{-1}(z')$. As A is closed, any point $z\in A\cap\pi^{-1}(z')$ with z' near x' has to be in a small neighborhood of one of the points $y\in A\cap\pi^{-1}(x')$. Choose $c_{d+1},\ldots,c_n$ such that the linear form $u(z'')$ separates all points in the fiber $A\cap\pi^{-1}(x')$. Then, for a root $u(z'')$ of $W_{u,x}\big(z'\,;T-u(x'')\big)$, the point z must be in a neighborhood of $y=x$, otherwise $u(z'')$ would be near $u(y'')\neq u(x'')$ and the Weierstrass polynomial $W_{u,x}(z'\,;T)$ would have a root away from 0, in contradiction with (4.10). Conversely, if $z\in A\cap\pi^{-1}(z')$ is near x, then $Q_{u,x}\big(z'\,;u(z'')-u(x'')\big)\neq0$ and $u(z'')$ is a root of $W_{u,x}\big(z'\,;T-u(x'')\big)$. From this, we infer that every polynomial $P(z'\,;T)\in\mathcal{O}_{\Delta',x'}[T]$ such that $P\big(z'\,;u(z'')\big)=0$ on (A,x) is a multiple of $W_{u,x}\big(z'\,;T-u(x'')\big)$, because the roots of the latter polynomial are simple for z' in the dense set $(\Delta'\smallsetminus S,x)$. In particular $\deg P<\deg W_{u,x}$ implies $P=0$ and

$$\delta(z')^qW_{k,x}\big(z'\,;B_k(z'\,;u(z''))/\delta(z')-x_k\big)$$

is a multiple of $W_{u,x}\big(z'\,;T-u(x'')\big)$. If we replace W_u, W_k by $W_{u,x}$, $W_{k,x}$ respectively, the proof of Lemma 4.18 shows that for every $f\in\mathcal{O}_{M,x}$ there is a polynomial $R\in\mathcal{O}_{\Delta',x'}[T]$ of degree $\deg R<\deg W_{u,x}$ such that

$$\begin{aligned}&\delta(z')^mf(z)=R\big(z'\,;u(z'')\big)\quad\text{modulo the ideal}\\ &\big(\ W_{u,x}\big(z'\,;u(z'')-u(x'')\big),\ \delta(z')z_k-B_k\big(z'\,;u(z'')\big)\ \big),\end{aligned}$$

and $f \in \mathcal{I}_{A,x}$ implies $R = 0$. Since $W_{u,x}$ differs from W_u only by an invertible element in $\mathcal{O}_{M,x}$, we conclude that

$$\Big(\sum \delta_\alpha c^\alpha\Big)\mathcal{I}_{A,x} = \delta^m \mathcal{I}_{A,x} \subset (G_{1,x}, \ldots, G_{N,x}).$$

This is true for a dense open set of coefficients $c_{d+1}, \ldots, c_n$, therefore

$$\delta_\alpha \mathcal{I}_{A,x} \subset (G_{1,x}, \ldots, G_{N,x}) \quad \text{for all} \quad \alpha. \qquad \square$$

(4.31) Theorem. *$A_{\rm sing}$ is an analytic subset of A.*

Proof. The statement is local. Assume first that $(A, 0)$ is an irreducible germ in $\mathbb{C}^n$. Let $g_1, \ldots, g_N$ be generators of the sheaf $\mathcal{I}_A$ on a neighborhood Ω of 0. Set $d = \dim A$. In a neighborhood of every point $x \in A_{\rm reg} \cap \Omega$, A can be defined by holomorphic equations $u_1(z) = \ldots = u_{n-d}(z) = 0$ such that $du_1, \ldots, du_{n-d}$ are linearly independant. As $u_1, \ldots, u_{n-d}$ are generated by $g_1, \ldots, g_N$, one can extract a subfamily $g_{j_1}, \ldots, g_{j_{n-d}}$ that has at least one non zero Jacobian determinant of rank $n - d$ at x. Therefore $A_{\rm sing} \cap \Omega$ is defined by the equations

$$\det\Big(\frac{\partial g_j}{\partial z_k}\Big)_{\substack{j\in J\\ k\in K}} = 0, \qquad J \subset \{1, \ldots, N\}, \quad K \subset \{1, \ldots, n\}, \quad |J| = |K| = n - d.$$

Assume now that $(A, 0) = \bigcup (A_l, 0)$ with $(A_l, 0)$ irreducible. The germ of an analytic set at a regular point is irreducible, thus every point which belongs simultaneously to at least two components is singular. Hence

$$(A_{\rm sing}, 0) = \bigcup (A_{l,{\rm sing}}, 0) \cup \bigcup_{k\neq l} (A_k \cap A_l, 0),$$

and $A_{\rm sing}$ is analytic. $\square$

Now, we give a characterization of regular points in terms of a simple algebraic property of the ring $\mathcal{O}_{A,x}$.

(4.32) Proposition. *Let (A, x) be a germ of analytic set of dimension d and let $\mathfrak{m}_{A,x} \subset \mathcal{O}_{A,x}$ be the maximal ideal of functions that vanish at x. Then $\mathfrak{m}_{A,x}$ cannot have less than d generators and $\mathfrak{m}_{A,x}$ has d generators if and only if x is a regular point.*

Proof. If $A \subset \mathbb{C}^n$ is a d-dimensional submanifold in a neighborhood of x, there are local coordinates centered at x such that A is given by the equations $z_{d+1} = \ldots = z_n$ near $z = 0$. Then $\mathcal{O}_{A,x} \simeq \mathcal{O}_d$ and $\mathfrak{m}_{A,x}$ is generated by $z_1, \ldots, z_d$. Conversely, assume that $\mathfrak{m}_{A,x}$ has s generators $g_1(z), \ldots, g_s(z)$ in $\mathcal{O}_{A,x} = \mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{A,x}$. Letting $x = 0$ for simplicity, we can write

$$z_j = \sum_{1\le k\le s} u_{jk}(z)g_k(z) + f_j(z), \qquad u_{jk} \in \mathcal{O}_n, \quad f_j \in \mathcal{I}_{A,0}, \quad 1 \le j \le n.$$

Then we find $dz_j = \sum c_{jk}(0)dg_k(0) + df_j(0)$, so that the rank of the system of differentials $\big(df_j(0)\big)_{1\le j\le n}$ is at least equal to $n-s$. Assume for example that $df_1(0), \ldots, df_{n-s}(0)$ are linearly independent. By the implicit function theorem, the equations $f_1(z) = \ldots = f_{n-s}(z) = 0$ define a germ of submanifold of dimension s containing $(A,0)$, thus $s \ge d$ and $(A,0)$ equals this submanifold if $s = d$. □

(4.33) Corollary. *Let $A \subset M$ be an analytic set of pure dimension d and let $B \subset A$ be an analytic subset of codimension $\ge p$ in A. Then, as an $\mathcal{O}_{A,x}$-module, the ideal $\mathcal{I}_{B,x}$ cannot be generated by less than p generators at any point $x \in B$, and by less than $p+1$ generators at any point $x \in B_{\rm reg} \cap A_{\rm sing}$.*

Proof. Suppose that $\mathcal{I}_{B,x}$ admits s-generators $(g_1, \ldots, g_s)$ at x. By coherence of $\mathcal{I}_B$ these germs also generate $\mathcal{I}_B$ in a neighborhood of x, so we may assume that x is a regular point of B. Then there are local coordinates $(z_1, \ldots, z_n)$ on M centered at x such that (B,x) is defined by $z_{k+1} = \ldots = z_n = 0$, where $k = \dim(B,x)$. Then the maximal ideal $\mathfrak{m}_{B,x} = \mathfrak{m}_{A,x}/\mathcal{I}_{B,x}$ is generated by $z_1, \ldots, z_k$, so that $\mathfrak{m}_{A,x}$ is generated by $(z_1, \ldots, z_k, g_1, \ldots, g_s)$. By Prop. 4.32, we get $k + s \ge d$, thus $s \ge d - k \ge p$, and we have strict inequalities when $x \in A_{\rm sing}$. □

§5. Complex Spaces

Much in the same way a manifold is constructed by piecing together open patches isomorphic to open sets in a vector space, a complex space is obtained by gluing together open patches isomorphic to analytic subsets. The general concept of analytic morphism (or holomorphic map between analytic sets) is first needed.

§5.1. Morphisms and Comorphisms

Let $A \subset \Omega \subset \mathbb{C}^n$ and $B \subset \Omega' \subset \mathbb{C}^p$ be analytic sets. A morphism from A to B is by definition a map $F : A \longrightarrow B$ such that for every $x \in A$ there is a neighborhood U of x and a holomorphic map $\tilde F : U \longrightarrow \mathbb{C}^p$ such that $\tilde F_{\restriction A\cap U} = F_{\restriction A\cap U}$. Equivalently, such a morphism can be defined as a continuous map $F : A \longrightarrow B$ such that for all $x \in A$ and $g \in \mathcal{O}_{B,F(x)}$ we have $g \circ F \in \mathcal{O}_{A,x}$. The induced ring morphism

$$(5.1) \qquad F_x^\star \; : \; \mathcal{O}_{B,F(x)} \ni g \longmapsto g \circ F \in \mathcal{O}_{A,x}$$

is called the *comorphism* of F at point x.

§5.1. Definition of Complex Spaces

(5.2) Definition. *A complex space X is a locally compact Hausdorff space, countable at infinity, together with a sheaf $\mathcal{O}_X$ of continuous functions on X, such that there exists an open covering (U_λ) of X and for each λ a homeomorphism $F_\lambda : U_\lambda \longrightarrow A_\lambda$ onto an analytic set $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{n_\lambda}$ such that the comorphism $F_\lambda^\star : \mathcal{O}_{A_\lambda} \longrightarrow \mathcal{O}_{X\restriction U_\lambda}$ is an isomorphism of sheaves of rings. $\mathcal{O}_X$ is called the structure sheaf of X.*

By definition a complex space X is locally isomorphic to an analytic set, so the concepts of holomorphic function on X, of analytic subset, of analytic morphism, etc ... are meaningful. If X is a complex space, Th. 4.31 implies that X_{sing} is an analytic subset of X.

(5.3) Theorem and definition. *For every complex space X, the set X_{reg} is a dense open subset of X, and consists of a disjoint union of connected complex manifolds X'_α. Let X_α be the closure of X'_α in X. Then (X_α) is a locally finite family of analytic subsets of X, and $X = \bigcup X_\alpha$. The sets X_α are called the global irreducible components of X.*

Observe that the germ at a given point of a global irreducible component can be reducible, as shows the example of the cubic curve $\Gamma : y^2 = x^2(1+x)$; the germ $(\Gamma, 0)$ has two analytic branches $y = \pm x\sqrt{1+x}$, however $\Gamma \setminus \{0\}$ is easily seen to be a connected smooth Riemann surface (the real points of γ corresponding to $-1 \le x \le 0$ form a path connecting the two branches). This example shows that the notion of *global irreducible component* is quite different from the notion of local irreducible component introduced in (4.4).

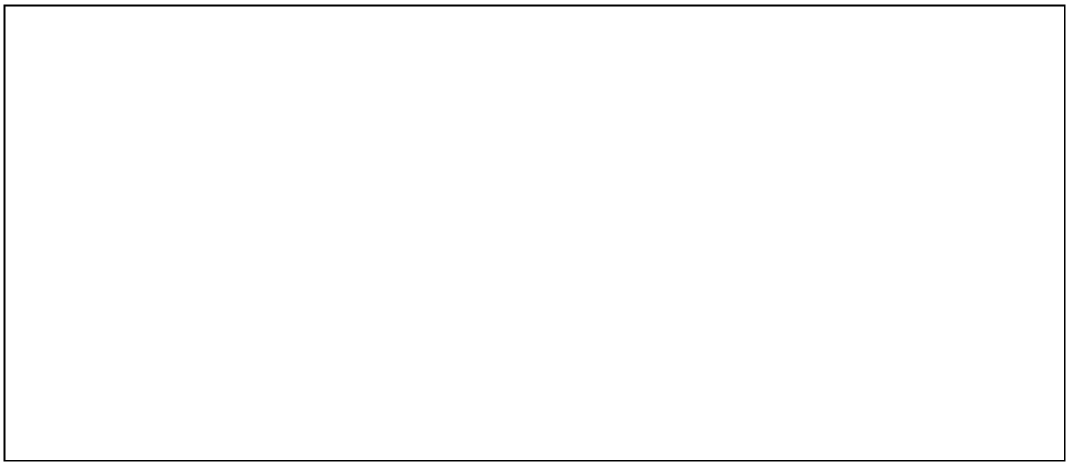

Fig. 2 The irreducible curve $y^2 = x^2(1+x)$ in $\mathbb{C}^2$.

Proof. By definition of $X_{\rm reg}$, the connected components X'_α are (disjoint) complex manifolds. Let us show that the germ of $X_\alpha = \overline{X}'_\alpha$ at any point $x \in X$ is analytic. We may assume that (X,x) is a germ of analytic set A in an open subset of $\mathbb{C}^n$. Let (A_l,x), $1 \le l \le N$, be the irreducible components of this germ and U a neighborhood of x such that $X \cap U = \bigcup A_l \cap U$. Let $\Omega_l \subset U$ be a neighborhood of x such that $A_{l,{\rm reg}} \cap \Omega_l$ is connected and dense in $A_l \cap \Omega_l$ (Prop. 4.24). Then $A'_l := X_{\rm reg} \cap A_l \cap \Omega_l$ equals $(A_{l,{\rm reg}} \cap \Omega_l) \smallsetminus \bigcup_{k\ne l} A_{l,{\rm reg}} \cap \Omega_l \cap A_k$. However, $A_{l,{\rm reg}} \cap \Omega_l \cap A_k$ is an analytic subset of $A_{l,{\rm reg}} \cap \Omega_l$, distinct from $A_{l,{\rm reg}} \cap \Omega_l$, otherwise $A_{l,{\rm reg}} \cap \Omega_l$ would be contained in A_k, thus $(A_l,x) \subset (A_k,x)$ by density. Remark 4.2 implies that A'_l is connected and dense in $A_{l,{\rm reg}} \cap \Omega_l$, hence in $A_l \cap \Omega_l$. Set $\Omega = \bigcap \Omega_l$ and let $(X_\alpha)_{\alpha\in J}$ be the family of global components which meet Ω (i.e. such that $X'_\alpha \cap \Omega \ne \emptyset$). As $X_{\rm reg} \cap \Omega = \bigcup A'_l \cap \Omega$, each X'_α, $\alpha \in J$, meets at least one set A'_l, and as $A'_l \subset X_{\rm reg}$ is connected, we have in fact $A'_l \subset X'_\alpha$. It follows that there exists a partition $(L_\alpha)_{\alpha\in J}$ of $\{1,\ldots,N\}$ such that $X'_\alpha \cap \Omega = \bigcup_{l\in L_\alpha} A'_l \cap \Omega$, $\alpha \in J$. Hence J is finite, $\operatorname{card} J \le N$, and

$$X_\alpha \cap \Omega = \overline{X}'_\alpha \cap \Omega = \bigcup_{l\in L_\alpha} \overline{A}'_l \cap \Omega = \bigcup_{l\in L_\alpha} A_l \cap \Omega$$

is analytic for all $\alpha \in J$. □

(5.4) Corollary. *If A, B are analytic subsets in a complex space X, then the closure $\overline{A \smallsetminus B}$ is an analytic subset, consisting of the union of all global irreducible components A_λ of A which are not contained in B.*

Proof. Let $C = \bigcup A_\lambda$ be the union of these components. Since (A_λ) is locally finite, C is analytic. Clearly $A \smallsetminus B = C \smallsetminus B = \bigcup A_\lambda \smallsetminus B$. The regular part A'_λ of each A_λ is a connected manifold and $A'_\lambda \cap B$ is a proper analytic subset (otherwise $A'_\lambda \subset B$ would imply $A_\lambda \subset B$). Thus $A'_\lambda \smallsetminus (A'_\lambda \cap B)$ is dense in A'_λ which is dense in A_λ, so $\overline{A \smallsetminus B} = \bigcup A_\lambda = C$. □

(5.5) Theorem. *For any family (A_λ) of analytic sets in a complex space X, the intersection $A = \bigcap A_\lambda$ is an analytic subset of X. Moreover, the intersection is stationary on every compact subset of X.*

Proof. It is sufficient to prove the last statement, namely that every point $x \in X$ has a neighborhood Ω such that $A \cap \Omega$ is already obtained as a finite intersection. However, since $\mathcal{O}_{X,x}$ is Noetherian, the family of germs of finite intersections has a minimum element (B,x), $B = \bigcap A_{\lambda_j}$, $1 \le j \le N$. Let $\tilde B$ be the union of the global irreducible components B_α of B which contain the point x ; clearly $(B,x) = (\tilde B, x)$. For any set A_λ in the family, the minimality of B implies $(B,x) \subset (A_\lambda, x)$. Let B'_α be the regular part of any global irreducible component B_α of $\tilde B$. Then $B'_\alpha \cap A_\lambda$ is a closed analytic subset of B'_α containing a non empty open subset (the intersection of B'_α with some neighborhood of x), so we must have $B'_\alpha \cap A_\lambda = B'_\alpha$. Hence $B_\alpha = \overline{B}'_\alpha \subset A_\lambda$ for all $B_\alpha \subset \tilde{B}$ and all A_λ, thus $\tilde{B} \subset A = \bigcap A_\lambda$. We infer

$$(B, x) = (\tilde{B}, x) \subset (A, x) \subset (B, x),$$

and the proof is complete. □

As a consequence of these general results, it is not difficult to show that a complex space always admits a (locally finite) stratification such that the strata are smooth manifolds.

(5.6) Proposition. *Let X be a complex space. Then there is a locally stationary increasing sequence of analytic subsets $Y_k \subset X$, $k \in \mathbb{N}$, such that Y_0 is a discrete set and such that $Y_k \smallsetminus Y_{k-1}$ is a smooth k-dimensional complex manifold for $k \geq 1$. Such a sequence is called a stratification of X, and the sets $Y_k \smallsetminus Y_{k-1}$ are called the strata (the strata may of course be empty for some indices $k < \dim X$).*

Proof. Let $\mathcal{F}$ be the family of irreducible analytic subsets $Z \subset X$ which can be obtained through a finite sequence of steps of the following types:

a) Z is an irreducible component of X;

b) Z is an irreducible component of the singular set $Z'_{\rm sing}$ of some member $Z' \in \mathcal{F}$;

c) Z is an irreducible component of some finite intersection of sets $Z_j \in \mathcal{F}$.

Since X has locally finite dimension and since steps b) or c) decrease the dimension of our sets Z, it is clear that $\mathcal{F}$ is a locally finite family of analytic sets in X. Let Y_k be the union of all sets $Z \in \mathcal{F}$ of dimension $\leq k$. It is easily seen that $\bigcup Y_k = X$ and that the irreducible components of $(Y_k)_{\rm sing}$ are contained in Y_{k-1} (these components are either intersections of components $Z_j \subset Y_k$ or parts of the singular set of some component $Z \subset Y_k$, so there are in either case obtained by step b) or c) above). Hence $Y_k \smallsetminus Y_{k-1}$ is a smooth manifold and it is of course k-dimensional, because the components of Y_k of dimension $< k$ are also contained in Y_{k-1} by definition.

(5.7) Theorem. *Let X be an irreducible complex space. Then every non constant holomorphic function f on X defines an open map $f : X \longrightarrow \mathbb{C}$.*

Proof. We show that the image $f(\Omega)$ of any neighborhood Ω of $x \in X$ contains a neighborhood of $f(x)$. Let (X_l, x) be an irreducible component of the germ (X, x) (embedded in $\mathbb{C}^n$) and $\Delta = \Delta' \times \Delta'' \subset \Omega$ a polydisk such that the projection $\pi : X_l \cap \Delta \longrightarrow \Delta'$ is a ramified covering. The function f is non constant on the dense open manifold $X_{\rm reg}$, so we may select a complex line $L \subset \Delta'$ through 0, not contained in the ramification locus of π, such that f

is non constant on the one dimensional germ $\pi^{-1}(L)$. Therefore we can find a germ of curve

$$(\mathbb{C}, 0) \ni t \longmapsto \gamma(t) \in (X, x)$$

such that $f \circ \gamma$ is non constant. This implies that the image of every neighborhood of $0 \in \mathbb{C}$ by $f \circ \gamma$ already contains a neighborhood of $f(x)$. □

(5.8) Corollary. *If X is a compact irreducible analytic space, then every holomorphic function $f \in \mathcal{O}(X)$ is constant.*

In fact, if $f \in \mathcal{O}(X)$ was non constant, $f(X)$ would be compact and also open in $\mathbb{C}$ by Th. 5.7, a contradiction. This result implies immediately the following consequence.

(5.9) Theorem. *Let X be a complex space such that the global holomorphic functions in $\mathcal{O}(X)$ separate the points of X. Then every compact analytic subset A of X is finite.*

Proof. A has a finite number of irreducible components A_λ which are also compact. Every function $f \in \mathcal{O}(X)$ is constant on A_λ, so A_λ must be reduced to a single point. □

§5.2. Coherent Sheaves over Complex Spaces

Let X be a complex space and $\mathcal{O}_X$ its structure sheaf. Locally, X can be identified to an analytic set A in an open set $\Omega \subset \mathbb{C}^n$, and we have $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{I}_A$. Thus $\mathcal{O}_X$ is coherent over the sheaf of rings $\mathcal{O}_\Omega$. It follows immediately that $\mathcal{O}_X$ is coherent over itself. Let $\mathcal{S}$ be a $\mathcal{O}_X$-module. If $\tilde{\mathcal{S}}$ denotes the extension of $\mathcal{S}_{\restriction A}$ to Ω obtained by setting $\tilde{\mathcal{S}}_x = 0$ for $x \in \Omega \smallsetminus A$, then $\tilde{\mathcal{S}}$ is a $\mathcal{O}_\Omega$-module, and it is easily seen that $\mathcal{S}_{\restriction A}$ is coherent over $\mathcal{O}_{X\restriction A}$ if and only if $\tilde{\mathcal{S}}$ is coherent over $\mathcal{O}_\Omega$. If Y is an analytic subset of X, then Y is locally given by an analytic subset B of A and the sheaf of ideals of Y in $\mathcal{O}_X$ is the quotient $\mathcal{I}_Y = \mathcal{I}_B/\mathcal{I}_A$; hence $\mathcal{I}_Y$ is coherent. Let us mention the following important property of supports.

(5.10) Theorem. *If $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module, the support of $\mathcal{S}$, defined as* $\operatorname{Supp} \mathcal{S} = \{x \in X \,;\, \mathcal{S}_x \neq 0\}$ *is an analytic subset of X.*

Proof. The result is local, thus after extending $\mathcal{S}$ by 0, we may as well assume that X is an open subset $\Omega \subset \mathbb{C}^n$. By (3.12), there is an exact sequence of sheaves

$$\mathcal{O}_U^{\oplus p} \xrightarrow{G} \mathcal{O}_U^{\oplus q} \xrightarrow{F} \mathcal{S}_{\restriction U} \longrightarrow 0$$

in a neighborhood U of any point. If $G : \mathcal{O}_x^{\oplus p} \longrightarrow \mathcal{O}_x^{\oplus q}$ is surjective it is clear that the linear map $G(x) : \mathbb{C}^p \longrightarrow \mathbb{C}^q$ must be surjective; conversely, if $G(x)$ is surjective, there is a q-dimensional subspace $E \subset \mathbb{C}^p$ on which the restriction of $G(x)$ is a bijection onto $\mathbb{C}^q$; then $G_{\restriction E} : \mathcal{O}_U \otimes_{\mathbb{C}} E \longrightarrow \mathcal{O}_U^{\oplus q}$ is bijective near x and G is surjective. The support of $\mathcal{S}_{\restriction U}$ is thus equal to the set of points $x \in U$ such that all minors of $G(x)$ of order q vanish. □

§6. Analytic Cycles and Meromorphic Functions

§6.1. Complete Intersections

Our goal is to study in more details the dimension of a subspace given by a set of equations. The following proposition is our starting point.

(6.1) Proposition. *Let X be a complex space of pure dimension p and A an analytic subset of X with* $\operatorname{codim}_X A \geq 2$. *Then every function $f \in \mathcal{O}(X \smallsetminus A)$ is locally bounded near A.*

Proof. The statement is local on X, so we may assume that X is an irreducible germ of analytic set in $(\mathbb{C}^n, 0)$. Let $(A_k, 0)$ be the irreducible components of $(A, 0)$. By a reasoning analogous to that of Prop. 4.26, we can choose coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ such that all projections

$$\pi : z \longmapsto (z_1, \ldots, z_p), \quad p = \dim X,$$
$$\pi_k : z \longmapsto (z_1, \ldots, z_{p_k}), \quad p_k = \dim A_k\,,$$

define ramified coverings $\pi : X \cap \Delta \longrightarrow \Delta'$, $\pi_k : A_k \cap \Delta \longrightarrow \Delta'_k$. By construction $\pi(A_k) \subset \Delta'$ is contained in the set B_k defined by some Weierstrass polynomials in the variables $z_{p_k+1}, \ldots, z_p$ and $\operatorname{codim}_{\Delta'} B_k = p - p_k \geq 2$. Let S be the ramification locus of π and $B = \bigcup B_k$. We have $\pi(A \cap \Delta) \subset B$. For $z' \in \Delta' \smallsetminus (S \cup B)$, we let

$$\sigma_k(z') = \text{elementary symmetric function of degree } k \text{ in } f(z', z''_\alpha),$$

where (z', z''_α) are the q points of X projecting on z'. Then σ_k is holomorphic on $\Delta' \smallsetminus (S \cup B)$ and locally bounded near every point of $S \smallsetminus B$, thus σ_k extends holomorphically to $\Delta' \smallsetminus B$ by Remark 4.2. Since $\operatorname{codim} B \geq 2$, σ_k extends to Δ' by Cor. 1.4.5. Now, f satisfies $f^q - \sigma_1 f^{q-1} + \ldots + (-1)^q \sigma_q = 0$, thus f is locally bounded on $X \cap \Delta$. □

(6.2) Theorem. *Let X be an irreducible complex space and $f \in \mathcal{O}(X)$, $f \not\equiv 0$. Then $f^{-1}(0)$ is empty or of pure dimension* $\dim X - 1$.

Proof. Let $A = f^{-1}(0)$. By Prop. 4.26, we know that $\dim A \leq \dim X - 1$. If A had an irreducible branch A_j of dimension $\leq \dim X - 2$, then in virtue of Prop. 6.1 the function $1/f$ would be bounded in a neighborhood of $A_j \smallsetminus \bigcup_{k\neq j} A_k$, a contradiction. □

(6.3) Corollary. *If $f_1, \ldots, f_p$ are holomorphic functions on an irreducible complex space X, then all irreducible components of $f_1^{-1}(0) \cap \ldots \cap f_p^{-1}(0)$ have codimension $\geq p$.* □

(6.4) Definition. *Let X be a complex space of pure dimension n and A an analytic subset of X of pure dimension. Then A is said to be a local (set theoretic) complete intersection in X if every point of A has a neighborhood Ω such that*

$$A \cap \Omega = \{x \in \Omega\,;\, f_1(x) = \ldots = f_p(x) = 0\}$$

with exactly $p = \operatorname{codim} A$ functions $f_j \in \mathcal{O}(\Omega)$.

(6.5) Remark. As a converse to Th. 6.2, one may ask whether every hypersurface A in X is locally defined by a single equation $f = 0$. In general the answer is negative. A simple counterexample for $\dim X = 3$ is obtained with the singular quadric $X = \{z_1z_2 + z_3z_4 = 0\} \subset \mathbb{C}^4$ and the plane $A = \{z_1 = z_3 = 0\} \subset X$. Then A cannot be defined by a single equation $f = 0$ near the origin, otherwise the plane $B = \{z_2 = z_4 = 0\}$ would be such that

$$f^{-1}(0) \cap B = A \cap B = \{0\},$$

in contradiction with Th. 6.2 (also, by Exercise 10.11, we would get the inequality $\operatorname{codim}_X A \cap B \leq 2$). However, the answer is positive when X is a manifold:

(6.6) Theorem. *Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, let (A, x) be an analytic germ of pure dimension $n-1$ and let A_j, $1 \leq j \leq N$, be its irreducible components.*

a) *The ideal of (A, x) is a principal ideal $\mathcal{I}_{A,x} = (g)$ where g is a product of irreducible germs g_j such that $\mathcal{I}_{A_j,x} = (g_j)$.*

b) *For every $f \in \mathcal{O}_{M,x}$ such that $f^{-1}(0) \subset (A, x)$, there is a unique decomposition $f = ug_1^{m_1} \ldots g_N^{m_N}$ where u is an invertible germ and m_j is the order of vanishing of f at any point $z \in A_{j,\mathrm{reg}} \smallsetminus \bigcup_{k\neq j} A_k$.*

Proof. a) In a suitable local coordinate system centered at x, the projection $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ realizes all A_j as ramified coverings

$$\pi : A_j \cap \Delta \longrightarrow \Delta' \subset \mathbb{C}^{n-1}, \qquad \text{ramification locus} = S_j \subset \Delta'.$$

The function

$$g_j(z', z_n) = \prod_{w \in A_j \cap \pi^{-1}(z')} (z_n - w_n), \qquad z' \in \Delta' \smallsetminus S_j$$

extends into a holomorphic function in $\mathcal{O}_{\Delta'}[z_n]$ and is irreducible at x. Set $g = \prod g_j \in \mathcal{I}_{A,x}$. For any $f \in \mathcal{I}_{A,x}$, the Weierstrass division theorem yields $f = gQ + R$ with $R \in \mathcal{O}_{n-1}[z_n]$ and $\deg R < \deg g$. As $R(z', z_n)$ vanishes when z_n is equal to w_n for each point $w \in A \cap \pi^{-1}(z')$, R has exactly $\deg g$ roots when $z' \in \Delta' \smallsetminus \big(\bigcup S_j \cup \bigcup \pi(A_j \cap A_k)\big)$, so $R = 0$. Hence $\mathcal{I}_{A,x} = (g)$ and similarly $\mathcal{I}_{A_j,x} = (g_j)$. Since $\mathcal{I}_{A_j}$ is coherent, g_j is also a generator of $\mathcal{I}_{A_j,z}$ for z near x and we infer that g_j has order 1 at any regular point $z \in A_{j,\mathrm{reg}}$.

b) As $\mathcal{O}_{M,x}$ is factorial, any $f \in \mathcal{O}_{M,x}$ can be written $f = u\, g_1^{m_1} \ldots g_N^{m_N}$ where u is either invertible or a product of irreducible elements distinct from the g_j's. In the latter case the hypersurface $u^{-1}(0)$ cannot be contained in (A, x), otherwise it would be a union of some of the components A_j and u would be divisible by some g_j. This proves b). □

(6.7) Definition. *Let X be an complex space of pure dimension n.*

a) *An analytic q-cycle Z on X is a formal linear combination $\sum \lambda_j A_j$ where (A_j) is a locally finite family of irreducible analytic sets of dimension q in X and $\lambda_j \in \mathbb{Z}$. The support of Z is $|Z| = \bigcup_{\lambda_j \neq 0} A_j$. The group of all q-cycles on X is denoted* $\mathrm{Cycl}^q(X)$. *Effective q-cycles are elements of the subset* $\mathrm{Cycl}^q_+(X)$ *of cycles such that all coefficients λ_j are ≥ 0 ; rational, real cycles are cycles with coefficients $\lambda_j \in \mathbb{Q}$, $\mathbb{R}$.*

b) *An analytic $(n-1)$-cycle is called a (Weil) divisor, and we set*

$$\mathrm{Div}(X) = \mathrm{Cycl}^{n-1}(X).$$

c) *Assume that* $\dim X_{\mathrm{sing}} \leq n-2$. *If $f \in \mathcal{O}(X)$ does not vanish identically on any irreducible component of X, we associate to f a divisor*

$$\mathrm{div}(f) = \sum m_j A_j \in \mathrm{Div}_+(X)$$

in the following way: the components A_j are the irreducible components of $f^{-1}(0)$ and the coefficient m_j is the vanishing order of f at every regular point in $X_{\mathrm{reg}} \cap A_{j,\mathrm{reg}} \smallsetminus \bigcup_{k \neq j} A_k$. It is clear that we have

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g).$$

d) *A Cartier divisor is a divisor $D = \sum \lambda_j A_j$ that is equal locally to a $\mathbb{Z}$-linear combination of divisors of the form* $\mathrm{div}(f)$.

It is easy to check that the collection of abelian groups $\mathrm{Cycl}^q(U)$ over all open sets $U \subset X$, together with the obvious restriction morphisms, satisfies axioms (1.4) of sheaves; observe however that the restriction of an irreducible component A_j to a smaller open set may subdivide in several components.

Hence we obtain sheaves of abelian groups Cycl^q and $\mathrm{Div} = \mathrm{Cycl}^{n-1}$ on X. The stalk Cycl^q_x is the free abelian group generated by the set of irreducible germs of q-dimensional analytic sets at the point x. These sheaves carry a natural partial ordering determined by the subsheaf of positive elements Cycl^q_+. We define the sup and inf of two analytic cycles $Z = \sum \lambda_j A_j$, $Z' = \sum \mu_j A_j$ by

$$(6.8)\quad \sup\{Z, Z'\} = \sum \sup\{\lambda_j, \mu_j\}\, A_j, \quad \inf\{Z, Z'\} = \sum \inf\{\lambda_j, \mu_j\}\, A_j\,;$$

it is clear that these operations are compatible with restrictions, i.e. they are defined as sheaf operations.

(6.9) Remark. When X is a manifold, Th. 6.6 shows that every effective $\mathbb{Z}$-divisor is locally the divisor of a holomorphic function; thus, for manifolds, the concepts of Weil and Cartier divisors coincide. This is not always the case in general: in Example 6.5, one can show that A is not a Cartier divisor (exercise 10.?).

§6.2. Divisors and Meromorphic Functions

Let X be a complex space. For $x \in X$, let $\mathcal{M}_{X,x}$ be the ring of quotients of $\mathcal{O}_{X,x}$, i.e. the set of formal quotients g/h, $g, h \in \mathcal{O}_{X,x}$, where h is not a zero divisor in $\mathcal{O}_{X,x}$, with the identification $g/h = g'/h'$ if $gh' = g'h$. We consider the disjoint union

$$(6.10)\quad \mathcal{M}_X = \coprod_{x \in X} \mathcal{M}_{X,x}$$

with the topology in which the open sets open sets are unions of sets of the type $\{G_x/H_x\,;\, x \in V\} \subset \mathcal{M}_X$ where V is open in X and $G, H \in \mathcal{O}_X(V)$. Then $\mathcal{M}_X$ is a sheaf over X, and the sections of $\mathcal{M}_X$ over an open set U are called *meromorphic functions* on U. By definition, these sections can be represented locally as quotients of holomorphic functions, but there need not exist such a global representation on U.

A point $x \in X$ is called a *pole* of a meromorphic function f on X if $f_x \notin \mathcal{O}_{X,x}$. Clearly, the set P_f of poles of f is a closed subset of X with empty interior: if $f = g/h$ on U, then $h \not\equiv 0$ on any irreducible component and $P_f \cap U \subset h^{-1}(0)$. For $x \notin P_f$, one can speak of *the value* $f(x)$. If the restriction of f to $X_{\mathrm{reg}} \smallsetminus P_f$ does not vanish identically on any irreducible component of (X, x), then $1/f$ is a meromorphic function in a neighborhood of x ; the set of poles of $1/f$ will be denoted Z_f and called the *zero set* of f. If f vanishes on some connected open subset of $X_{\mathrm{reg}} \smallsetminus P_f$, then f vanishes identically (outside P_f) on the global irreducible component X_α containing this set; we agree that these components X_α are contained in Z_f. For every point x in the complement of $Z_f \cap P_f$, we have either $f_x \in \mathcal{O}_{X,x}$ or $(1/f)_x \in \mathcal{O}_{X,x}$, thus

f defines a holomorphic map $X \smallsetminus (Z_f \cap P_f) \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ with values in the projective line. In general, no value (finite or infinite) can be assigned to f at a point $x \in Z_f \cap P_f$, as shows the example of the function $f(z) = z_2/z_1$ in $\mathbb{C}^2$. The set $Z_f \cap P_f$ is called the *indeterminacy set* of f.

(6.11) Theorem. *For every meromorphic function f on X, the sets P_f, Z_f and the indeterminacy set $Z_f \cap P_f$ are analytic subsets.*

Proof. Let $\mathcal{J}_x$ be the ideal of germs $u \in \mathcal{O}_{X,x}$ such that $uf_x \in \mathcal{O}_{X,x}$. Let us write $f = g/h$ on a small open set U. Then $\mathcal{J}_{\restriction U}$ appears as the projection on the first factor of the sheaf of relations $\mathcal{R}(g,h) \subset \mathcal{O}_U \times \mathcal{O}_U$, so $\mathcal{J}$ is a coherent sheaf of ideals. Now

$$P_f = \big\{x \in X\,;\, \mathcal{J}_x = \mathcal{O}_{X,x}\big\} = \mathrm{Supp}\ \mathcal{O}_X/\mathcal{J},$$

thus P_f is analytic by Th. 5.10. Similarly, the projection of $\mathcal{R}(g,h)$ on the second factor defines a sheaf of ideals $\mathcal{J}'$ such that $Z_f = \mathrm{Supp}\ \mathcal{O}_X/\mathcal{J}'$. □

When X has pure dimension n and $\dim X_{\mathrm{sing}} \le n-2$, Def. 6.7 c) can be extended to meromorphic functions: if $f = g/h$ locally, we set

$$(6.12)\quad \mathrm{div}(f) = \mathrm{div}(g) - \mathrm{div}(h).$$

By 6.7 c), we immediately see that this definition does not depend on the choice of the local representant g/h. Furthermore, *Cartier divisors* are precisely those divisors which are associated locally to meromorphic functions.

Assume from now on that M is a connected n-dimensional complex manifold. Then, for every point $x \in M$, the ring $\mathcal{O}_{M,x} \simeq \mathcal{O}_n$ is factorial. This property makes the study of meromorphic functions much easier.

(6.13) Theorem. *Let f be a non zero meromorphic function on a manifold M, $\dim_{\mathbb{C}} M = n$. Then the sets Z_f, P_f are purely $(n-1)$-dimensional, and the indeterminacy set $Z_f \cap P_f$ is purely $(n-2)$-dimensional.*

Proof. For every point $a \in M$, the germ f_a can be written g_a/h_a where $g_a, h_a \in \mathcal{O}_{M,a}$ are relatively prime holomorphic germs. By Th. 1.12, the germs g_x, h_x are still relatively prime for x in a neighborhood U of a. Thus the ideal $\mathcal{J}$ associated to f coincides with (h) on U, and we have

$$P_f \cap U = \mathrm{Supp}\ \mathcal{O}_U/(h) = h^{-1}(0), \qquad Z_f \cap U = g^{-1}(0).$$

Th. 6.2 implies our contentions: if g_λ and h_μ are the irreducible components of g, h, then $Z_f \cap P_f = \bigcup g_\lambda^{-1}(0) \cap h_\mu^{-1}(0)$ is $(n-2)$-dimensional. As we will see in the next section, Th. 6.13 does not hold on an arbitrary complex space. □

Let (A_j), resp. (B_j), be the global irreducible components of Z_f, resp. P_f. In a neighborhood V_j of the $(n-1)$-dimensional analytic set

$$A'_j = A_j \smallsetminus \big(P_f \cup \bigcup_{k\neq j} A_k\big)$$

f is holomorphic and $V \cap f^{-1}(0) = A'_j$. As $A'_{j,\mathrm{reg}}$ is connected, we must have $\mathrm{div}(f_{\restriction V_j}) = m_j A'_j$ for some constant multiplicity m_j equal to the vanishing order of f along $A'_{j,\mathrm{reg}}$. Similarly, $1/f$ is holomorphic in a neighborhood W_j of

$$B'_j = B_j \smallsetminus \big(Z_f \cup \bigcup_{k\neq j} B_k\big)$$

and we have $\mathrm{div}(f_{\restriction V}) = -p_j B'_j$ where p_j is the vanishing order of $1/f$ along $B'_{j,\mathrm{reg}}$. At a point $x \in M$ the germs $A_{j,x}$ and $B_{j,x}$ may subdivide in irreducible local components $A_{j,\lambda,x}$ and $B_{j,\lambda,x}$. If $g_{j,\lambda}$ and $h_{j,\lambda}$ are local generators of the corresponding ideals, we may a priori write

$$f_x = u\,g/h \quad \text{where} \quad g = \prod g_{j,\lambda}^{m_{j,\lambda}}, \quad h = \prod h_{j,\lambda}^{p_{j,\lambda}}$$

and where u is invertible. Then necessarily $m_{j,\lambda} = m_j$ and $p_{j,\lambda} = p_j$ for all λ, and we see that the global divisor of f on M is

$$(6.14) \quad \mathrm{div}(f) = \sum m_j A_j - \sum p_j B_j.$$

Let us denote by $\mathcal{M}^\star$ the multiplicative sheaf of germs of non zero meromorphic functions, and by $\mathcal{O}^\star$ the sheaf of germs of invertible holomorphic functions. Then we have an exact sequence of sheaves

$$(6.15) \quad 1 \longrightarrow \mathcal{O}^\star \longrightarrow \mathcal{M}^\star \xrightarrow{\mathrm{div}} \mathrm{Div} \longrightarrow 0.$$

Indeed, the surjectivity of div is a consequence of Th. 6.6. Moreover, any meromorphic function that has a positive divisor must be holomorphic by the fact that $\mathcal{O}_n$ is factorial. Hence a meromorphic function f with $\mathrm{div}(f) = 0$ is an invertible holomorphic function.

§7. Normal Spaces and Normalization

§7.1. Weakly Holomorphic Functions

The goal of this section is to show that the singularities of X can be studied by enlarging the structure sheaf $\mathcal{O}_X$ into a sheaf $\tilde{\mathcal{O}}_X$ of so-called weakly holomorphic functions.

(7.1) Definition. *Let X be a complex space. A weakly holomorphic function f on X is a holomorphic function on X_{reg} such that every point of X_{sing} has a neighborhood V for which f is bounded on $X_{\mathrm{reg}} \cap V$. We denote by $\tilde{\mathcal{O}}_{X,x}$ the ring of germs of weakly holomorphic functions over neighborhoods of x and $\tilde{\mathcal{O}}_X$ the associated sheaf.*

Clearly, $\tilde{\mathcal{O}}_{X,x}$ is a ring containing $\mathcal{O}_{X,x}$. If (X_j, x) are the irreducible components of (X, x), there is a fundamental system of neighborhoods V of x such that $X_{\mathrm{reg}} \cap V$ is a disjoint union of connected open sets

$$X_{j,\mathrm{reg}} \cap V \smallsetminus \bigcup_{k \neq j} X_k \cap X_{j,\mathrm{reg}} \cap V$$

which are dense in $X_{j,\mathrm{reg}} \cap V$. Therefore any bounded holomorphic function on $X_{\mathrm{reg}} \cap V$ extends to each component $X_{j,\mathrm{reg}} \cap V$ and we see that

$$\tilde{\mathcal{O}}_{X,x} = \bigoplus \tilde{\mathcal{O}}_{X_j,x}.$$

The first important fact is that weakly holomorphic functions are always meromorphic and possess "universal denominators".

(7.2) Theorem. *For every point $x \in X$, there is a neighborhood V of x and $h \in \mathcal{O}_X(V)$ such that $h^{-1}(0)$ is nowhere dense in V and $h_y \tilde{\mathcal{O}}_{X,y} \subset \mathcal{O}_{X,y}$ for all $y \in V$; such a function h is called a universal denominator on V. In particular $\tilde{\mathcal{O}}_X$ is contained in the ring $\mathcal{M}_X$ of meromorphic functions.*

Proof. First assume that (X, x) is irreducible and that we have a ramified covering $\pi : X \cap \Delta \longrightarrow \Delta'$ with ramification locus S. We claim that the discriminant $\delta(z')$ of a primitive element $u(z'') = c_{d+1} z_{d+1} + \cdots + c_n z_n$ is a universal denominator on $X \cap \Delta$. To see this, we imitate the proof of Lemma 4.15. Let $f \in \tilde{\mathcal{O}}_{X,y}$, $y \in X \cap \Delta$. Then we solve the equation

$$f(z) = \sum_{0 \le j \le q} b_j(z') u(z'')^j$$

in a neighborhood of y. For $z' \in \Delta' \smallsetminus S$, let us denote by (z', z''_α), $1 \le \alpha \le q$, the points in the fiber $X \cap \pi^{-1}(z')$. Among these, only q' are close to y, where q' is the sum of the sheet numbers of the irreducible components of (X, y) by the projection π. The other points (z', z''_α), say $q' < \alpha \le q$, are in neighborhoods of the points of $\pi^{-1}(y') \smallsetminus \{y\}$. We take $\big(b_j(z')\big)$ to be the solution of the linear system

$$\sum_{0 \le j \le q} b_j(z') u(z''_\alpha)^j = \begin{cases} f(z', z''_\alpha) & \text{for } 1 \le \alpha \le q', \\ 0 & \text{for } q' < \alpha \le n. \end{cases}$$

The solutions $b_j(z')$ are holomorphic on $\Delta' \smallsetminus S$ near y'. Since the determinant is $\delta(z')^{1/2}$, we see that δb_j is bounded, thus $\delta b_j \in \mathcal{O}_{\Delta',y'}$ and $\delta_y f \in \mathcal{O}_{X,y}$.

Now, assume that $(X,x) \subset (\mathbb{C}^n,0)$ has irreducible components (X_j,x). We can find for each j a neighborhood Ω_j of 0 in $\mathbb{C}^n$ and a function $\delta_j \in \mathcal{O}_n(\Omega_j)$ which is a universal denominator on $X_j \cap \Omega_j$. After adding to δ_j a function which is identically zero on (X_j,x) and non zero on (X_k,x), $k \neq j$, we may assume that $\delta_j^{-1}(0) \cap X_k \cap \Omega$ is nowhere dense in $X_k \cap \Omega$ for all j and k and some small $\Omega \subset \bigcap \Omega_j$. Then $\delta = \prod \delta_j$ is a universal denominator on each component $X_j \cap \Omega$. For some possibly smaller Ω, select a function $v_j \in \mathcal{O}_n(\Omega)$ such that v_j vanishes identically on $\bigcup_{k\neq j} X_k \cap \Omega$ and $v_j^{-1}(0)$ is nowhere dense in $X_j \cap \Omega$, and set $h = \delta \sum v_k$. For any germ $f \in \tilde{\mathcal{O}}_{X,y}$, $y \in X \cap \Omega$, there is a germ $g_j \in \mathcal{O}_{\Omega,y}$ with $\delta f = g_j$ on (X_j,y). We have $h = \delta v_j$ on $X_j \cap \Omega$, so $h^{-1}(0)$ is nowhere dense in $X \cap \Omega$ and

$$hf = v_j \delta f = v_j g_j = \sum v_k g_k \quad \text{on each} \quad (X_j,y).$$

Since $\sum v_k g_k \in \mathcal{O}_{\Omega,y}$, we get $h\tilde{\mathcal{O}}_{X,y} \subset \mathcal{O}_{X,y}$. □

(7.3) Theorem. *If (X,x) is irreducible, $\tilde{\mathcal{O}}_{X,x}$ is the integral closure of $\mathcal{O}_{X,x}$ in its quotient field $\mathcal{M}_{X,x}$. Moreover, every germ $f \in \tilde{\mathcal{O}}_{X,x}$ admits a limit*

$$\lim_{X_{\mathrm{reg}} \ni z \to x} f(z).$$

Observe that $\mathcal{O}_{X,x}$ is an entire ring, so the ring of quotients $\mathcal{M}_{X,x}$ is actually a field. A simple illustration of the theorem is obtained with the irreducible germ of curve $X : z_1^3 = z_2^2$ in $(\mathbb{C}^2,0)$. Then X can be parametrized by $z_1 = t^2$, $z_2 = t^3$, $t \in \mathbb{C}$, and $\mathcal{O}_{X,0} = \mathbb{C}\{z_1,z_2\}/(z_1^3 - z_2^2) = \mathbb{C}\{t^2,t^3\}$ consists of all convergent series $\sum a_n t^n$ with $a_1 = 0$. The function $z_2/z_1 = t$ is weakly holomorphic on X and satisfies the integral equation $t^2 - z_1 = 0$. Here we have $\tilde{\mathcal{O}}_{X,0} = \mathbb{C}\{t\}$.

Proof. a) Let $f = g/h$ be an element in $\mathcal{M}_{X,x}$ satisfying an integral equation

$$f^m + a_1 f^{m-1} + \ldots + a_m = 0, \qquad a_k \in \mathcal{O}_{X,x}.$$

Set $A = h^{-1}(0)$. Then f is holomorphic on $X \smallsetminus A$ near x, and Lemma 4.10 shows that f is bounded on a neighborhood of x. By Remark 4.2, f can be extended as a holomorphic function on X_{reg} in a neighborhood of x, thus $f \in \tilde{\mathcal{O}}_{X,x}$.

b) Let $f \in \tilde{\mathcal{O}}_{X,x}$ and let $\pi : X \cap \Delta \longrightarrow \Delta'$ be a ramified covering in a neighborhood of x, with ramification locus S. As in the proof of Th. 6.1, f satisfies an equation

$$f^q - \sigma_1 f^{q-1} + \cdots + (-1)^q \sigma_q = 0, \qquad \sigma_k \in \mathcal{O}(\Delta') \ ;$$

indeed the elementary symmetric functions $\sigma_k(z')$ are holomorphic on $\Delta' \setminus S$ and bounded, so they extend holomorphically to Δ'. Hence $\widetilde{\mathcal{O}}_{X,x}$ is integral over $\mathcal{O}_{X,x}$ and we already know that $\widetilde{\mathcal{O}}_{X,x} \subset \mathcal{M}_{X,x}$.

c) Finally, the cluster set $\bigcap_{V \ni x} \overline{f(X_{\rm reg} \cap V)}$ is connected, because there is a fundamental system of neighborhoods V of x such that $X_{\rm reg} \cap V$ is connected, and any intersection of a decreasing sequence of compact connected sets is connected. However the limit set is contained in the finite set of roots of equation b) at point $x' \in \Delta'$, so it must be reduced to one element. □

§7.2. Normal Spaces

Normal spaces are spaces for which all weakly holomorphic functions are actually holomorphic. These spaces will be seen later to have "simpler" singularities than general analytic spaces.

(7.4) Definition. *A complex space X is said to be normal at a point x if (X,x) is irreducible and $\widetilde{\mathcal{O}}_{X,x} = \mathcal{O}_{X,x}$, that is, $\mathcal{O}_{X,x}$ is integrally closed in its field of quotients. The set of normal (resp. non-normal) points will be denoted $X_{\rm norm}$ (resp. $X_{\text{n-n}}$). The space X itself is said to be normal if X is normal at every point.*

Observe that any regular point x is normal: in fact $\mathcal{O}_{X,x} \simeq \mathcal{O}_n$ is then factorial, hence integrally closed. Therefore $X_{\text{n-n}} \subset X_{\rm sing}$.

(7.5) Theorem. *The non-normal set $X_{\text{n-n}}$ is an analytic subset of X. In particular, $X_{\rm norm}$ is open in X.*

Proof. We give here a beautifully simple proof due to (Grauert and Remmert 1984). Let h be a universal denominator on a neighborhood V of a given point and let $\mathcal{J} = \sqrt{h\mathcal{O}_X}$ be the sheaf of ideals of $h^{-1}(0)$ by Hilbert's Nullstellensatz. Finally, let $\mathcal{F} = \hom_{\mathcal{O}}(\mathcal{J}, \mathcal{J})$ be the sheaf of $\mathcal{O}_X$-endomorphisms of $\mathcal{J}$. Since $\mathcal{J}$ is coherent, so is $\mathcal{F}$ (cf. Exercise 10.?). Clearly, the homotheties of $\mathcal{J}$ give an injection $\mathcal{O}_X \subset \mathcal{F}$ over V. We claim that there is a natural injection $\mathcal{F} \subset \widetilde{\mathcal{O}}_X$. In fact, any endomorphism of $\mathcal{J}$ yields by restriction a homomorphism $h\mathcal{O}_X \longrightarrow \mathcal{O}_X$, and by $\mathcal{O}_X$-linearity such a homomorphism is obtained by multiplication by an element in $h^{-1}\mathcal{O}_X$. Thus $\mathcal{F} \subset h^{-1}\mathcal{O}_X \subset \mathcal{M}_X$. Since each stalk $\mathcal{J}_x$ is a finite $\mathcal{O}_{X,x}$-module containing non-zero divisors, it follows that that any meromorphic germ f such that $f\mathcal{J}_x \subset \mathcal{J}_x$ is integral over $\mathcal{O}_{X,x}$ (Lang 1965, Chapter IX, §1), hence $\mathcal{F}_x \subset \widetilde{\mathcal{O}}_{X,x}$. Thus we have inclusions $\mathcal{O}_X \subset \mathcal{F} \subset \widetilde{\mathcal{O}}_X$. Now, we assert that

$$X_{\text{n-n}} \cap V = \{x \in V\,;\, \mathcal{F}_x \neq \mathcal{O}_{X,x}\} = \mathcal{F}/\mathcal{O}_X.$$

This will imply the theorem by 5.10. To prove the equality, we first observe that $\mathcal{F}_x \neq \mathcal{O}_{X,x}$ implies $\widetilde{\mathcal{O}}_{X,x} \neq \mathcal{O}_{X,x}$, thus $x \in X_{\text{n-n}}$. Conversely, assume that

x is non normal, that is, $\tilde{\mathcal{O}}_{X,x} \neq \mathcal{O}_{X,x}$. Let k be the smallest integer such that $\mathcal{I}_x^k \tilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$; such an integer exists since $\mathcal{I}_x^l \tilde{\mathcal{O}}_{X,x} \subset h\tilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$ for l large. Then there is an element $w \in \mathcal{I}_x^{k-1}\tilde{\mathcal{O}}_{X,x}$ such that $w \notin \mathcal{O}_{X,x}$. We have $w\mathcal{I}_x \subset \mathcal{O}_{X,x}$; moreover, as w is locally bounded near X_{sing}, any germ wg in $w\mathcal{I}_x$ satisfies $\lim w(z)g(z) = 0$ when $z \in X_{\mathrm{reg}}$ tends to a point of the zero variety $h^{-1}(0)$ of $\mathcal{I}_x$. Hence $w\mathcal{I}_x \subset \mathcal{I}_x$, i.e. $w \in \mathcal{F}_x$, but $w \notin \mathcal{O}_{X,x}$, so $\mathcal{F}_x \neq \mathcal{O}_{X,x}$. □

(7.6) Theorem. *If $x \in X$ is a normal point, then (X_{sing}, x) has codimension at least 2 in (X, x).*

Proof. We suppose that $\Sigma = X_{\mathrm{sing}}$ has codimension 1 in a neighborhood of x and try to get a contradiction. By restriction to a smaller neighborhood, we may assume that X itself is normal and irreducible (since X_{norm} is open), $\dim X = n$, that Σ has pure dimension $n-1$ and that the ideal sheaf $\mathcal{I}_\Sigma$ has global generators $(g_1, \ldots, g_k)$. Then $\Sigma \subset \bigcup g_j^{-1}(0)$; both sets have pure dimension $n-1$ and thus singular sets of dimension $\leq n-2$. Hence there is a point $a \in \Sigma$ that is regular on Σ and on $\bigcup g_j^{-1}(0)$, in particular there is a neighborhood V of a such that $g_1^{-1}(0) \cap V = \ldots = g_k^{-1}(0) \cap V = \Sigma \cap V$ is a smooth $(n-1)$-dimensional manifold. Since $\operatorname{codim}_X \Sigma = 1$ and a is a singular point of X, $\mathcal{I}_{\Sigma,a}$ cannot have less than 2 generators in $\mathcal{O}_{X,a}$ by Cor. 4.33. Take $(g_1, \ldots, g_l)$, $l \geq 2$, to be a minimal subset of generators. Then $f = g_2/g_1$ cannot belong to $\mathcal{O}_{X,a}$, but f is holomorphic on $V \smallsetminus \Sigma$. We may assume that there is a sequence $a_\nu \in V \smallsetminus \Sigma$ converging to a such that $f(a_\nu)$ remains bounded (otherwise reverse g_1 and g_2 and pass to a subsequence). Since $g_1^{-1}(0) \cap V = \Sigma \cap V$, Hilbert's Nullstellensatz gives an integer m such that $\mathcal{I}_{\Sigma,a}^m \subset g_1\mathcal{O}_{X,a}$, hence $f_a\mathcal{I}_{\Sigma,a}^m \subset \mathcal{O}_{X,a}$. We take m to be the smallest integer such that the latter inclusion holds. Then there is a product $g^\alpha = g_1^{\alpha_1} \ldots g_l^{\alpha_l}$ with $|\alpha| = m-1$ such that $fg^\alpha \notin \mathcal{O}_{X,a}$ but $fg^\alpha g_j \in \mathcal{O}_{X,a}$ for each j. Since the sequence $f(a_\nu)$ is bounded we conclude that $fg^\alpha g_j$ vanishes at a. The zero set of this function has dimension $n-1$ and is contained in $\bigcup g_k^{-1}(0) \cap V = \Sigma \cap V$ so it must contain the germ (Σ, a). Hence $fg^\alpha g_j \in \mathcal{I}_{\Sigma,a}$ and $fg^\alpha \mathcal{I}_{\Sigma,a} \subset \mathcal{I}_{\Sigma,a}$. As $\mathcal{I}_{\Sigma,a}$ is a finitely generated $\mathcal{O}_{X,a}$-module, this implies $fg^\alpha \in \tilde{\mathcal{O}}_{X,a} = \mathcal{O}_{X,a}$, a contradiction. □

(7.7) Corollary. *A complex curve is normal if and only if it is regular.*

(7.8) Corollary. *Let X be a normal complex space and Y an analytic subset of X such that $\dim(Y, x) \leq \dim(X, x) - 2$ for any $x \in X$. Then any holomorphic function on $X \smallsetminus Y$ can be extended to a holomorphic function on X.*

Proof. By Cor. 1.4.5, every holomorphic function f on $X_{\mathrm{reg}} \smallsetminus Y$ extends to X_{reg}. Since $\operatorname{codim} X_{\mathrm{sing}} \geq 2$, Th. 6.1 shows that f is locally bounded near X_{sing}. Therefore f extends to X by definition of a normal space. □

§7.3. The Oka Normalization Theorem

The important normalization theorem of (Oka 1950) shows that $\tilde{\mathcal{O}}_X$ can be used to define the structure sheaf of a new analytic space $\tilde{X}$ which is normal and is obtained by "simplifying" the singular set of X. More precisely:

(7.9) Definition. *Let X be a complex space. A normalization (Y, π) of X is a normal complex space Y together with a holomorphic map $\pi : Y \longrightarrow X$ such that the following conditions are satisfied.*

a) *$\pi : Y \longrightarrow X$ is proper and has finite fibers*;

b) *if Σ is the set of singular points of X and $A = \pi^{-1}(\Sigma)$, then $Y \smallsetminus A$ is dense in Y and $\pi : Y \smallsetminus A \longrightarrow X \smallsetminus \Sigma = X_{\mathrm{reg}}$ is an analytic isomorphism.*

It follows from b) that $Y \smallsetminus A \subset Y_{\mathrm{reg}}$. Thus Y is obtained from X by a suitable "modification" of its singular points. Observe that Y_{reg} may be larger than $Y \smallsetminus A$, as is the case in the following two examples.

(7.10) Examples.

a) Let $X = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ be the complex curve $z_1 z_2 = 0$ in $\mathbb{C}^2$. Then the normalization of X is the disjoint union $Y = \mathbb{C} \times \{1, 2\}$ of two copies of $\mathbb{C}$, with the map $\pi(t_1) = (t_1, 0)$, $\pi(t_2) = (0, t_2)$. The set $A = \pi^{-1}(0, 0)$ consists of exactly two points.

b) The cubic curve $X : z_1^3 = z_2^2$ is normalized by the map $\pi : \mathbb{C} \longrightarrow X$, $t \longmapsto (t^2, t^3)$. Here π is a homeomorphism but π^{-1} is not analytic at $(0, 0)$. □

We first show that the normalization is essentially unique up to isomorphism and postpone the proof of its existence for a while.

(7.11) Lemma. *If (Y_1, π_1) and (Y_2, π_2) are normalizations of X, there is a unique analytic isomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\pi_1 = \pi_2 \circ \varphi$.*

Proof. Let Σ be the set of singular points of X and $A_j = \pi_j^{-1}(\Sigma)$, $j = 1, 2$. Let $\varphi' : Y_1 \smallsetminus A_1 \longrightarrow Y_2 \smallsetminus A_2$ be the analytic isomorphism $\pi_2^{-1} \circ \pi_1$. We assert that φ' can be extended into a map $\varphi : Y_1 \longrightarrow Y_2$. In fact, let $a \in A_1$ and $s = \pi_1(a) \in \Sigma$. Then $\pi_2^{-1}(s)$ consists of a finite set of points $y_j \in Y_2$. Take disjoint neighborhoods U_j of y_j such that U_j is an analytic subset in an open set $\Omega_j \subset\subset \mathbb{C}^N$. Since π_2 is proper, there is a neighborhood V of s in X such that $\pi_2^{-1}(V) \subset \bigcup U_j$ and by continuity of π_1 a neighborhood W of a such that $\pi_1(W) \subset V$. Then $\varphi' = \pi_2^{-1} \circ \pi_1$ maps $W \smallsetminus A_1$ into $\bigcup U_j$ and can be seen as a bounded holomorphic map into $\mathbb{C}^N$ through the embeddings $U_j \subset \Omega_j \subset\subset \mathbb{C}^N$. Since Y_1 is normal, φ' extends to W, and the extension takes values in $\bigcup \overline{U}_j$ which is contained in Y_2 (shrink U_j if necessary). Thus

φ' extends into a map $\varphi : Y_1 \longrightarrow Y_2$ and similarly φ'^{-1} extends into a map $\psi : Y_2 \longrightarrow Y_1$. By density of $Y_j \smallsetminus A_j$, we have $\psi \circ \varphi = \mathrm{Id}_{Y_1}$, $\varphi \circ \psi = \mathrm{Id}_{Y_2}$. □

(7.12) Oka normalization theorem. *Let X be any complex space. Then X has a normalization (Y, π).*

Proof. Because of the previous lemma, it suffices to prove that any point $x \in X$ has a neighborhood U such that U admits a normalization; all these local normalizations will then glue together. Hence we may suppose that X is an analytic set in an open set of $\mathbb{C}^n$. Moreover, if (X, x) splits into irreducible components (X_j, x) and if (Y_j, π_j) is a normalization of $X_j \cap U$, then the disjoint union $Y = \coprod Y_j$ with $\pi = \coprod \pi_j$ is easily seen to be a normalization of $X \cap U$. We may therefore assume that (X, x) is irreducible. Let h be a universal denominator in a neighborhood of x. Then $\widetilde{\mathcal{O}}_{X,x}$ is isomorphic to its image $h\widetilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$, so it is a finitely generated $\mathcal{O}_{X,x}$-module. Let $(f_1, \ldots, f_m)$ be a finite set of generators of $\mathcal{O}_{X,x}$. After shrinking X again, we may assume the following two points:

- X is an analytic set in an open set $\Omega \subset \mathbb{C}^n$, (X, x) is irreducible and X_{reg} is connected;
- f_j is holomorphic in X_{reg}, can be written $f_j = g_j/h$ on X with g_j, h in $\mathcal{O}_n(\Omega)$ and satisfies an integral equation $P_j(z\,;\, f_j(z)) = 0$ where $P_j(z\,;\, T)$ is a unitary polynomial with holomorphic coefficients on X.

Set $X' = X \smallsetminus h^{-1}(0)$. Consider the holomorphic map

$$F : X_{\mathrm{reg}} \longrightarrow \Omega \times \mathbb{C}^m, \qquad z \longmapsto \big(z, f_1(z), \ldots, f_m(z)\big)$$

and the image $Y' = F(X')$. We claim that the closure Y of Y' in $\Omega \times \mathbb{C}^m$ is an analytic set. In fact, the set

$$Z = \big\{(z, w) \in \Omega \times \mathbb{C}^m\,;\, z \in X\,,\ h(z)w_j = g_j(z)\big\}$$

is analytic and $Y' = Z \smallsetminus \{h(z) = 0\}$, so we may apply Cor. 5.4. Observe that Y' is contained in the set defined by $P_j(z\,;\, w_j) = 0$, thus so is its closure Y. The first projection $\Omega \times \mathbb{C}^m \longrightarrow \Omega$ gives a holomorphic map $\pi : Y \longrightarrow X$ such that $\pi \circ F = \mathrm{Id}$ on X', hence also on X_{reg}. If $\Sigma = X_{\mathrm{sing}}$ and $A = \pi^{-1}(\Sigma)$, the restriction $\pi : Y \smallsetminus A \longrightarrow X \smallsetminus \Sigma = X_{\mathrm{reg}}$ is thus an analytic isomorphism and F is its inverse. Since (X, x) is irreducible, each f_j has a limit ℓ_j at x by Th. 7.3 and the fiber $\pi^{-1}(x)$ is reduced to the single point $y = (x, \ell)$. The other fibers $\pi^{-1}(z)$ are finite because they are contained in the finite set of roots of the equations $P_j(z\,;\, w_j) = 0$. The same argument easily shows that π is proper (use Lemma 4.10).

Next, we show that Y is normal at the point $y = \pi^{-1}(x)$. In fact, for any bounded holomorphic function u on (Y_{reg}, y) the function $u \circ F$ is bounded and holomorphic on (X_{reg}, x). Hence $u \circ F \in \widetilde{\mathcal{O}}_{X,x} = \mathcal{O}_{X,x}[f_1, \ldots, f_m]$ and we can write $u \circ F(z) = Q(z\,;\, f_1(z), \ldots, f_m(z)) = Q \circ F(z)$ where $Q(z\,;\, w) =$

$\sum a_\alpha(z)w^\alpha$ is a polynomial in w with coefficients in $\mathcal{O}_{X,x}$. Thus u coincides with Q on (Y_{reg}, y), and as Q is holomorphic on $(X,x)\times\mathbb{C}^m \supset (Y,y)$, we conclude that $u \in \mathcal{O}_{Y,y}$. Therefore $\tilde{\mathcal{O}}_{Y,y} = \mathcal{O}_{Y,y}$.

Finally, by Th. 7.5, there is a neighborhood $V \subset Y$ of y such that every point of V is normal. As π is proper, we can find a neighborhood U of x with $\pi^{-1}(U) \subset V$. Then $\pi : \pi^{-1}(U) \longrightarrow U$ is the required normalization in a neighborhood of x. □

The proof of Th. 7.12 shows that the fiber $\pi^{-1}(x)$ has exactly one point y_j for each irreducible component (X_j, x) of (X,x). As a one-to-one proper map is a homeomorphism, we get in particular:

(7.13) Corollary. *If X is a locally irreducible complex space, the normalization $\pi : Y \longrightarrow X$ is a homeomorphism.* □

(7.14) Remark. In general, for any open set $U \subset X$, we have an isomorphism

$$(7.15)\qquad \pi^\star : \tilde{\mathcal{O}}_X(U) \xrightarrow{\simeq} \mathcal{O}_Y\big(\pi^{-1}(U)\big),$$

whose inverse is given by the comorphism of $\pi^{-1} : X_{\mathrm{reg}} \longrightarrow Y$; note that $\tilde{\mathcal{O}}_Y(U) = \mathcal{O}_Y(U)$ since Y is normal. Taking the direct limit over all neighborhoods U of a given point $x \in X$, we get an isomorphism

$$(7.15')\qquad \pi^\star : \tilde{\mathcal{O}}_{X,x} \longrightarrow \bigoplus_{y_j \in \pi^{-1}(x)} \mathcal{O}_{Y,y_j}.$$

In other words, $\tilde{\mathcal{O}}_X$ is isomorphic to the direct image sheaf $\pi_\star\mathcal{O}_Y$, see (1.12). We will prove later on the deep fact that the direct image of a coherent sheaf by a proper holomorphic map is always coherent (Grauert 1960, see 9.?.1). Hence $\tilde{\mathcal{O}}_X = \pi_\star\mathcal{O}_Y$ is a coherent sheaf over $\mathcal{O}_X$.

§8. Holomorphic Mappings and Extension Theorems

§8.1. Rank of a Holomorphic Mapping

Our goal here is to introduce the general concept of the rank of a holomorphic map and to relate the rank to the dimension of the fibers. As in the smooth case, the rank is shown to satisfy semi-continuity properties.

(8.1) Lemma. *Let $F : X \longrightarrow Y$ be a holomorphic map from a complex space X to a complex space Y.*

a) *If F is finite, i.e. proper with finite fibers, then* $\dim X \le \dim Y$.

b) *If F is finite and surjective, then* $\dim X = \dim Y$.

Proof. a) Let $x \in X$, (X_j, x) an irreducible component and $m = \dim(X_j, x)$. If (Y_k, y) are the irreducible components of Y at $y = F(x)$, then (X_j, x) is contained in $\bigcup F^{-1}(Y_k)$, hence (X_j, x) is contained in one of the sets $F^{-1}(Y_k)$. If $p = \dim(Y_k, y)$, there is a ramified covering π from some neighborhood of y in Y_k onto a polydisk in $\Delta' \subset \mathbb{C}^p$. Replacing X by some neighborhood of x in X_j and F by the finite map $\pi \circ F_{\restriction X_j} : X_j \longrightarrow \Delta'$, we may suppose that $Y = \Delta'$ and that X is irreducible, $\dim X = m$. Let $r = \operatorname{rank} dF_{x_0}$ be the maximum of the rank of the differential of F on X_{reg}. Then $r \leq \min\{m, p\}$ and the rank of dF is constant equal to r on a neighborhood U of x_0. The constant rank theorem implies that the fibers $F^{-1}(y) \cap U$ are $(m-r)$-dimensional submanifolds, hence $m - r = 0$ and $m = r \leq p$.

b) We only have to show that $\dim X \geq \dim Y$. Fix a regular point $y \in Y$ of maximal dimension. By taking the restriction $F : F^{-1}(U) \longrightarrow U$ to a small neighborhood U of y, we may assume that Y is an open subset of $\mathbb{C}^p$. If $\dim X < \dim Y$, then X is a union of analytic manifolds of dimension $< \dim Y$ and Sard's theorem implies that $F(X)$ has zero Lebesgue measure in Y, a contradiction. □

(8.2) Proposition. *For any holomorphic map $F : X \longrightarrow Y$, the fiber dimension* $\dim\big(F^{-1}(F(x)), x\big)$ *is an upper semi-continuous function of x.*

Proof. Without loss of generality, we may suppose that X is an analytic set in $\Omega \subset \mathbb{C}^n$, that $F(X)$ is contained in a small neighborhood of $F(x)$ in Y which is embedded in $\mathbb{C}^N$, and that $x = 0$, $F(x) = 0$. Set $A = F^{-1}(0)$ and $s = \dim(A, 0)$. We can find a linear form ξ_1 on $\mathbb{C}^n$ such that $\dim(A \cap \xi_1^{-1}(0), 0) = s - 1$; in fact we need only select a point $x_j \neq 0$ on each irreducible component $(A_j, 0)$ of $(A, 0)$ and take $\xi_1(x_j) \neq 0$. By induction, we can find linearly independent forms $\xi_1, \ldots, \xi_s$ on $\mathbb{C}^n$ such that

$$\dim\big(A \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0), 0\big) = s - j$$

for all $j = 1, \ldots, s$; in particular 0 is an isolated point in the intersection when $j = s$. After a change of coordinates, we may suppose that $\xi_j(z) = z_j$. Fix r'' so small that the ball $\overline{B}'' \subset \mathbb{C}^{n-s}$ of center 0 and radius r'' satisfies $A \cap (\{0\} \times \overline{B}'') = \{0\}$. Then A is disjoint from the compact set $\{0\} \times \partial B''$, so there exists a small ball $B' \subset \mathbb{C}^s$ of center 0 such that $A \cap (\overline{B}' \times \partial B'') = \emptyset$, i.e. F does not vanish on the compact set $K = X \cap (\overline{B}' \times \partial B'')$. Set $\varepsilon = \min_K |F|$. Then for $|y| < \varepsilon$ the fiber $F^{-1}(y)$ does not intersect $\overline{B}' \times \partial B''$. This implies that the projection map $\pi : F^{-1}(y) \cap (B' \times B'') \longrightarrow B'$ is proper. The fibers of π are then compact analytic subsets of B'', so they are finite by 5.9. Lemma 8.1 a) implies

$$\dim F^{-1}(y) \cap (B' \times B'') \leq \dim B' = s = \dim(A, 0) = \dim(F^{-1}(0), 0). \quad \square$$

Let X be a pure dimensional complex space and $F : X \longrightarrow Y$ a holomorphic map. For any point $x \in X$, we define the *rank of F at x* by

$$(8.3) \quad \rho_F(x) = \dim(X, x) - \dim\big(F^{-1}(F(x)), x\big).$$

By the above proposition, ρ_F is a lower semi-continuous function on X. In particular, if ρ_F is maximum at some point x_0, it must be constant in a neighborhood of x_0. The maximum $\overline{\rho}(F) = \max_X \rho_F$ is thus attained on $X_{\rm reg}$ or on any dense open subset $X' \subset X_{\rm reg}$. If X is not pure dimensional, we define $\overline{\rho}(F) = \max_\alpha \overline{\rho}(F_{\restriction X_\alpha})$ where (X_α) are the irreducible components of X. For a map $F : X \longrightarrow \mathbb{C}^N$, the constant rank theorem implies that $\overline{\rho}(F)$ is equal to the maximum of the rank of the jacobian matrix dF at points of $X_{\rm reg}$ (or of X').

(8.4) Proposition. *If $F : X \longrightarrow Y$ is a holomorphic map and Z an analytic subset of X, then $\overline{\rho}(F_{\restriction Z}) \le \overline{\rho}(F)$.*

Proof. Since each irreducible component of Z is contained in an irreducible component of X, we may assume X irreducible. Let $\pi : \tilde{X} \longrightarrow X$ be the normalization of X and $\tilde{Z} = \pi^{-1}(Z)$. Since π is finite and surjective, the fiber of $F \circ \pi$ at point x has the same dimension than the fiber of F at $\pi(x)$ by Lemma 8.1 b). Therefore $\overline{\rho}(F \circ \pi) = \overline{\rho}(F)$ and $\overline{\rho}(F \circ \pi_{\restriction \tilde{Z}}) = \overline{\rho}(F_{\restriction Z})$, so we may assume X normal. By induction on $\dim X$, we may also suppose that Z has pure codimension 1 in X (every point of Z has a neighborhood $V \subset X$ such that $Z \cap V$ is contained in a pure one codimensional analytic subset of V). But then $Z_{\rm reg} \cap X_{\rm reg}$ is dense in $Z_{\rm reg}$ because $\operatorname{codim} X_{\rm sing} \ge 2$. Thus we are reduced to the case when X is a manifold and Z a submanifold, and this case is clear if we consider the rank of the jacobian matrix. □

(8.5) Theorem. *Let $F : X \longrightarrow Y$ be a holomorphic map. If Y is pure dimensional and $\overline{\rho}(F) < \dim Y$, then $F(X)$ has empty interior in Y.*

Proof. Taking the restriction of F to $F^{-1}(Y_{\rm reg})$, we may assume that Y is a manifold. Since X is a countable union of compact sets, so is $F(X)$, and Baire's theorem shows that the result is local for X. By Prop. 8.4 and an induction on $\dim X$, $F(X_{\rm sing})$ has empty interior in Y. The set $Z \subset X_{\rm reg}$ of points where the jacobian matrix of F has rank $< \overline{\rho}(F)$ is an analytic subset hence, by induction again, $F(Z)$ has empty interior. The constant rank theorem finally shows that every point $x \in X_{\rm reg} \smallsetminus Z$ has a neighborhood V such that $F(V)$ is a submanifold of dimension $\overline{\rho}(F)$ in Y, thus $F(V)$ has empty interior and Baire's theorem completes the proof. □

(8.6) Corollary. *Let $F : X \longrightarrow Y$ be a surjective holomorphic map. Then* $\dim Y = \overline{\rho}(F)$.

Proof. By the remark before Prop. 8.4, there is a regular point $x_0 \in X$ such that the jacobian matrix of F has rank $\overline{\rho}(F)$. Hence, by the constant rank theorem $\dim Y \geq \overline{\rho}(F)$. Conversely, let Y_α be an irreducible component of Y of dimension equal to $\dim Y$, and $Z = F^{-1}(Y_\alpha) \subset X$. Then $F(Z) = Y_\alpha$ and Th. 8.5 implies $\overline{\rho}(F) \geq \overline{\rho}(F_{\restriction Z}) \geq \dim Y_\alpha$. □

§8.2. Remmert and Remmert-Stein Theorems

We are now ready to prove two important results: the extension theorem for analytic subsets due to (Remmert and Stein 1953) and the theorem of (Remmert 1956,1957) which asserts that the image of a complex space under a proper holomorphic map is an analytic set. These will be obtained by a simultaneous induction on the dimension.

(8.7) Remmert-Stein theorem. *Let X be a complex space, A an analytic subset of X and Z an analytic subset of $X \smallsetminus A$. Suppose that there is an integer $p \geq 0$ such that $\dim A \leq p$, while $\dim(Z,x) > p$ for all $x \in Z$. Then the closure $\overline{Z}$ of Z in X is an analytic subset.*

(8.8) Remmert's proper mapping theorem. *Let $F : X \longrightarrow Y$ be a proper holomorphic map. Then $F(X)$ is an analytic subset of Y.*

Proof. We let (8.7_m) denote statement (8.7) for $\dim Z \leq m$ and (8.8_m) denote statement (8.8) for $\dim X \leq m$. We proceed by induction on m in two steps:

Step 1. (8.7_m) and (8.8_{m-1}) imply (8.8_m).
Step 2. (8.8_{m-1}) implies (8.7_m).

As (8.8_m) is obvious for $m = 0$, our statements will then be valid for all m, i.e. for all complex spaces of bounded dimension. However, Th. 8.7 is local on X and Th. 8.8 is local on Y, so the general case is immediately reduced to the finite dimensional case.

Proof of step 1. The analyticity of $F(X)$ is a local question in Y. Since $F : F^{-1}(U) \longrightarrow U$ is proper for any open set $U \subset Y$ and $F^{-1}(U) \subset\subset X$ if $U \subset\subset Y$, we may suppose that Y is embedded in an open set $\Omega \subset \mathbb{C}^n$ and that X only has finitely many irreducible components X_α. Then we have $F(X) = \bigcup F(X_\alpha)$ and we are reduced to the case when X is irreducible, $\dim X = m$ and $Y = \Omega$.

First assume that X is a manifold and that the rank of dF is constant. The constant rank theorem implies that every point in X has a neighborhood V such that $F(V)$ is a closed submanifold in a neighborhood W of $F(x)$ in Y. For any point $y \in Y$, the fiber $F^{-1}(y)$ can be covered by finitely many neighborhoods V_j of points $x_j \in F^{-1}(y)$ such that $F(V_j)$ is a closed submanifold in a neighborhood W_j of y. Then there is a neighborhood of y

$W \subset \bigcap W_j$ such that $F^{-1}(W) \subset \bigcup V_j$, so $F(X) \cap W = \bigcup F(V_j) \cap W$ is a finite union of closed submanifolds in W and $F(X)$ is analytic in Y.

Now suppose that X is a manifold, set $r = \overline{\rho}(F)$ and let $Z \subset X$ be the analytic subset of points x where the rank of dF_x is $< r$. Since $\dim Z < m = \dim X$, the hypothesis (8.8_{m-1}) shows that $F(Z)$ is analytic. We have $\dim F(Z) = \overline{\rho}(F_{\restriction Z}) < r$. If $F(Z) = F(X)$, then $F(X)$ is analytic. Otherwise $A = F^{-1}\big(F(Z)\big)$ is a proper analytic subset of X, dF has constant rank on $X \smallsetminus A \subset X \smallsetminus Z$ and the morphism $F : X \smallsetminus A \longrightarrow Y \smallsetminus F(Z)$ is proper. Hence the image $F(X \smallsetminus A)$ is analytic in $Y \smallsetminus F(Z)$. Since $\dim F(X \smallsetminus A) = r \le m$ and $\dim F(Z) < r$, hypothesis (8.7_m) implies that $F(X) = \overline{F(X \smallsetminus A)}$ is analytic in Y. When X is not a manifold, we apply the same reasoning with $Z = X_{\rm sing}$ in order to be reduced to the case of $F : X \smallsetminus A \longrightarrow Y \smallsetminus F(Z)$ where $X \smallsetminus A$ is a manifold. □

Proof of step 2. Since Th. 8.7 is local on X, we may suppose that X is an open set $\Omega \subset \mathbb{C}^n$. Then we use induction on p to reduce the situation to the case when A is a p-dimensional submanifold (if this case is taken for granted, the closure of Z in $\Omega \smallsetminus A_{\rm sing}$ is analytic and we conclude by the induction hypothesis). By a local analytic change of coordinates, we may assume that $0 \in A$ and that $A = \Omega \cap L$ where L is a vector subspace of $\mathbb{C}^n$ of dimension p. By writing $Z = \bigcup_{p<s\le m} Z_s$ where Z_s is an analytic subset of $\Omega \smallsetminus Y$ of pure dimension s, we may suppose that Z has pure dimension s, $p < s \le m$. We are going to show that $\overline{Z}$ is analytic in a neighborhood of 0.

Let ξ_1 be a linear form on $\mathbb{C}^n$ which is not identically zero on L nor on any irreducible component of Z (just pick a point x_ν on each component and take $\xi_1(x_\nu) \ne 0$ for all ν). Then $\dim L \cap \xi_1^{-1}(0) = p - 1$ and the analytic set $Z \cap \xi_1^{-1}(0)$ has pure dimension $s - 1$. By induction, there exist linearly independent forms $\xi_1, \ldots, \xi_s$ such that

$$\begin{aligned} &\dim L \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0) = p - j, \qquad 1 \le j \le p, \\ (8.9)\qquad &\dim Z \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0) = s - j, \qquad 1 \le j \le s. \end{aligned}$$

By adding a suitable linear combination of $\xi_1, \ldots, \xi_p$ to each ξ_j, $p < j \le s$, we may take $\xi_{j\restriction L} = 0$ for $p < j \le s$. After a linear change of coordinates, we may suppose that $\xi_j(z) = z_j$, $L = \mathbb{C}^p \times \{0\}$ and $A = \Omega \cap (\mathbb{C}^p \times \{0\})$. Let $\xi = (\xi_1, \ldots, \xi_s) : \mathbb{C}^n \longrightarrow \mathbb{C}^s$ be the projection onto the first s variables. As Z is closed in $\Omega \smallsetminus A$, $Z \cup A$ is closed in Ω. Moreover, our construction gives $(Z \cup A) \cap \xi^{-1}(0) = \big(Z \cap \xi^{-1}(0)\big) \cup \{0\}$ and the case $j = s$ of (8.9) shows that $Z \cap \xi^{-1}(0)$ is a locally finite sequence in $\Omega \cap (\{0\} \times \mathbb{C}^{n-s}) \smallsetminus \{0\}$. Therefore, we can find a small ball $\overline{B}''$ of center 0 in $\mathbb{C}^{n-s}$ such that $Z \cap (\{0\} \times \partial B'') = \emptyset$. As $\{0\} \times \partial B''$ is compact and disjoint from the closed set $Z \cup A$, there is a small ball B' of center 0 in $\mathbb{C}^s$ such that $(Z \cup A) \cap (\overline{B}' \times \partial B'') = \emptyset$. This implies that the projection $\xi : (Z \cup A) \cap (B' \times B'') \longrightarrow B'$ is proper. Set $A' = B' \cap (\mathbb{C}^p \times \{0\})$. Then the restriction

Fig. 3 Projection $\pi : Z \cap ((B' \smallsetminus A') \times B'') \longrightarrow B' \smallsetminus A'$.

$$\pi = \xi : Z \cap (B' \times B'') \smallsetminus (A' \times B'') \longrightarrow B' \smallsetminus A'$$

is proper, and $Z \cap (B' \times B'')$ is analytic in $(B' \times B'') \smallsetminus A$, so π has finite fibers by Th. 5.9. By definition of the rank we have $\overline{\rho}(\pi) = s$. Let $S_1 = Z_{\mathrm{sing}} \cap \pi^{-1}(B' \smallsetminus A')$ and $S'_1 = \pi(S_1)$; further, let S_2 be the set of points $x \in Z \cap \pi^{-1}\big(B' \smallsetminus (A' \cup S'_1)\big) \subset Z_{\mathrm{reg}}$ such that $d\pi_x$ has rank $< s$ and $S'_2 = \pi(S_2)$. We have $\dim S_j \leq s - 1 \leq m - 1$. Hypothesis $(8.8)_{m-1}$ implies that S'_1 is analytic in $B' \smallsetminus A'$ and that S'_2 is analytic in $B' \smallsetminus (A' \cup S'_1)$. By Remark 4.2, $B' \smallsetminus (A' \cup S'_1 \cup S'_2)$ is connected and every bounded holomorphic function on this set extends to B'. As π is a (non ramified) covering over $B' \smallsetminus (A' \cup S'_1 \cup S'_2)$, the sheet number is a constant q.

Let $\lambda(z) = \sum_{j>s} \lambda_j z_j$ be a linear form on $\mathbb{C}^n$ in the coordinates of index $j > s$. For $z' \in B' \smallsetminus (A' \cup S'_1 \cup S'_2)$, we let $\sigma_j(z')$ be the elementary symmetric functions in the q complex numbers $\lambda(z)$ corresponding to $z \in \pi^{-1}(z')$. Then these functions can be extended as bounded holomorphic functions on B' and we get a polynomial $P_\lambda(z' ; T)$ such that $P_\lambda\big(z' ; \lambda(z'')\big)$ vanishes identically on $Z \smallsetminus \pi^{-1}(A' \cup S'_1 \cup S'_2)$. Since π is finite, $Z \cap \pi^{-1}(A' \cup S'_1 \cup S'_2)$ is a union of three (non necessarily closed) analytic subsets of dimension $\leq s - 1$, thus has empty interior in Z. It follows that the closure $\overline{Z} \cap (B' \times B'')$ is contained in the analytic set $W \subset B' \times B''$ equal to the common zero set of all functions

$P_\lambda\big(z'\,;\,\lambda(z'')\big)$. Moreover, by construction,

$$Z \smallsetminus \pi^{-1}(A' \cup S_1' \cup S_2') = W \smallsetminus \pi^{-1}(A' \cup S_1' \cup S_2').$$

As in the proof of Cor. 5.4, we easily conclude that $\overline{Z} \cap (B' \times B'')$ is equal to the union of all irreducible components of W that are not contained in $\pi^{-1}(A' \cup S_1' \cup S_2')$. Hence $\overline{Z}$ is analytic. □

Finally, we give two interesting applications of the Remmert-Stein theorem. We assume here that the reader knows what is the complex projective space $\mathbb{P}^n$. For more details, see Sect. 5.15.

(8.10) Chow's theorem (Chow 1949)**.** *Let A be an analytic subset of the complex projective space $\mathbb{P}^n$. Then A is algebraic, i.e. A is the common zero set of finitely many homogeneous polynomials $P_j(z_0, \ldots, z_n)$, $1 \le j \le N$.*

Proof. Let $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{P}^n$ be the natural projection and $Z = \pi^{-1}(A)$. Then Z is an analytic subset of $\mathbb{C}^{n+1} \smallsetminus \{0\}$ which is invariant by homotheties and $\dim Z = \dim A + 1 \ge 1$. The Remmert-Stein theorem implies that $\overline{Z} = Z \cup \{0\}$ is an analytic subset of $\mathbb{C}^{n+1}$. Let $f_1, \ldots, f_N$ be holomorphic functions on a small polydisk $\Delta \subset \mathbb{C}^{n+1}$ of center 0 such that $\overline{Z} \cap \Delta = \bigcap f_j^{-1}(0)$. The Taylor series at 0 gives an expansion $f_j = \sum_{k=0}^{+\infty} P_{j,k}$ where $P_{j,k}$ is a homogeneous polynomial of degree k. We claim that $\overline{Z}$ coincides with the common zero W set of the polynomials $P_{j,k}$. In fact, we clearly have $W \cap \Delta \subset \bigcap f_j^{-1}(0) = \overline{Z} \cap \Delta$. Conversely, for $z \in \overline{Z} \cap \Delta$, the invariance of Z by homotheties shows that $f_j(tz) = \sum P_{j,k}(z) t^k$ vanishes for every complex number t of modulus < 1, so all coefficients $P_{j,k}(z)$ vanish and $z \in W \cap \Delta$. By homogeneity $\overline{Z} = W$; since $\mathbb{C}[z_0, \ldots, z_n]$ is Noetherian, W can be defined by finitely many polynomial equations. □

(8.11) E.E. Levi's continuation theorem. *Let X be a normal complex space and A an analytic subset such that* $\dim(A, x) \le \dim(X, x) - 2$ *for all $x \in A$. Then every meromorphic function on $X \smallsetminus A$ has a meromorphic extension to X.*

Proof. We may suppose X irreducible, $\dim X = n$. Let f be a meromorphic function on $X \smallsetminus A$. By Th. 6.13, the pole set P_f has pure dimension $(n-1)$, so the Remmert-Stein theorem implies that $\overline{P}_f$ is analytic in X. Fix a point $x \in A$. There is a connected neighborhood V of x and a non zero holomorphic function $h \in \mathcal{O}_X(V)$ such that $\overline{P}_f \cap V$ has finitely many irreducible components $\overline{P}_{f,j}$ and $\overline{P}_f \cap V \subset h^{-1}(0)$. Select a point x_j in $\overline{P}_{f,j} \smallsetminus (X_{\mathrm{sing}} \cup (\overline{P}_f)_{\mathrm{sing}} \cup A)$. As x_j is a regular point on X and on $\overline{P}_f$, there is a local coordinate $z_{1,j}$ at x_j defining an equation of $\overline{P}_{f,j}$, such that $z_{1,j}^{m_j} f \in \mathcal{O}_{X,x_j}$ for some integer m_j. Since h vanishes along P_f, we have $h^{m_j} f \in \mathcal{O}_{X,x}$. Thus, for $m = \max\{m_j\}$, the pole set P_g of $g = h^m f$ in $V \smallsetminus A$ does not contain x_j. As P_g is $(n-1)$-dimensional and contained in

$P_f \cap V$, it is a union of irreducible components $\overline{P}_{f,j} \smallsetminus A$. Hence P_g must be empty and g is holomorphic on $V \smallsetminus A$. By Cor. 7.8, g has an extension to a holomorphic function $\tilde{g}$ on V. Then $\tilde{g}/h^m$ is the required meromorphic extension of f on V. □

§9. Complex Analytic Schemes

Our goal is to introduce a generalization of the notion of complex space given in Def. 5.2. A complex space is a space locally isomorphic to an analytic set A in an open subset $\Omega \subset \mathbb{C}^n$, together with the sheaf of rings $\mathcal{O}_A = (\mathcal{O}_\Omega/\mathcal{I}_A)_{\restriction A}$. Our desire is to enrich the structure sheaf $\mathcal{O}_A$ by replacing $\mathcal{I}_A$ with a possibly smaller ideal $\mathcal{J}$ defining the same zero variety $V(\mathcal{J}) = A$. In this way holomorphic functions are described not merely by their values on A, but also possibly by some "transversal derivatives" along A.

§9.1. Ringed Spaces

We start by an abstract notion of ringed space on an arbitrary topological space.

(9.1) Definition. *A ringed space is a pair $(X, \mathcal{R}_X)$ consisting of a topological space X and of a sheaf of rings $\mathcal{R}_X$ on X, called the structure sheaf. A morphism*

$$F : (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

of ringed spaces is a pair $(f, F^\star)$ where $f : X \longrightarrow Y$ is a continuous map and

$$F^\star \ :\ f^{-1}\mathcal{R}_Y \longrightarrow \mathcal{R}_X, \qquad F^\star_x \ :\ (\mathcal{R}_Y)_{f(x)} \longrightarrow (\mathcal{R}_X)_x$$

a homomorphism of sheaves of rings on X, called the comorphism of F.

If $F : (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$ and $G : (Y, \mathcal{R}_Y) \longrightarrow (Z, \mathcal{R}_Z)$ are morphisms of ringed spaces, the composite $G \circ F$ is the pair consisting of the map $g \circ f : X \longrightarrow Z$ and of the comorphism $(G \circ F)^\star = F^\star \circ f^{-1}G^\star$:

$$\begin{aligned} F^\star \circ f^{-1}G^\star \ &:\ f^{-1}g^{-1}\mathcal{R}_Z \xrightarrow{f^{-1}G^\star} f^{-1}\mathcal{R}_Y \xrightarrow{F^\star} \mathcal{R}_X, \\ F^\star_x \circ G^\star_{f(x)} \ &:\ (\mathcal{R}_Z)_{g\circ f(x)} \longrightarrow (\mathcal{R}_Y)_{f(x)} \longrightarrow (\mathcal{R}_X)_x. \end{aligned} \tag{9.2}$$

§9.2. Definition of Complex Analytic Schemes

We begin by a description of what will be the local model of an analytic scheme. Let $\Omega \subset \mathbb{C}^n$ be an open subset, $\mathcal{J} \subset \mathcal{O}_\Omega$ a coherent sheaf of ideals and $A = V(\mathcal{J})$ the analytic set in Ω defined locally as the zero set of a system of generators of $\mathcal{J}$. By Hilbert's Nullstellensatz 4.22 we have $\mathcal{I}_A = \sqrt{\mathcal{J}}$, but $\mathcal{I}_A$ differs in general from $\mathcal{J}$. The sheaf of rings $\mathcal{O}_\Omega/\mathcal{J}$ is supported on A, i.e. $(\mathcal{O}_\Omega/\mathcal{J})_x = 0$ if $x \notin A$. Ringed spaces of the type $(A, \mathcal{O}_\Omega/\mathcal{J})$ will be used as the local models of analytic schemes.

(9.3) Definition. *A morphism*

$$F = (f, F^\star) : (A, \mathcal{O}_\Omega/\mathcal{J}_{\restriction A}) \longrightarrow (A', \mathcal{O}_{\Omega'}/\mathcal{J}'_{\restriction A'})$$

is said to be analytic if for every point $x \in A$ there exists a neighborhood W_x of x in Ω and a holomorphic function $\Phi : W_x \longrightarrow \Omega'$ such that $f_{\restriction A\cap W_x} = \Phi_{\restriction A\cap W_x}$ and such that the comorphism

$$F_x^\star : (\mathcal{O}_{\Omega'}/\mathcal{J}')_{f(x)} \longrightarrow (\mathcal{O}_\Omega/\mathcal{J})_x$$

is induced by $\Phi^\star : \mathcal{O}_{\Omega', f(x)} \ni u \longmapsto u \circ \Phi \in \mathcal{O}_{\Omega,x}$ with $\Phi^\star \mathcal{J}' \subset \mathcal{J}$.

(9.4) Example. Take $\Omega = \mathbb{C}^n$ and $\mathcal{J} = (z_n^2)$. Then A is the hyperplane $\mathbb{C}^{n-1} \times \{0\}$, and the sheaf $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$ can be identified with the sheaf of rings of functions $u + z_n u'$, $u, u' \in \mathcal{O}_{\mathbb{C}^{n-1}}$, with the relation $z_n^2 = 0$. In particular, z_n is a nilpotent element of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$. A morphism F of $(A, \mathcal{O}_{\mathbb{C}^n}/\mathcal{J})$ into itself is induced (at least locally) by a holomorphic map $\Phi = (\widetilde{\Phi}, \Phi_n)$ defined on a neighborhood of A in $\mathbb{C}^n$ with values in $\mathbb{C}^n$, such that $\Phi(A) \subset A$, i.e. $\Phi_{n\restriction A} = 0$. We see that F is completely determined by the data

$$\begin{aligned} f(z_1, \ldots, z_{n-1}) &= \widetilde{\Phi}(z_1, \ldots, z_{n-1}, 0), & f &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}, \\ f'(z_1, \ldots, z_{n-1}) &= \frac{\partial \Phi}{\partial z_n}(z_1, \ldots, z_{n-1}, 0), & f' &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^n, \end{aligned}$$

which can be chosen arbitrarily.

(9.5) Definition. *A complex analytic scheme is a ringed space $(X, \mathcal{O}_X)$ over a separable Hausdorff topological space X, satisfying the following property: there exist an open covering (U_λ) of X and isomorphisms of ringed spaces*

$$G_\lambda : (U_\lambda, \mathcal{O}_{X\restriction U_\lambda}) \longrightarrow (A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$$

where A_λ is the zero set of a coherent sheaf of ideals $\mathcal{J}_\lambda$ on an open subset $\Omega_\lambda \subset \mathbb{C}^{N_\lambda}$, such that every transition morphism $G_\lambda \circ G_\mu^{-1}$ is a holomorphic isomorphism from $g_\mu(U_\lambda \cap U_\mu) \subset A_\mu$ onto $g_\lambda(U_\lambda \cap U_\mu) \subset A_\lambda$, equipped with the respective structure sheaves $\mathcal{O}_{\Omega_\mu}/\mathcal{J}_{\mu\restriction A_\mu}$, $\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda}$.

We shall often consider the maps G_λ as identifications and write simply $U_\lambda = A_\lambda$. A morphism $F : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of analytic schemes obtained by gluing patches $(A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$ and $(A'_\mu, \mathcal{O}_{\Omega'_\mu}/\mathcal{J}'_{\mu\,A'_\mu})$, respectively, is a morphism F of ringed spaces such that for each pair (λ, μ), the restriction of F from $A_\lambda \cap f^{-1}(A'_\mu) \subset X$ to $A'_\mu \subset Y$ is holomorphic in the sense of Def. 9.3.

§9.3. Nilpotent Elements and Reduced Schemes

Let $(X, \mathcal{O}_X)$ be an analytic scheme. The set of *nilpotent elements* is the sheaf of ideals of $\mathcal{O}_X$ defined by

$$(9.6)\quad \mathcal{N}_X = \{u \in \mathcal{O}_X\,;\ u^k = 0 \ \text{ for some } \ k \in \mathbb{N}\}.$$

Locally, we have $\mathcal{O}_{X\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda}$, thus

$$(9.7)\quad \mathcal{N}_{X\restriction A_\lambda} = (\sqrt{\mathcal{J}_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda},$$

$$(9.8)\quad (\mathcal{O}_X/\mathcal{N}_X)_{\restriction A_\lambda} \simeq (\mathcal{O}_{\Omega_\lambda}/\sqrt{\mathcal{J}_\lambda})_{\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{I}_{A_\lambda})_{\restriction A_\lambda} = \mathcal{O}_{A_\lambda}.$$

The scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if $\mathcal{N}_X = 0$. The associated ringed space $(X, \mathcal{O}_X/\mathcal{N}_X)$ is reduced by construction; it is called the *reduced scheme* of $(X, \mathcal{O}_X)$. We shall often denote the original scheme by the letter X merely, the associated reduced scheme by X_{red}, and let $\mathcal{O}_{X,\mathrm{red}} = \mathcal{O}_X/\mathcal{N}_X$. There is a canonical morphism $X_{\mathrm{red}} \to X$ whose comorphism is the reduction morphism

$$(9.9)\quad \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\mathrm{red}}(U) = (\mathcal{O}_X/\mathcal{N}_X)(U), \qquad \forall U \text{ open set in } X.$$

By (9.8), the notion of reduced scheme is equivalent to the notion of complex space introduced in Def. 5.2. It is easy to see that a morphism F of reduced schemes X, Y is completely determined by the set-theoretic map $f : X \longrightarrow Y$.

§9.4. Coherent Sheaves on Analytic Schemes

If $(X, \mathcal{O}_X)$ is an analytic scheme, a sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules is said to be *coherent* if it satisfies the same properties as those already stated when X is a manifold:

(9.10) $\mathcal{S}$ is locally finitely generated over $\mathcal{O}_X$;

(9.10′) for any open set $U \subset X$ and any sections $G_1, \ldots, G_q \in \mathcal{S}(U)$, the relation sheaf $\mathcal{R}(G_1, \ldots, G_q) \subset \mathcal{O}^{\oplus q}_{X\restriction U}$ is locally finitely generated.

Locally, we have $\mathcal{O}_{X\restriction A_\lambda} = \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda$, so if $i_\lambda : A_\lambda \to \Omega_\lambda$ is the injection, the direct image $\mathcal{S}_\lambda = (i_\lambda)_\star(\mathcal{S}_{\restriction A_\lambda})$ is a module over $\mathcal{O}_{\Omega_\lambda}$ such that $\mathcal{J}_\lambda.\mathcal{S}_\lambda = 0$. It is clear that $\mathcal{S}_{\restriction\Omega_\lambda}$ is coherent if and only if $\mathcal{S}_\lambda$ is coherent as a module over $\mathcal{O}_{\Omega_\lambda}$. It follows immediately that the Oka theorem and its consequences 3.16–20 are still valid over analytic schemes.

§9.5. Subschemes

Let X be an analytic scheme and $\mathcal{J}$ a coherent sheaf of ideals in $\mathcal{O}_X$. The image of $\mathcal{J}$ in $\mathcal{O}_{X,\mathrm{red}}$ is a coherent sheaf of ideals, and its zero set Y is clearly an analytic subset of X_{red}. We can make Y into a scheme by introducing the structure sheaf

$$(9.11)\qquad \mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})_{\restriction Y},$$

and we have a scheme morphism $F : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ such that f is the inclusion and $F^\star : f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ the obvious map of $\mathcal{O}_{X\restriction Y}$ onto its quotient $\mathcal{O}_Y$. The scheme $(Y, \mathcal{O}_Y)$ will be denoted $V(\mathcal{J})$. When the analytic set Y is given, the structure sheaf of $V(\mathcal{J})$ depends of course on the choice of the equations of Y in the ideal $\mathcal{J}$; in general $\mathcal{O}_Y$ has nilpotent elements.

§9.6. Inverse Images of Coherent Sheaves

Let $F : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of analytic schemes and $\mathcal{S}$ a coherent sheaf over Y. The sheaf theoretic inverse image $f^{-1}\mathcal{S}$, whose stalks are $(f^{-1}\mathcal{S})_x = \mathcal{S}_{f(x)}$, is a sheaf of modules over $f^{-1}\mathcal{O}_Y$. We define the *analytic inverse image* $F^\star\mathcal{S}$ by

$$(9.12)\qquad F^\star\mathcal{S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{S}, \qquad (F^\star\mathcal{S})_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{S}_{f(x)}.$$

Here the tensor product is taken with respect to the comorphism $F^\star : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, which yields a ring morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. If $\mathcal{S}$ is given over $U \subset Y$ by a local presentation

$$\mathcal{O}^{\oplus p}_{Y\restriction U} \xrightarrow{\ A\ } \mathcal{O}^{\oplus q}_{Y\restriction U} \longrightarrow \mathcal{S}_{\restriction U} \longrightarrow 0$$

where A is a $(q \times p)$-matrix with coefficients in $\mathcal{O}_Y(U)$, our definition shows that $F^\star\mathcal{S}$ is a coherent sheaf over $\mathcal{O}_X$, given over $f^{-1}(U)$ by the local presentation

$$(9.13)\qquad \mathcal{O}^{\oplus p}_{X\restriction f^{-1}(U)} \xrightarrow{\ F^\star A\ } \mathcal{O}^{\oplus q}_{X\restriction f^{-1}(U)} \longrightarrow F^\star\mathcal{S}_{\restriction f^{-1}(U)} \longrightarrow 0.$$

§9.7. Products of Analytic Schemes

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be analytic schemes, and let $(A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda)$, $(B_\mu, \mathcal{O}_{\Omega'_\mu}/\mathcal{J}'_\mu)$ be local models of X, Y, respectively. The *product scheme* $(X \times Y, \mathcal{O}_{X\times Y})$ is obtained by gluing the open patches

$$(9.14)\qquad \Big(A_\lambda \times B_\mu\ ,\ \mathcal{O}_{\Omega_\lambda\times\Omega'_\mu}\big/\big(\mathrm{pr}_1^{-1}\mathcal{J}_\lambda + \mathrm{pr}_2^{-1}\mathcal{J}'_\mu\big)\mathcal{O}_{\Omega_\lambda\times\Omega'_\mu}\Big).$$

In other words, if A_λ, B_μ are the subschemes of Ω_λ, Ω'_μ defined by the equations $g_{\lambda,j}(x) = 0$, $g'_{\mu,k}(y) = 0$, where $(g_{\lambda,j})$ and $(g'_{\mu,k})$ are generators of

$\mathcal{J}_\lambda$ and $\mathcal{J}'_\mu$ respectively, then $A_\lambda \times B_\mu$ is equipped with the structure sheaf $\mathcal{O}_{\Omega_\lambda \times \Omega'_\mu} / \big(g_{\lambda,j}(x), g'_{\mu,k}(y)\big)$.

Now, let $\mathcal{S}$ be a coherent sheaf over $\mathcal{O}_X$ and let $\mathcal{S}'$ be a coherent sheaf over $\mathcal{O}_Y$. The (analytic) *external tensor product* $\mathcal{S} \boxtimes \mathcal{S}'$ is defined to be

$$(9.15) \quad \mathcal{S} \boxtimes \mathcal{S}' = \mathrm{pr}_1^\star \mathcal{S} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^\star \mathcal{S}'.$$

If we go back to the definition of the inverse image, we see that the stalks of $\mathcal{S} \boxtimes \mathcal{S}'$ are given by

$$(9.15') \quad (\mathcal{S} \boxtimes \mathcal{S}')_{(x,y)} = \mathcal{O}_{X\times Y,(x,y)} \otimes_{\mathcal{O}_{X,x}\otimes \mathcal{O}_{Y,y}} (\mathcal{S}_x \otimes_{\mathbb{C}} \mathcal{S}'_y)\ ,$$

in particular $(\mathcal{S} \boxtimes \mathcal{S}')_{(x,y)}$ *does not coincide* with the sheaf theoretic tensor product $\mathcal{S}_x \otimes \mathcal{S}'_y$ which is merely a module over $\mathcal{O}_{X,x} \otimes \mathcal{O}_{Y,y}$. If $\mathcal{S}$ and $\mathcal{S}'$ are given by local presentations

$$\mathcal{O}_{X\restriction U}^{\oplus p} \xrightarrow{A} \mathcal{O}_{X\restriction U}^{\oplus q} \longrightarrow \mathcal{S}_{\restriction U} \longrightarrow 0, \qquad \mathcal{O}_{Y\restriction U'}^{p'} \xrightarrow{B} \mathcal{O}_{Y\restriction U'}^{q'} \longrightarrow \mathcal{S}'_{\restriction U'} \longrightarrow 0,$$

then $\mathcal{S} \boxtimes \mathcal{S}'$ is the coherent sheaf given by

$$\mathcal{O}_{X\times Y\restriction U\times U'}^{pq' \oplus qp'} \xrightarrow{(A(x)\otimes \mathrm{Id}, \mathrm{Id}\otimes B(y))} \mathcal{O}_{X\times Y\restriction U\times U'}^{qq'} \longrightarrow (\mathcal{S} \boxtimes \mathcal{S}')_{\restriction U\times U'} \longrightarrow 0.$$

§9.8. Zariski Embedding Dimension

If x is a point of an analytic scheme $(X, \mathcal{O}_X)$, the *Zariski embedding dimension* of the germ (X,x) is the smallest integer N such that (X,x) can be embedded in $\mathbb{C}^N$, i.e. such that there exists a patch of X near x isomorphic to $(A, \mathcal{O}_\Omega/\mathcal{J})$ where Ω is an open subset of $\mathbb{C}^N$. This dimension is denoted

$$(9.16) \quad \mathrm{embdim}(X,x) = \text{ smallest such } N.$$

Consider the maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ of functions which vanish at point x. If (X,x) is embedded in $(\Omega,x) = (\mathbb{C}^N, 0)$, then $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is generated by $z_1, \ldots, z_N$, so $d = \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \le N$. Let $s_1, \ldots, s_d$ be germs in $\mathfrak{m}_{\Omega,x}$ which yield a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \simeq \mathfrak{m}_{\Omega,x}/(\mathfrak{m}_{\Omega,x}^2 + \mathcal{J}_x)$. We can write

$$z_j = \sum_{1\le k\le d} c_{jk} s_k + u_j + f_j, \quad c_{jk} \in \mathbb{C}, \quad u_j \in \mathfrak{m}_{\Omega,x}^2, \quad f_j \in \mathcal{J}_x, \quad 1 \le j \le n.$$

Then we find $dz_j = \sum c_{jk}\, ds_k(x) + df_j(x)$, so that the rank of the system of differentials $\big(df_j(x)\big)$ is at least $N-d$. Assume for example that $df_1(x), \ldots, df_{N-d}(x)$ are linearly independant . By the implicit function theorem, the equations $f_1 = \ldots = f_{N-d} = 0$ define a germ of smooth subvariety $(Z,x) \subset (\Omega,x)$ of dimension d which contains (X,x). We have $\mathcal{O}_Z = \mathcal{O}_\Omega/(f_1, \ldots, f_{N-d})$ in a neighborhood of x, thus

$$\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{J} \simeq \mathcal{O}_Z/\mathcal{J}' \quad \text{ where } \mathcal{J}' = \mathcal{J}/(f_1, \ldots, f_{N-d}).$$

This shows that (X, x) can be imbedded in $\mathbb{C}^d$, and we get

(9.17) $\quad \operatorname{embdim}(X, x) = \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2.$

(9.18) Remark. For a given dimension $n = \dim(X, x)$, the embedding dimension d can be arbitrarily large. Consider for example the curve $\Gamma \subset \mathbb{C}^N$ parametrized by $\mathbb{C} \ni t \longmapsto (t^N, t^{N+1}, \ldots, t^{2N-1})$. Then $\mathcal{O}_{\Gamma,0}$ is the ring of convergent series in $\mathbb{C}\{t\}$ which have no terms $t, t^2, \ldots, t^{N-1}$, and $\mathfrak{m}_{\Gamma,0}/\mathfrak{m}_{\Gamma,0}^2$ admits precisely $z_1 = t^N, \ldots, z_N = t^{2N-1}$ as a basis. Therefore $n = 1$ but $d = N$ can be as large as we want.

§10. Bimeromorphic maps, Modifications and Blow-ups

It is a very frequent situation in analytic or algebraic geometry that two complex spaces have isomorphic dense open subsets but are nevertheless different along some analytic subset. These ideas are made precise by the notions of modification and bimeromorphic map. This will also lead us to generalize the notion of meromorphic function to maps between analytic schemes. If $(X, \mathcal{O}_X)$ is an analytic scheme, $\mathcal{M}_X$ denotes the sheaf of meromorphic functions on X, defined at the beginning of § 6.2.

(10.1) Definition. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be analytic schemes. An analytic morphism $F : X \to Y$ is said to be a modification if F is proper and if there exists a nowhere dense closed analytic subset $B \subset Y$ such that the restriction $F : X \smallsetminus F^{-1}(B) \to Y \smallsetminus B$ is an isomorphism.*

(10.2) Definition. *If $F : X \to Y$ is a modification, then the comorphism $F^\star : f^\star\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphism $F^\star : f^\star\mathcal{M}_Y \to \mathcal{M}_X$ for the sheaves of meromorphic functions on X and Y.*

Proof. Let $v = g/h$ be a section of $\mathcal{M}_Y$ on a small open set Ω where u is actually given as a quotient of functions $g, h \in \mathcal{O}_Y(\Omega)$. Then $F^\star u = (g \circ F)/(h \circ F)$ is a section of $\mathcal{M}_X$ on $F^{-1}(\Omega)$, for $h \circ F$ cannot vanish identically on any open subset W of $F^{-1}(\Omega)$ (otherwise h would vanish on the open subset $F(W \smallsetminus F^{-1}(B))$ of $\Omega \smallsetminus B$). Thus the extension of the comorphism to sheaves of meromorphic functions is well defined. Our claim is that this is an isomorphism. The injectivity of $F^\star$ is clear: $F^\star u = 0$ implies $g \circ F = 0$, which implies $g = 0$ on $\Omega \smallsetminus B$ and thus $g = 0$ on Ω because B is nowhere dense. In order to prove surjectivity, we need only show that every section $u \in \mathcal{O}_X(F^{-1}(\Omega))$ is in the image of $\mathcal{M}_Y(\Omega)$ by $F^\star$. For this, we may shrink Ω into a relatively compact subset $\Omega' \subset\subset \Omega$ and thus assume that u is bounded (here we use the properness of F through the fact that $F^{-1}(\Omega')$ is relatively compact in $F^{-1}(\Omega)$). Then $v = u \circ F^{-1}$ defines a bounded holomorphic function on $\Omega \smallsetminus B$. By Th. 7.2, it follows that v is weakly holomorphic for the reduced structure of Y. Our claim now follows from the following Lemma. $\square$

(10.3) Lemma. *If $(X, \mathcal{O}_X)$ is an analytic scheme, then every holomorphic function v in the complement of a nowhere dense analytic subset $B \subset Y$ which is weakly holomorphic on X_{red} is meromorphic on X.*

Proof. It is enough to argue with the germ of v at any point $x \in Y$, and thus we may suppose that $(Y, \mathcal{O}_Y) = (A, \mathcal{O}_\Omega/\mathcal{I})$ is embedded in $\mathbb{C}^N$. Because v is weakly holomorphic, we can write $v = g/h$ in Y_{red}, for some germs of holomorphic functions g, h. Let $\widetilde{g}$ and $\widetilde{h}$ be extensions of g, h to $\mathcal{O}_{\Omega,x}$. Then there is a neighborhood U of x such that $\widetilde{g} - v\widetilde{h}$ is a nilpotent section of $cO_\Omega(U \smallsetminus B)$ which is in $\mathcal{I}$ on

(10.4) Definition. *A meromorphic map $F : X \dashrightarrow Y$ is a scheme morphism $F : X \smallsetminus A \to Y$ defined in the complement of a nowhere dense analytic subset $A \subset X$, such that the closure of the graph of F in $X \times Y$ is an analytic subset (for the reduced complex space structure of $X \times Y$).*

§11. Exercises

11.1. Let $\mathcal{A}$ be a sheaf on a topological space X. If the sheaf space $\widetilde{\mathcal{A}}$ is Hausdorff, show that $\mathcal{A}$ satisfies the following *unique continuation principle*: any two sections $s, s' \in \mathcal{A}(U)$ on a connected open set U which coincide on some non empty open subset $V \subset U$ must coincide identically on U. Show that the converse holds if X is Hausdorff and locally connected.

11.2. Let $\mathcal{A}$ be a sheaf of abelian groups on X and let $s \in \mathcal{A}(X)$. The support of s, denoted Supp s, is defined to be $\{x \in X\,;\, s(x) \neq 0\}$. Show that Supp s is a closed subset of X. The support of $\mathcal{A}$ is defined to be Supp $\mathcal{A} = \{x \in X\,;\, \mathcal{A}_x \neq 0\}$. Show that Supp $\mathcal{A}$ is not necessarily closed: if Ω is an open set in X, consider the sheaf $\mathcal{A}$ such that $\mathcal{A}(U)$ is the set of continuous functions $f \in \mathcal{C}(U)$ which vanish on a neighborhood of $U \cap (X \smallsetminus \Omega)$.

11.3. Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules. We define a presheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ such that $\mathcal{H}(U)$ is the module of all sheaf-homomorphisms $\mathcal{F}_{\restriction U} \to \mathcal{G}_{\restriction U}$ which are $\mathcal{A}$-linear.

a) Show that $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is a sheaf and that there is a canonical homomorphism $\varphi_x : \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \longrightarrow \hom_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ for every $x \in X$.

b) If $\mathcal{F}$ is locally finitely generated, then φ_x is injective, and if $\mathcal{F}$ has local finite presentations as in (3.12), then φ_x is bijective.

c) Suppose that $\mathcal{A}$ is a coherent sheaf of rings and that $\mathcal{F}$, $\mathcal{G}$ are coherent modules over $\mathcal{A}$. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{A}$-module.

Hint: observe that the result is true if $\mathcal{F} = \mathcal{A}^{\oplus p}$ and use a local presentation of $\mathcal{F}$ to get the conclusion.

11.4. Let $f : X \to Y$ be a continuous map of topological spaces. Given sheaves of abelian groups $\mathcal{A}$ on X and $\mathcal{B}$ on Y, show that there is a natural isomorphism

$$\hom_X(f^{-1}\mathcal{B}, \mathcal{A}) = \hom_Y(\mathcal{B}, f_\star\mathcal{A}).$$

Hint: use the natural morphisms (2.17).

11.5. Show that the sheaf of polynomials over $\mathbb{C}^n$ is a coherent sheaf of rings (with either the ordinary topology or the Zariski topology on $\mathbb{C}^n$). Extend this result to the case of regular algebraic functions on an algebraic variety.
Hint: check that the proof of the Oka theorem still applies.

11.6. Let P be a non zero polynomial on $\mathbb{C}^n$. If P is irreducible in $\mathbb{C}[z_1, \ldots, z_n]$, show that the hypersurface $H = P^{-1}(0)$ is globally irreducible as an analytic set. In general, show that the irreducible components of H are in a one-to-one correspondence with the irreducible factors of P.
Hint: for the first part, take coordinates such that $P(0, \ldots, 0, z_n)$ has degree equal to P; if H splits in two components H_1, H_2, then P can be written as a product P_1P_2 where the roots of $P_j(z', z_n)$ correspond to points in H_j.

11.7. Prove the following facts:

a) For every algebraic variety A of pure dimension p in $\mathbb{C}^n$, there are coordinates $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ such that $\pi : A \to \mathbb{C}^p$, $z \mapsto z''$ is proper with finite fibers, and such that A is entirely contained in a cone

$$|z''| \le C(|z'| + 1).$$

Hint: imitate the proof of Cor. 4.11.

b) Conversely if an analytic set A of pure dimension p in $\mathbb{C}^n$ is contained in a cone $|z''| \le C(|z'| + 1)$, then A is algebraic.
Hint: first apply (5.9) to conclude that the projection $\pi : A \to \mathbb{C}^p$ is finite. Then repeat the arguments used in the final part of the proof of Th. 4.19.

c) Deduce from a), b) that an algebraic set in $\mathbb{C}^n$ is irreducible if and only if it is irreducible as an analytic set.

11.8. Let $\Gamma : f(x, y) = 0$ be a germ of analytic curve in $\mathbb{C}^2$ through $(0, 0)$ and let $(\Gamma_j, 0)$ be the irreducible components of $(\Gamma, 0)$. Suppose that $f(0, y) \not\equiv 0$. Show that the roots y of $f(x, y) = 0$ corresponding to points of Γ near 0 are given by *Puiseux expansions* of the form $y = g_j(x^{1/q_j})$, where $g_j \in \mathcal{O}_{\mathbb{C},0}$ and where q_j is the sheet number of the projection $\Gamma_j \to \mathbb{C}$, $(x, y) \mapsto x$.
Hint: special case of the parametrization obtained in (4.27).

11.9. The goal of this exercise is to prove the existence and the analyticity of the *tangent cone* to an arbitrary analytic germ $(A, 0)$ in $\mathbb{C}^n$. Suppose that A is defined by holomorphic equations $f_1 = \ldots = f_N = 0$ in a ball $\Omega = B(0, r)$. Then the (set theoretic) tangent cone to A at 0 is the set $C(A, 0)$ of all limits of sequences $t_\nu^{-1} z_\nu$ with $z_\nu \in A$ and $\mathbb{C}^\star \ni t_\nu$ converging to 0.

a) Let E be the set of points $(z, t) \in \Omega \times \mathbb{C}^\star$ such that $z \in t^{-1}A$. Show that the closure $\overline{E}$ in $\Omega \times \mathbb{C}$ is analytic.

Hint: observe that $E = A \setminus (\Omega \times \{0\})$ where $A = \{f_j(tz) = 0\}$ and apply Cor. 5.4.

b) Show that $C(A,0)$ is a conic set and that $\overline{E} \cap (\Omega \times \{0\}) = C(A,0) \times \{0\}$ and conclude. Infer from this that $C(A,0)$ is an algebraic subset of $\mathbb{C}^n$.

11.10. Give a new proof of Theorem 5.5 based on the coherence of ideal sheaves and on the strong noetherian property.

11.11. Let X be an analytic space and let A, B be analytic subsets of pure dimensions. Show that $\operatorname{codim}_X(A \cap B) \le \operatorname{codim}_X A + \operatorname{codim}_X B$ if A or B is a local complete intersection, but that the equality does not necessarily hold in general.
Hint: see Remark (6.5).

11.12. Let Γ be the curve in $\mathbb{C}^3$ parametrized by $\mathbb{C} \ni t \longmapsto (x,y,z) = (t^3, t^4, t^5)$. Show that the ideal sheaf $\mathfrak{I}_\Gamma$ is generated by the polynomials $xz - y^2$, $x^3 - yz$ and $x^2y - z^2$, and that the germ $(\Gamma, 0)$ is not a (sheaf theoretic) complete intersection.
Hint: Γ is smooth except at the origin. Let $f(x,y,z) = \sum a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ be a convergent power series near 0. Show that $f \in \mathfrak{I}_{\Gamma,0}$ if and only if all weighted homogeneous components $f_k = \sum_{3\alpha+4\beta+5\gamma=k} a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ are in $\mathfrak{I}_{\Gamma,0}$. By means of suitable substitutions, reduce the proof to the case when $f = f_k$ is homogeneous with all non zero monomials satisfying $\alpha \le 2$, $\beta \le 1$, $\gamma \le 1$; then check that there is at most one such monomial in each weighted degree ≤ 15 the product of a power of x by a homogeneous polynomial of weighted degree ≤ 8.