Chapter IV
Sheaf Cohomology and Spectral Sequences

One of the main topics of this book is the computation of various cohomology groups arising in algebraic geometry. The theory of sheaves provides a general framework in which many cohomology theories can be treated in a unified way. The cohomology theory of sheaves will be constructed here by means of Godement's simplicial flabby resolution. However, we have emphasized the analogy with Alexander-Spanier cochains in order to give a simple definition of the cup product. In this way, all the basic properties of cohomology groups (long exact sequences, Mayer Vietoris exact sequence, Leray's theorem, relations with Cech cohomology, De Rham-Weil isomorphism theorem) can be derived in a very elementary way from the definitions. Spectral sequences and hypercohomology groups are then introduced, with two principal examples in view: the Leray spectral sequence and the Hodge-Frölicher spectral sequence. The basic results concerning cohomology groups with constant or locally constant coefficients (invariance by homotopy, Poincaré duality, Leray-Hirsch theorem) are also included, in order to present a self-contained approach of algebraic topology.

1. Basic Results of Homological Algebra

Let us first recall briefly some standard notations and results of homological algebra that will be used systematically in the sequel. Let R be a commutative ring with unit. A *differential module* (K, d) is a R-module K together with an endomorphism $d : K \to K$, called the *differential*, such that $d \circ d = 0$. The modules of *cycles* and of *boundaries* of K are defined respectively by

(1.1) $\quad Z(K) = \ker d, \qquad B(K) = \operatorname{Im} d.$

Our hypothesis $d \circ d = 0$ implies $B(K) \subset Z(K)$. The *homology group* of K is by definition the quotient module

(1.2) $\quad H(K) = Z(K)/B(K).$

A *morphism of differential modules* $\varphi : K \longrightarrow L$ is a R-homomorphism $\varphi : K \longrightarrow L$ such that $d \circ \varphi = \varphi \circ d$; here we denote by the same symbol d the differentials of K and L. It is then clear that $\varphi\big(Z(K)\big) \subset Z(L)$ and $\varphi\big(B(K)\big) \subset B(L)$. Therefore, we get an induced morphism on homology groups, denoted

(1.3) $\quad H(\varphi) : H(K) \longrightarrow H(L).$

It is easily seen that H is a functor, i.e. $H(\psi \circ \varphi) = H(\psi) \circ H(\varphi)$. We say that two morphisms $\varphi, \psi : K \longrightarrow L$ are *homotopic* if there exists a R-linear map $h : K \longrightarrow L$ such that

(1.4) $\quad d \circ h + h \circ d = \psi - \varphi.$

Then h is said to be a *homotopy* between φ and ψ. For every cocycle $z \in Z(K)$, we infer $\psi(z) - \varphi(z) = dh(z)$, hence the maps $H(\varphi)$ and $H(\psi)$ coincide. The module K itself is said to be homotopic to 0 if Id_K is homotopic to 0; then $H(K) = 0$.

(1.5) Snake lemma. *Let*

$$0 \longrightarrow K \xrightarrow{\ \varphi\ } L \xrightarrow{\ \psi\ } M \longrightarrow 0$$

be a short exact sequence of morphisms of differential modules. Then there exists a homomorphism $\partial : H(M) \longrightarrow H(K)$, called the connecting homomorphism, and a homology exact sequence

$$\begin{tikzcd} H(K) \arrow[r, "H(\varphi)"] & H(L) \arrow[r, "H(\psi)"] & H(M) \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

Moreover, to any commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{K} \arrow[r] & \widetilde{L} \arrow[r] & \widetilde{M} \arrow[r] & 0 \end{tikzcd}$$

is associated a commutative diagram of homology exact sequences

$$\begin{tikzcd} H(K) \arrow[r] \arrow[d] & H(L) \arrow[r] \arrow[d] & H(M) \arrow[r, "\partial"] \arrow[d] & H(K) \arrow[r] \arrow[d] & \cdots \\ H(\widetilde{K}) \arrow[r] & H(\widetilde{L}) \arrow[r] & H(\widetilde{M}) \arrow[r, "\partial"] & H(\widetilde{K}) \arrow[r] & \cdots. \end{tikzcd}$$

Proof. We first define the connecting homomorphism ∂ : let $m \in Z(M)$ represent a given cohomology class $\{m\}$ in $H(M)$. Then

$$\partial\{m\} = \{k\} \in H(K)$$

is the class of any element $k \in \varphi^{-1} d \psi^{-1}(m)$, as obtained through the following construction:

$$\begin{tikzcd} & l \in L \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "d"] & m \in M \arrow[d, mapsto, "d"] \\ k \in K \arrow[r, mapsto, "\varphi"] & dl \in L \arrow[r, mapsto, "\psi"] & 0 \in M. \end{tikzcd}$$

The element l is chosen to be a preimage of m by the surjective map ψ ; as $\psi(dl) = d(m) = 0$, there exists a unique element $k \in K$ such that $\varphi(k) = dl$. The element k is actually a cocycle in $Z(K)$ because φ is injective and

$$\varphi(dk) = d\varphi(k) = d(dl) = 0 \implies dk = 0.$$

The map ∂ will be well defined if we show that the cohomology class $\{k\}$ depends only on $\{m\}$ and not on the choices made for the representatives m and l. Consider another representative $m' = m + dm_1$. Let $l_1 \in L$ such that $\psi(l_1) = m_1$. Then l has to be replaced by an element $l' \in L$ such that

$$\psi(l') = m + dm_1 = \psi(l + dl_1).$$

It follows that $l' = l + dl_1 + \varphi(k_1)$ for some $k_1 \in K$, hence

$$dl' = dl + d\varphi(k_1) = \varphi(k) + \varphi(dk_1) = \varphi(k'),$$

therefore $k' = k + dk_1$ and k' has the same cohomology class as k.

Now, let us show that $\ker\partial = \operatorname{Im} H(\psi)$. If $\{m\}$ is in the image of $H(\psi)$, we can take $m = \psi(l)$ with $dl = 0$, thus $\partial\{m\} = 0$. Conversely, if $\partial\{m\} = \{k\} = 0$, we have $k = dk_1$ for some $k_1 \in K$, hence $dl = \varphi(k) = d\varphi(k_1)$, $z := l - \varphi(k_1) \in Z(L)$ and $m = \psi(l) = \psi(z)$ is in $\operatorname{Im} H(\psi)$. We leave the verification of the other equalities $\operatorname{Im} H(\varphi) = \ker H(\psi)$, $\operatorname{Im}\partial = \ker H(\varphi)$ and of the commutation statement to the reader. □

In most applications, the differential modules come with a natural $\mathbb{Z}$-grading. A homological complex is a graded differential module $K_\bullet = \bigoplus_{q\in\mathbb{Z}} K_q$ together with a differential d of degree -1, i.e. $d = \bigoplus d_q$ with $d_q : K_q \longrightarrow K_{q-1}$ and $d_{q-1} \circ d_q = 0$. Similarly, a cohomological complex is a graded differential module $K^\bullet = \bigoplus_{q\in\mathbb{Z}} K^q$ with differentials $d^q : K^q \longrightarrow K^{q+1}$ such that $d^{q+1} \circ d^q = 0$ (superscripts are always used instead of subscripts in that case). The corresponding (co)cycle, (co)boundary and (co)homology modules inherit a natural $\mathbb{Z}$-grading. In the case of cohomology, say, these modules will be denoted

$$Z^\bullet(K^\bullet) = \bigoplus Z^q(K^\bullet), \quad B^\bullet(K^\bullet) = \bigoplus B^q(K^\bullet), \quad H^\bullet(K^\bullet) = \bigoplus H^q(K^\bullet).$$

Unless otherwise stated, morphisms of complexes are assumed to be of degree 0, i.e. of the form $\varphi^\bullet = \bigoplus \varphi^q$ with $\varphi^q : K^q \longrightarrow L^q$. Any short exact sequence

$$0 \longrightarrow K^\bullet \xrightarrow{\varphi^\bullet} L^\bullet \xrightarrow{\psi^\bullet} M^\bullet \longrightarrow 0$$

gives rise to a corresponding *long exact sequence* of cohomology groups

$$(1.6)\quad H^q(K^\bullet) \xrightarrow{H^q(\varphi^\bullet)} H^q(L^\bullet) \xrightarrow{H^q(\psi^\bullet)} H^q(M^\bullet) \xrightarrow{\partial^q} H^{q+1}(K^\bullet) \xrightarrow{H^{q+1}(\varphi^\bullet)} \cdots$$

and there is a similar homology long exact sequence with a connecting homomorphism ∂_q of degree -1. When dealing with commutative diagrams of such sequences, the following simple lemma is often useful; the proof consists in a straightforward diagram chasing.

(1.7) Five lemma. *Consider a commutative diagram of R-modules*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow\varphi_1 & & \downarrow\varphi_2 & & \downarrow\varphi_3 & & \downarrow\varphi_4 & & \downarrow\varphi_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the rows are exact sequences. If φ_2 and φ_4 are injective and φ_1 surjective, then φ_3 is injective. If φ_2 and φ_4 is surjective and φ_5 injective, then φ_3 is surjective. In particular, φ_3 is an isomorphism as soon as $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ are isomorphisms.

2. The Simplicial Flabby Resolution of a Sheaf

Let X be a topological space and let $\mathcal{A}$ be a sheaf of abelian groups on X (see § II-2 for the definition). All the sheaves appearing in the sequel are assumed implicitly to be sheaves of *abelian groups*, unless otherwise stated. The first useful notion is that of resolution.

(2.1) Definition. *A* (*cohomological*) *resolution of $\mathcal{A}$ is a differential complex of sheaves $(\mathcal{L}^\bullet, d)$ with $\mathcal{L}^q = 0$, $d^q = 0$ for $q < 0$, such that there is an exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{d^q} \mathcal{L}^{q+1} \longrightarrow \cdots .$$

If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a morphism of sheaves and $(\mathcal{M}^\bullet, d)$ a resolution of $\mathcal{B}$, a morphism of resolutions $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$ is a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{j} & \mathcal{L}^0 & \xrightarrow{d^0} & \mathcal{L}^1 & \longrightarrow \cdots \longrightarrow & \mathcal{L}^q & \xrightarrow{d^q} & \mathcal{L}^{q+1} & \longrightarrow \cdots \\ & & \downarrow\varphi & & \downarrow\varphi^0 & & \downarrow\varphi^1 & & \downarrow\varphi^q & & \downarrow\varphi^{q+1} & \\ 0 & \longrightarrow & \mathcal{B} & \xrightarrow{j} & \mathcal{M}^0 & \xrightarrow{d^0} & \mathcal{M}^1 & \longrightarrow \cdots \longrightarrow & \mathcal{M}^q & \xrightarrow{d^q} & \mathcal{M}^{q+1} & \longrightarrow \cdots . \end{array}$$

(2.2) Example. Let X be a differentiable manifold and $\mathcal{E}^q$ the sheaf of germs of C^∞ differential forms of degree q with real values. The exterior derivative d defines a resolution $(\mathcal{E}^\bullet, d)$ of the sheaf $\mathbb{R}$ of locally constant functions with real values. In fact Poincaré's lemma asserts that d is locally exact in degree $q \geq 1$, and it is clear that the sections of $\ker d^0$ on connected open sets are constants. □

In the sequel, we will be interested by special resolutions in which the sheaves $\mathcal{L}^q$ have no local "rigidity". For that purpose, we introduce flabby sheaves, which have become a standard tool in sheaf theory since the publication of Godement's book (Godement 1957).

(2.3) Definition. *A sheaf $\mathcal{F}$ is called flabby if for every open subset U of X, the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is onto, i.e. if every section of $\mathcal{F}$ on U can be extended to X.*

Let $\pi : \mathcal{A} \longrightarrow X$ be a sheaf on X. We denote by $\mathcal{A}^{[0]}$ the sheaf of germs of sections $X \longrightarrow \mathcal{A}$ which are *not necessarily continuous.* In other words, $\mathcal{A}^{[0]}(U)$ is the set of all maps $f : U \longrightarrow \mathcal{A}$ such that $f(x) \in \mathcal{A}_x$ for all $x \in U$, or equivalently $\mathcal{A}^{[0]}(U) = \prod_{x\in U} \mathcal{A}_x$. It is clear that $\mathcal{A}^{[0]}$ is flabby and there is a canonical injection

$$j : \mathcal{A} \longrightarrow \mathcal{A}^{[0]}$$

defined as follows: to any $s \in \mathcal{A}_x$ we associate the germ $\widetilde{s} \in \mathcal{A}^{[0]}_x$ equal to the continuous section $y \longmapsto \widetilde{s}(y)$ near x such that $\widetilde{s}(x) = s$. In the sequel we merely denote $\widetilde{s} : y \longmapsto s(y)$ for simplicity. The sheaf $\mathcal{A}^{[0]}$ is called the *canonical flabby sheaf* associated to $\mathcal{A}$. We define inductively

$$\mathcal{A}^{[q]} = (\mathcal{A}^{[q-1]})^{[0]}.$$

The stalk $\mathcal{A}^{[q]}_x$ can be considered as the set of equivalence classes of maps $f : X^{q+1} \longrightarrow \mathcal{A}$ such that $f(x_0, \ldots, x_q) \in \mathcal{A}_{x_q}$, with two such maps identified if they coincide on a set of the form

$$(2.4) \quad x_0 \in V, \quad x_1 \in V(x_0), \quad \ldots\,, \quad x_q \in V(x_0, \ldots, x_{q-1}),$$

where V is an open neighborhood of x and $V(x_0, \ldots, x_j)$ an open neighborhood of x_j, depending on $x_0, \ldots, x_j$. This is easily seen by induction on q, if we identify a map $f : X^{q+1} \to \mathcal{A}$ to the map $X \to \mathcal{A}^{[q-1]}$, $x_0 \mapsto f_{x_0}$ such that $f_{x_0}(x_1, \ldots, x_q) = f(x_0, x_1, \ldots, x_q)$. Similarly, $\mathcal{A}^{[q]}(U)$ is the set of equivalence classes of functions $X^{q+1} \ni (x_0, \ldots, x_q) \longmapsto f(x_0, \ldots, x_q) \in \mathcal{A}_{x_q}$, with two such functions identified if they coincide on a set of the form

$$(2.4') \quad x_0 \in U, \quad x_1 \in V(x_0), \quad \ldots\,, \quad x_q \in V(x_0, \ldots, x_{q-1}).$$

Here, we may of course suppose $V(x_0, \ldots, x_{q-1}) \subset \ldots \subset V(x_0, x_1) \subset V(x_0) \subset U$. We define a differential $d^q : \mathcal{A}^{[q]} \longrightarrow \mathcal{A}^{[q+1]}$ by

$$(2.5) \quad (d^q f)(x_0, \ldots, x_{q+1}) = \sum_{0\le j\le q} (-1)^j f(x_0, \ldots, \widehat{x_j}, \ldots, x_{q+1}) + (-1)^{q+1} f(x_0, \ldots, x_q)(x_{q+1}).$$

The meaning of the last term is to be understood as follows: the element $s = f(x_0, \ldots, x_q)$ is a germ in $\mathcal{A}_{x_q}$, therefore s defines a continuous section $x_{q+1} \mapsto s(x_{q+1})$ of $\mathcal{A}$ in a neighborhood $V(x_0, \ldots, x_q)$ of x_q. In low degrees, we have the formulas

$$(2.6)\qquad \begin{aligned} (js)(x_0) &= s(x_0), \quad s \in \mathcal{A}_x, \\ (d^0 f)(x_0, x_1) &= f(x_1) - f(x_0)(x_1), \quad f \in \mathcal{A}_x^{[0]}, \\ (d^1 f)(x_0, x_1, x_2) &= f(x_1, x_2) - f(x_0, x_2) + f(x_0, x_1)(x_2), \quad f \in \mathcal{A}_x^{[1]}. \end{aligned}$$

(2.7) Theorem (Godement 1957). *The complex $(\mathcal{A}^{[\bullet]}, d)$ is a resolution of the sheaf $\mathcal{A}$, called the simplicial flabby resolution of $\mathcal{A}$.*

Proof. For $s \in \mathcal{A}_x$, the associated continuous germ obviously satisfies $s(x_0)(x_1) = s(x_1)$ for $x_0 \in V$, $x_1 \in V(x_0)$ small enough. The reader will easily infer from this that $d^0 \circ j = 0$ and $d^{q+1} \circ d^q = 0$. In order to verify that $(\mathcal{A}^{[\bullet]}, d)$ is a resolution of $\mathcal{A}$, we show that the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{A}_x \xrightarrow{\ j\ } \mathcal{A}_x^{[0]} \xrightarrow{d^0} \cdots \longrightarrow \mathcal{A}_x^{[q]} \xrightarrow{d^q} \mathcal{A}_x^{[q+1]} \longrightarrow \cdots$$

is homotopic to zero for every point $x \in X$. Set $\mathcal{A}^{[-1]} = \mathcal{A}$, $d^{-1} = j$ and

$$\begin{aligned} h^0 &: \mathcal{A}_x^{[0]} \longrightarrow \mathcal{A}_x, & h^0(f) &= f(x) \in \mathcal{A}_x, \\ h^q &: \mathcal{A}_x^{[q]} \longrightarrow \mathcal{A}_x^{[q-1]}, & h^q(f)(x_0, \ldots, x_{q-1}) &= f(x, x_0, \ldots, x_{q-1}). \end{aligned}$$

A straightforward computation shows that $(h^{q+1} \circ d^q + d^{q-1} \circ h^q)(f) = f$ for all $q \in \mathbb{Z}$ and $f \in \mathcal{A}_x^{[q]}$. $\square$

If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a sheaf morphism, it is clear that φ induces a morphism of resolutions

$$(2.8)\qquad \varphi^{[\bullet]} : \mathcal{A}^{[\bullet]} \longrightarrow \mathcal{B}^{[\bullet]}.$$

For every short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of sheaves, we get a corresponding short exact sequence of sheaf complexes

$$(2.9)\qquad \mathcal{A}^{[\bullet]} \longrightarrow \mathcal{B}^{[\bullet]} \longrightarrow \mathcal{C}^{[\bullet]}.$$

3. Cohomology Groups with Values in a Sheaf

3.A. Definition and Functorial Properties

If $\pi : \mathcal{A} \to X$ is a sheaf of abelian groups, the *cohomology groups* of $\mathcal{A}$ on X are (in a vague sense) algebraic invariants which describe the rigidity properties of the global sections of $\mathcal{A}$.

(3.1) Definition. *For every $q \in \mathbb{Z}$, the q-th cohomology group of X with values in $\mathcal{A}$ is*

$$\begin{aligned} H^q(X, \mathcal{A}) &= H^q\big(\mathcal{A}^{[\bullet]}(X)\big) = \\ &= \ker\big(d^q : \mathcal{A}^{[q]}(X) \to \mathcal{A}^{[q+1]}(X)\big) / \operatorname{Im}(d^{q-1} : \mathcal{A}^{[q-1]}(X) \to \mathcal{A}^{[q]}(X)\big) \end{aligned}$$

with the convention $\mathcal{A}^{[q]} = 0$, $d^q = 0$, $H^q(X,\mathcal{A}) = 0$ *when* $q < 0$.

For any subset $S \subset X$, we denote by $\mathcal{A}_{\restriction S}$ the *restriction* of $\mathcal{A}$ to S, i.e. the sheaf $\mathcal{A}_{\restriction S} = \pi^{-1}(S)$ equipped with the projection $\pi_{\restriction S}$ onto S. Then we write $H^q(S, \mathcal{A}_{\restriction S}) = H^q(S,\mathcal{A})$ for simplicity. When U is open, we see that $(\mathcal{A}^{[q]})_{\restriction U}$ coincides with $(\mathcal{A}_{\restriction U})^{[q]}$, thus we have $H^q(U,\mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}(U)\big)$. It is easy to show that every exact sequence of sheaves $0 \to \mathcal{A} \to \mathcal{L}^0 \to \mathcal{L}^1$ induces an exact sequence

$$(3.2) \quad 0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{L}^0(X) \longrightarrow \mathcal{L}^1(X).$$

If we apply this to $\mathcal{L}^q = \mathcal{A}^{[q]}$, $q = 0, 1$, we conclude that

$$(3.3) \quad H^0(X,\mathcal{A}) = \mathcal{A}(X).$$

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a sheaf morphism; (2.8) shows that there is an induced morphism

$$(3.4) \quad H^q(\varphi) : H^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{B})$$

on cohomology groups. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. Then we have an exact sequence of *groups*

$$0 \longrightarrow \mathcal{A}^{[0]}(X) \longrightarrow \mathcal{B}^{[0]}(X) \longrightarrow \mathcal{C}^{[0]}(X) \longrightarrow 0$$

because $\mathcal{A}^{[0]}(X) = \prod_{x\in X} \mathcal{A}_x$. Similarly, (2.9) yields for every q an exact sequence of *groups*

$$0 \longrightarrow \mathcal{A}^{[q]}(X) \longrightarrow \mathcal{B}^{[q]}(X) \longrightarrow \mathcal{C}^{[q]}(X) \longrightarrow 0.$$

If we take (3.3) into account, the snake lemma implies:

(3.5) Theorem. *To any exact sequence of sheaves* $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ *is associated a long exact sequence of cohomology groups*

$$\begin{array}{l} 0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{B}(X) \longrightarrow \mathcal{C}(X) \longrightarrow H^1(X,\mathcal{A}) \longrightarrow \cdots \\ \cdots \longrightarrow H^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{B}) \longrightarrow H^q(X,\mathcal{C}) \longrightarrow H^{q+1}(X,\mathcal{A}) \longrightarrow \cdots. \end{array}$$

(3.6) Corollary. *Let* $\mathcal{B} \to \mathcal{C}$ *be a surjective sheaf morphism and let* $\mathcal{A}$ *be its kernel. If* $H^1(X,\mathcal{A}) = 0$, *then* $\mathcal{B}(X) \longrightarrow \mathcal{C}(X)$ *is surjective.* $\square$

3.B. Exact Sequence Associated to a Closed Subset

Let S be a closed subset of X and $U = X \setminus S$. For any sheaf $\mathcal{A}$ on X, the presheaf

$$\Omega \longmapsto \mathcal{A}(S \cap \Omega), \quad \Omega \subset X \ \text{ open}$$

with the obvious restriction maps satisfies axioms (II-2.4′) and (II-2.4″), so it defines a sheaf on X which we denote by $\mathcal{A}^S$. This sheaf should not be confused with the restriction sheaf $\mathcal{A}_{\restriction S}$, which is a sheaf on S. We easily find

$$(3.7) \quad (\mathcal{A}^S)_x = \mathcal{A}_x \quad \text{if } \ x \in S, \quad (\mathcal{A}^S)_x = 0 \quad \text{if } \ x \in U.$$

Observe that these relations would completely fail if S were not closed. The restriction morphism $f \mapsto f_{\restriction S}$ induces a surjective sheaf morphism $\mathcal{A} \to \mathcal{A}^S$. We let $\mathcal{A}_U$ be its kernel, so that we have the relations

$$(3.8) \quad (\mathcal{A}_U)_x = 0 \quad \text{if } \ x \in S, \quad (\mathcal{A}_U)_x = \mathcal{A}_x \quad \text{if } \ x \in U.$$

From the definition, we obtain in particular

$$(3.9) \quad \mathcal{A}^S(X) = \mathcal{A}(S), \quad \mathcal{A}_U(X) = \{\text{sections of } \mathcal{A}(X) \text{ vanishing on } S\}.$$

Theorem 3.5 applied to the exact sequence $0 \to \mathcal{A}_U \to \mathcal{A} \to \mathcal{A}^S \to 0$ on X gives a long exact sequence

$$(3.9) \quad \begin{array}{l} 0 \longrightarrow \mathcal{A}_U(X) \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{A}(S) \longrightarrow H^1(X, \mathcal{A}_U) \cdots \\ \longrightarrow H^q(X, \mathcal{A}_U) \longrightarrow H^q(X, \mathcal{A}) \longrightarrow H^q(X, \mathcal{A}^S) \longrightarrow H^{q+1}(X, \mathcal{A}_U) \cdots \end{array}$$

3.C. Mayer-Vietoris Exact Sequence

Let U_1, U_2 be open subsets of X and $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. For any sheaf $\mathcal{A}$ on X and any q we have an exact sequence

$$0 \longrightarrow \mathcal{A}^{[q]}(U) \longrightarrow \mathcal{A}^{[q]}(U_1) \oplus \mathcal{A}^{[q]}(U_2) \longrightarrow \mathcal{A}^{[q]}(V) \longrightarrow 0$$

where the injection is given by $f \longmapsto (f_{\restriction U_1}, f_{\restriction U_2})$ and the surjection by $(g_1, g_2) \longmapsto g_{2\restriction V} - g_{1\restriction V}$; the surjectivity of this map follows immediately from the fact that $\mathcal{A}^{[q]}$ is flabby. An application of the snake lemma yields:

(3.11) Theorem. *For any sheaf $\mathcal{A}$ on X and any open sets $U_1, U_2 \subset X$, set $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. Then there is an exact sequence*

$$H^q(U, \mathcal{A}) \longrightarrow H^q(U_1, \mathcal{A}) \oplus H^q(U_2, \mathcal{A}) \longrightarrow H^q(V, \mathcal{A}) \longrightarrow H^{q+1}(U, \mathcal{A}) \cdots \square$$

4. Acyclic Sheaves

Given a sheaf $\mathcal{A}$ on X, it is usually very important to decide whether the cohomology groups $H^q(U,\mathcal{A})$ vanish for $q \geq 1$, and if this is the case, for which type of open sets U. Note that one cannot expect to have $H^0(U,\mathcal{A}) = 0$ in general, since a sheaf always has local sections.

(4.1) Definition. *A sheaf $\mathcal{A}$ is said to be acyclic on an open subset U if $H^q(U,\mathcal{A}) = 0$ for $q \geq 1$.*

4.A. Case of Flabby Sheaves

We are going to show that flabby sheaves are acyclic. First we need the following simple result.

(4.2) Proposition. *Let $\mathcal{A}$ be a sheaf with the following property: for every section f of $\mathcal{A}$ on an open subset $U \subset X$ and every point $x \in X$, there exists a neighborhood Ω of x and a section $h \in \mathcal{A}(\Omega)$ such that $h = f$ on $U \cap \Omega$. Then $\mathcal{A}$ is flabby.*

A consequence of this proposition is that flabbiness is a local property: a sheaf $\mathcal{A}$ is flabby on X if and only if it is flabby on a neighborhood of every point of X.

Proof. Let $f \in \mathcal{A}(U)$ be given. Consider the set of pairs (v, V) where v in $\mathcal{B}(V)$ is an extension of f on an open subset $V \supset U$. This set is inductively ordered, so there exists a maximal extension (v, V) by Zorn's lemma. The assumption shows that V must be equal to X. $\square$

(4.3) Proposition. *Let $0 \longrightarrow \mathcal{A} \xrightarrow{j} \mathcal{B} \xrightarrow{p} \mathcal{C} \longrightarrow 0$ be an exact sequence of sheaves. If $\mathcal{A}$ is flabby, the sequence of groups*

$$0 \longrightarrow \mathcal{A}(U) \xrightarrow{j} \mathcal{B}(U) \xrightarrow{p} \mathcal{C}(U) \longrightarrow 0$$

is exact for every open set U. If $\mathcal{A}$ and $\mathcal{B}$ are flabby, then $\mathcal{C}$ is flabby.

Proof. Let $g \in \mathcal{C}(U)$ be given. Consider the set E of pairs (v, V) where V is an open subset of U and $v \in \mathcal{B}(V)$ is such that $p(v) = g$ on V. It is clear that E is inductively ordered, so E has a maximal element (v, V), and we will prove that $V = U$. Otherwise, let $x \in U \smallsetminus V$ and let h be a section of $\mathcal{B}$ in a neighborhood of x such that $p(h_x) = g_x$. Then $p(h) = g$ on a neighborhood Ω of x, thus $p(v - h) = 0$ on $V \cap \Omega$ and $v - h = j(u)$ with $u \in \mathcal{A}(V \cap \Omega)$. If $\mathcal{A}$ is flabby, u has an extension $\widetilde{u} \in \mathcal{A}(X)$ and we can define a section $w \in \mathcal{B}(V \cup \Omega)$ such that $p(w) = g$ by

$$w = v \ \text{ on } \ V, \qquad w = h + j(\widetilde{u}) \ \text{ on } \ \Omega,$$

contradicting the maximality of (v, V). Therefore $V = U$, $v \in \mathcal{B}(U)$ and $p(v) = g$ on U. The first statement is proved. If $\mathcal{B}$ is also flabby, v has an extension $\widetilde{v} \in \mathcal{B}(X)$ and $\widetilde{g} = p(\widetilde{v}) \in \mathcal{C}(X)$ is an extension of g. Hence $\mathcal{C}$ is flabby. □

(4.4) Theorem. *A flabby sheaf $\mathcal{A}$ is acyclic on all open sets $U \subset X$.*

Proof. Let $\mathcal{Z}^q = \ker\big(d^q : \mathcal{A}^{[q]} \to \mathcal{A}^{[q+1]}\big)$. Then $\mathcal{Z}^0 = \mathcal{A}$ and we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{Z}^q \longrightarrow \mathcal{A}^{[q]} \xrightarrow{d^q} \mathcal{Z}^{q+1} \longrightarrow 0$$

because $\operatorname{Im} d^q = \ker d^{q+1} = \mathcal{Z}^{q+1}$. Proposition 4.3 implies by induction on q that all sheaves $\mathcal{Z}^q$ are flabby, and yields exact sequences

$$0 \longrightarrow \mathcal{Z}^q(U) \longrightarrow \mathcal{A}^{[q]}(U) \xrightarrow{d^q} \mathcal{Z}^{q+1}(U) \longrightarrow 0.$$

For $q \geq 1$, we find therefore

$$\begin{aligned}\ker\big(d^q : \mathcal{A}^{[q]}(U) \to \mathcal{A}^{[q+1]}(U)\big) &= \mathcal{Z}^q(U)\\ &= \operatorname{Im}\big(d^{q-1} : \mathcal{A}^{[q-1]}(U) \to \mathcal{A}^{[q]}(U)\big),\end{aligned}$$

that is, $H^q(U, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}(U)\big) = 0$. □

4.B. Soft Sheaves over Paracompact Spaces

We now discuss another general situation which produces acyclic sheaves. Recall that a topological space X is said to be *paracompact* if X is Hausdorff and if every open covering of X has a locally finite refinement. For instance, it is well known that every metric space is paracompact. A paracompact space X is always *normal*; in particular, for any locally finite open covering (U_α) of X there exists an open covering (V_α) such that $\overline{V}_\alpha \subset U_\alpha$. We will also need another closely related concept.

(4.5) Definition. *We say that a subspace S is strongly paracompact in X if S is Hausdorff and if the following property is satisfied: for every covering (U_α) of S by open sets in X, there exists another such covering (V_β) and a neighborhood W of S such that each set $W \cap \overline{V}_\beta$ is contained in some U_α, and such that every point of S has a neighborhood intersecting only finitely many sets V_β.*

It is clear that a strongly paracompact subspace S is itself paracompact. Conversely, the following result is easy to check:

(4.6) Lemma. *A subspace S is strongly paracompact in X as soon as one of the following situations occurs:*

a) *X is paracompact and S is closed;*

b) *S has a fundamental family of paracompact neighborhoods in X ;*

c) *S is paracompact and has a neighborhood homeomorphic to some product $S \times T$, in which S is embedded as a slice $S \times \{t_0\}$.* □

(4.7) Theorem. *Let $\mathcal{A}$ be a sheaf on X and S a strongly paracompact subspace of X. Then every section f of $\mathcal{A}$ on S can be extended to a section of $\mathcal{A}$ on some open neighborhood Ω of $\mathcal{A}$.*

Proof. Let $f \in \mathcal{A}(S)$. For every point $z \in S$ there exists an open neighborhood U_z and a section $\widetilde{f}_z \in \mathcal{A}(U_z)$ such that $\widetilde{f}_z(z) = f(z)$. After shrinking U_z, we may assume that $\widetilde{f}_z$ and f coincide on $S \cap U_z$. Let (V_α) be an open covering of S that is locally finite near S and W a neighborhood of S such that $W \cap \overline{V}_\alpha \subset U_{z(\alpha)}$ (Def. 4.5). We let

$$\Omega = \big\{x \in W \cap \bigcup V_\alpha\,;\ \widetilde{f}_{z(\alpha)}(x) = \widetilde{f}_{z(\beta)}(x),\ \forall\alpha,\beta \text{ with } x \in \overline{V}_\alpha \cap \overline{V}_\beta\big\}.$$

Then $(\Omega \cap V_\alpha)$ is an open covering of Ω and all pairs of sections $\widetilde{f}_{z(\alpha)}$ coincide in pairwise intersections. Thus there exists a section F of $\mathcal{A}$ on Ω which is equal to $\widetilde{f}_{z(\alpha)}$ on $\Omega \cap V_\alpha$. It remains only to show that Ω is a neighborhood of S. Let $z_0 \in S$. There exists a neighborhood U' of z_0 which meets only finitely many sets $V_{\alpha_1}, \ldots, V_{\alpha_p}$. After shrinking U', we may keep only those V_{α_l} such that $z_0 \in \overline{V}_{\alpha_l}$. The sections $\widetilde{f}_{z(\alpha_l)}$ coincide at z_0, so they coincide on some neighborhood U'' of this point. Hence $W \cap U'' \subset \Omega$, so Ω is a neighborhood of S. □

(4.8) Corollary. *If X is paracompact, every section $f \in \mathcal{A}(S)$ defined on a closed set S extends to a neighborhood Ω of S.* □

(4.9) Definition. *A sheaf $\mathcal{A}$ on X is said to be soft if every section f of $\mathcal{A}$ on a closed set S can be extended to X, i.e. if the restriction map $\mathcal{A}(X) \longrightarrow \mathcal{A}(S)$ is onto for every closed set S.*

(4.10) Example. On a paracompact space, every flabby sheaf $\mathcal{A}$ is soft: this is a consequence of Cor. 4.8.

(4.11) Example. On a paracompact space, the Tietze-Urysohn extension theorem shows that the sheaf $\mathcal{C}_X$ of germs of continuous functions on X is a soft sheaf of rings. However, observe that $\mathcal{C}_X$ is not flabby as soon as X is not discrete.

(4.12) Example. If X is a paracompact differentiable manifold, the sheaf $\mathcal{E}_X$ of germs of C^∞ functions on X is a soft sheaf of rings. □

Until the end of this section, we assume that X is a *paracompact topological space*. We first show that softness is a local property.

(4.13) Proposition. *A sheaf $\mathcal{A}$ is soft on X if and only if it is soft in a neighborhood of every point $x \in X$.*

Proof. If $\mathcal{A}$ is soft on X, it is soft on any closed neighborhood of a given point. Conversely, let $(U_\alpha)_{\alpha\in I}$ be a locally finite open covering of X which refine some covering by neighborhoods on which $\mathcal{A}$ is soft. Let (V_α) be a finer covering such that $\overline{V}_\alpha \subset U_\alpha$, and $f \in \mathcal{A}(S)$ be a section of $\mathcal{A}$ on a closed subset S of X. We consider the set E of pairs (g, J), where $J \subset I$ and where g is a section over $F_J := S \cup \bigcup_{\alpha\in J} \overline{V}_\alpha$, such that $g = f$ on S. As the family $(\overline{V}_\alpha)$ is locally finite, a section of $\mathcal{A}$ over F_J is continuous as soon it is continuous on S and on each $\overline{V}_\alpha$. Then $(f, \emptyset) \in E$ and E is inductively ordered by the relation

$$(g', J') \longrightarrow (g'', J'') \quad \text{if} \quad J' \subset J'' \quad \text{and} \quad g' = g'' \quad \text{on} \quad F_{J'}$$

No element (g, J), $J \neq I$, can be maximal: the assumption shows that $g_{\restriction F_J \cap \overline{V}_\alpha}$ has an extension to $\overline{V}_\alpha$, thus such a g has an extension to $F_{J\cup\{\alpha\}}$ for any $\alpha \notin J$. Hence E has a maximal element (g, I) defined on $F_I = X$. □

(4.14) Proposition. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. If $\mathcal{A}$ is soft, the map $\mathcal{B}(S) \to \mathcal{C}(S)$ is onto for any closed subset S of X. If $\mathcal{A}$ and $\mathcal{B}$ are soft, then $\mathcal{C}$ is soft.*

By the above inductive method, this result can be proved in a way similar to its analogue for flabby sheaves. We therefore obtain:

(4.15) Theorem. *On a paracompact space, a soft sheaf is acyclic on all closed subsets.* □

(4.16) Definition. *The support of a section $f \in \mathcal{A}(X)$ is defined by*

$$\operatorname{Supp} f = \{x \in X\,;\, f(x) \neq 0\}.$$

Supp f is always a closed set: as $\mathcal{A} \to X$ is a local homeomorphism, the equality $f(x) = 0$ implies $f = 0$ in a neighborhood of x.

(4.17) Theorem. *Let $(U_\alpha)_{\alpha\in I}$ be an open covering of X. If $\mathcal{A}$ is soft and $f \in \mathcal{A}(X)$, there exists a partition of f subordinate to (U_α), i.e. a family of sections $f_\alpha \in \mathcal{A}(X)$ such that* (Supp f_α) *is locally finite,* Supp $f_\alpha \subset U_\alpha$ *and $\sum f_\alpha = f$ on X.*

Proof. Assume first that (U_α) is locally finite. There exists an open covering (V_α) such that $\overline{V}_\alpha \subset U_\alpha$. Let $(f_\alpha)_{\alpha\in J}$, $J \subset I$, be a maximal family of sections

$f_\alpha \in \mathcal{A}(X)$ such that $\operatorname{Supp} f_\alpha \subset U_\alpha$ and $\sum_{\alpha\in J} f_\alpha = f$ on $S = \bigcup_{\alpha\in J} \overline{V}_\alpha$. If $J \neq I$ and $\beta \in I \smallsetminus J$, there exists a section $f_\beta \in \mathcal{A}(X)$ such that

$$f_\beta = 0 \quad \text{on} \quad X \smallsetminus U_\beta \quad \text{and} \quad f_\beta = f - \sum_{\alpha\in J} f_\alpha \quad \text{on} \quad S \cup \overline{V}_\beta$$

because $(X \smallsetminus U_\beta) \cup S \cup \overline{V}_\beta$ is closed and $f - \sum f_\alpha = 0$ on $(X \smallsetminus U_\alpha) \cap S$. This is a contradiction unless $J = I$.

In general, let (V_j) be a locally finite refinement of (U_α), such that $V_j \subset U_{\rho(j)}$, and let (f'_j) be a partition of f subordinate to (V_j). Then $f_\alpha = \sum_{j\in\rho^{-1}(\alpha)} f'_j$ is the required partition of f. □

Finally, we discuss a special situation which occurs very often in practice. Let $\mathcal{R}$ be a sheaf of commutative rings on X ; the rings $\mathcal{R}_x$ are supposed to have a unit element. Assume that $\mathcal{A}$ is a sheaf of modules over $\mathcal{R}$. It is clear that $\mathcal{A}^{[0]}$ is a $\mathcal{R}^{[0]}$-module, and thus also a $\mathcal{R}$-module. Therefore all sheaves $\mathcal{A}^{[q]}$ are $\mathcal{R}$-modules and the cohomology groups $H^q(U,\mathcal{A})$ have a natural structure of $\mathcal{R}(U)$-module.

(4.18) Lemma. *If $\mathcal{R}$ is soft, every sheaf $\mathcal{A}$ of $\mathcal{R}$-modules is soft.*

Proof. Every section $f \in \mathcal{A}(S)$ defined on a closed set S has an extension to some open neighborhood Ω. Let $\psi \in \mathcal{R}(X)$ be such that $\psi = 1$ on S and $\psi = 0$ on $X \smallsetminus \Omega$. Then ψf, defined as 0 on $X \smallsetminus \Omega$, is an extension of f to X. □

(4.19) Corollary. *Let $\mathcal{A}$ be a sheaf of $\mathcal{E}_X$-modules on a paracompact differentiable manifold X. Then $H^q(X,\mathcal{A}) = 0$ for all $q \geq 1$.*

5. Čech Cohomology

5.A. Definitions

In many important circumstances, cohomology groups with values in a sheaf $\mathcal{A}$ can be computed by means of the complex of Čech cochains, which is directly related to the spaces of sections of $\mathcal{A}$ on sufficiently fine coverings of X. This more concrete approach was historically the first one used to define sheaf cohomology (Leray 1950, Cartan 1950); however Čech cohomology does not always coincide with the "good" cohomology on non paracompact spaces. Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of X. For the sake of simplicity, we denote

$$U_{\alpha_0\alpha_1\ldots\alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}.$$

The group $C^q(\mathcal{U},\mathcal{A})$ of *Čech q-cochains* is the set of families

$$c = (c_{\alpha_0\alpha_1\ldots\alpha_q}) \in \prod_{(\alpha_0,\ldots,\alpha_q)\in I^{q+1}} \mathcal{A}(U_{\alpha_0\alpha_1\ldots\alpha_q}).$$

The group structure on $C^q(\mathfrak{U},\mathcal{A})$ is the obvious one deduced from the addition law on sections of $\mathcal{A}$. The *Čech differential* $\delta^q : C^q(\mathfrak{U},\mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{U},\mathcal{A})$ is defined by the formula

$$(5.1) \quad (\delta^q c)_{\alpha_0\ldots\alpha_{q+1}} = \sum_{0\le j\le q+1} (-1)^j\, c_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{q+1}\ \restriction U_{\alpha_0\ldots\alpha_{q+1}}},$$

and we set $C^q(\mathfrak{U},\mathcal{A}) = 0$, $\delta^q = 0$ for $q < 0$. In degrees 0 and 1, we get for example

$$(5.2) \qquad q=0, \quad c=(c_\alpha), \quad (\delta^0 c)_{\alpha\beta} = c_\beta - c_{\alpha\ \restriction U_{\alpha\beta}},$$

$$(5.2') \qquad q=1, \quad c=(c_{\alpha\beta}), \quad (\delta^1 c)_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta\ \restriction U_{\alpha\beta\gamma}}.$$

Easy verifications left to the reader show that $\delta^{q+1}\circ\delta^q = 0$. We get therefore a cochain complex $\big(C^\bullet(\mathfrak{U},\mathcal{A}),\delta\big)$, called the *complex of Čech cochains* relative to the covering $\mathfrak{U}$.

(5.3) Definition. *The Čech cohomology group of $\mathcal{A}$ relative to $\mathfrak{U}$ is*

$$\check{H}^q(\mathfrak{U},\mathcal{A}) = H^q\big(C^\bullet(\mathfrak{U},\mathcal{A})\big).$$

Formula (5.2) shows that the set of Čech 0-cocycles is the set of families $(c_\alpha) \in \prod \mathcal{A}(U_\alpha)$ such that $c_\beta = c_\alpha$ on $U_\alpha \cap U_\beta$. Such a family defines in a unique way a global section $f \in \mathcal{A}(X)$ with $f_{\restriction U_\alpha} = c_\alpha$. Hence

$$(5.4) \quad \check{H}^0(\mathfrak{U},\mathcal{A}) = \mathcal{A}(X).$$

Now, let $\mathcal{V} = (V_\beta)_{\beta\in J}$ be another open covering of X that is finer than $\mathfrak{U}$; this means that there exists a map $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for every $\beta \in J$. Then we can define a morphism $\rho^\bullet : C^\bullet(\mathfrak{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{V},\mathcal{A})$ by

$$(5.5) \quad (\rho^q c)_{\beta_0\ldots\beta_q} = c_{\rho(\beta_0)\ldots\rho(\beta_q)\ \restriction V_{\beta_0\ldots\beta_q}}\ ;$$

the commutation property $\delta\rho^\bullet = \rho^\bullet\delta$ is immediate. If $\rho' : J \to I$ is another refinement map such that $V_\beta \subset U_{\rho'(\beta)}$ for all β, the morphisms $\rho^\bullet$, $\rho'^\bullet$ are homotopic. To see this, we define a map $h^q : C^q(\mathfrak{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{V},\mathcal{A})$ by

$$(h^q c)_{\beta_0\ldots\beta_{q-1}} = \sum_{0\le j\le q-1} (-1)^j c_{\rho(\beta_0)\ldots\rho(\beta_j)\rho'(\beta_j)\ldots\rho'(\beta_{q-1})\ \restriction V_{\beta_0\ldots\beta_{q-1}}}.$$

The homotopy identity $\delta^{q-1}\circ h^q + h^{q+1}\circ\delta^q = \rho'^q - \rho^q$ is easy to verify. Hence $\rho^\bullet$ and $\rho'^\bullet$ induce a map depending only on $\mathfrak{U}$, $\mathcal{V}$:

$$(5.6) \quad H^q(\rho^\bullet) = H^q(\rho'^\bullet)\ :\ \check{H}^q(\mathfrak{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{V},\mathcal{A}).$$

Now, we want to define a *direct limit* $\check{H}^q(X,\mathcal{A})$ of the groups $\check{H}^q(\mathcal{U},\mathcal{A})$ by means of the refinement mappings (5.6). In order to avoid set theoretic difficulties, the coverings used in this definition will be considered as subsets of the power set $\mathcal{P}(X)$, so that the collection of all coverings becomes actually a set.

(5.7) Definition. *The Čech cohomology group $\check{H}^q(X,\mathcal{A})$ is the direct limit*

$$\check{H}^q(X,\mathcal{A}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathcal{A})$$

when $\mathcal{U}$ runs over the collection of all open coverings of X. Explicitly, this means that the elements of $\check{H}^q(X,\mathcal{A})$ are the equivalence classes in the disjoint union of the groups $\check{H}^q(\mathcal{U},\mathcal{A})$, with an element in $\check{H}^q(\mathcal{U},\mathcal{A})$ and another in $\check{H}^q(\mathcal{V},\mathcal{A})$ identified if their images in $\check{H}^q(\mathcal{W},\mathcal{A})$ coincide for some refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{V}$.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a sheaf morphism, we have an obvious induced morphism $\varphi^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{B})$, and therefore we find a morphism

$$H^q(\varphi^\bullet) : \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{U},\mathcal{B}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. We have an exact sequence of groups

$$(5.8)\quad 0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q(\mathcal{U},\mathcal{C}),$$

but in general the last map is not surjective, because every section in $\mathcal{C}(U_{\alpha_0,\ldots,\alpha_q})$ need not have a lifting in $\mathcal{B}(U_{\alpha_0,\ldots,\alpha_q})$. The image of $C^\bullet(\mathcal{U},\mathcal{B})$ in $C^\bullet(\mathcal{U},\mathcal{C})$ will be denoted $C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C})$ and called the complex of *liftable cochains* of $\mathcal{C}$ in $\mathcal{B}$. By construction, the sequence

$$(5.9)\quad 0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow 0$$

is exact, thus we get a corresponding long exact sequence of cohomology

$$(5.10)\quad \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{U},\mathcal{B}) \longrightarrow \check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \cdots.$$

If $\mathcal{A}$ is flabby, Prop. 4.3 shows that we have $C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) = C^q(\mathcal{U},\mathcal{C})$, hence $\check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) = \check{H}^q(\mathcal{U},\mathcal{C})$.

(5.11) Proposition. *Let $\mathcal{A}$ be a sheaf on X. Assume that either*

a) *$\mathcal{A}$ is flabby, or :*

b) *X is paracompact and $\mathcal{A}$ is a sheaf of modules over a soft sheaf of rings $\mathcal{R}$ on X.*

Then $\check{H}^q(\mathcal{U},\mathcal{A}) = 0$ for every $q \geq 1$ and every open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X.

Proof. b) Let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity in $\mathcal{R}$ subordinate to $\mathcal{U}$ (Prop. 4.17). We define a map $h^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{U},\mathcal{A})$ by

$$(5.12)\quad (h^q c)_{\alpha_0\ldots\alpha_{q-1}} = \sum_{\nu\in I} \psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}}$$

where $\psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}}$ is extended by 0 on $U_{\alpha_0\ldots\alpha_{q-1}} \cap \complement U_\nu$. It is clear that

$$(\delta^{q-1}h^q c)_{\alpha_0\ldots\alpha_q} = \sum_{\nu\in I} \psi_\nu \big(c_{\alpha_0\ldots\alpha_q} - (\delta^q c)_{\nu\alpha_0\ldots\alpha_q}\big),$$

i.e. $\delta^{q-1}h^q + h^{q+1}\delta^q = \mathrm{Id}$. Hence $\delta^q c = 0$ implies $\delta^{q-1}h^q c = c$ if $q \geq 1$.

a) First we show that the result is true for the sheaf $\mathcal{A}^{[0]}$. One can find a family of sets $L_\nu \subset U_\nu$ such that (L_ν) is a partition of X. If ψ_ν is the characteristic function of L_ν, Formula (5.12) makes sense for any cochain $c \in C^q(\mathcal{U},\mathcal{A}^{[0]})$ because $\mathcal{A}^{[0]}$ is a module over the ring $\mathbb{Z}^{[0]}$ of germs of arbitrary functions $X \to \mathbb{Z}$. Hence $\check{H}^q(\mathcal{U},\mathcal{A}^{[0]}) = 0$ for $q \geq 1$. We shall prove this property for all flabby sheaves by induction on q. Consider the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{[0]} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{C} = \mathcal{A}^{[0]}/\mathcal{A}$. By the remark after (5.10), we have exact sequences

$$\mathcal{A}^{[0]}(X) \longrightarrow \mathcal{C}(X) \longrightarrow \check{H}^1(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^1(\mathcal{U},\mathcal{A}^{[0]}) = 0,$$
$$\check{H}^q(\mathcal{U},\mathcal{C}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}^{[0]}) = 0.$$

Then $\mathcal{A}^{[0]}(X) \longrightarrow \mathcal{C}(X)$ is surjective by Prop. 4.3, thus $\check{H}^1(\mathcal{U},\mathcal{A}) = 0$. By 4.3 again, $\mathcal{C}$ is flabby; the induction hypothesis $\check{H}^q(\mathcal{U},\mathcal{C}) = 0$ implies that $\check{H}^{q+1}(\mathcal{U},\mathcal{A}) = 0$. □

5.B. Leray's Theorem for Acyclic Coverings

We first show the existence of a natural morphism from Čech cohomology to ordinary cohomology. Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be a covering of X. Select a map $\lambda : X \to I$ such that $x \in U_{\lambda(x)}$ for every $x \in X$. To every cochain $c \in C^q(\mathcal{U},\mathcal{A})$ we associate the section $\lambda^q c = f \in \mathcal{A}^{[q]}(X)$ such that

$$(5.13)\quad f(x_0,\ldots,x_q) = c_{\lambda(x_0)\ldots\lambda(x_q)}(x_q) \in \mathcal{A}_{x_q}\ ;$$

note that the right hand side is well defined as soon as

$$x_0 \in X,\quad x_1 \in U_{\lambda(x_0)},\quad \ldots\ ,\quad x_q \in U_{\lambda(x_0)\ldots\lambda(x_{q-1})}.$$

A comparison of (2.5) and (5.13) immediately shows that the section of $\mathcal{A}^{[q+1]}(X)$ associated to $\delta^q c$ is

$$\sum_{0\leq j\leq q+1} (-1)^j\, c_{\lambda(x_0)\ldots\widehat{\lambda(x_j)}\ldots\lambda(x_{q+1})}(x_{q+1}) = (d^q f)(x_0,\ldots,x_{q+1}).$$

In this way we get a morphism of complexes $\lambda^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow \mathcal{A}^{[\bullet]}(X)$. There is a corresponding morphism

$$(5.14)\quad H^q(\lambda^\bullet) : \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(X,\mathcal{A}).$$

If $\mathcal{V} = (V_\beta)_{\beta\in J}$ is a refinement of $\mathcal{U}$ such that $V_\beta \subset U_{\rho(\beta)}$ and $x \in V_{\mu(x)}$ for all x, β, we get a commutative diagram

$$\begin{tikzcd} \check{H}^q(\mathcal{U},\mathcal{A}) \arrow[rr, "H^q(\rho^\bullet)"] \arrow[dr, "H^q(\lambda^\bullet)"'] & & \check{H}^q(\mathcal{V},\mathcal{A}) \arrow[dl, "H^q(\mu^\bullet)"] \\ & H^q(X,\mathcal{A}) & \end{tikzcd}$$

with $\lambda = \rho\circ\mu$. In particular, (5.6) shows that the map $H^q(\lambda^\bullet)$ in (5.14) does not depend on the choice of λ : if λ' is another choice, then $H^q(\lambda^\bullet)$ and $H^q(\lambda'^\bullet)$ can be both factorized through the group $\check{H}^q(\mathcal{V},\mathcal{A})$ with $V_x = U_{\lambda(x)} \cap U_{\lambda'(x)}$ and $\mu = \mathrm{Id}_X$. By the universal property of direct limits, we get an induced morphism

$$(5.15)\quad \check{H}^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{A}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}^{[\bullet]}(X) \arrow[r] & \mathcal{B}^{[\bullet]}(X) \arrow[r] & \mathcal{C}^{[\bullet]}(X) \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are given by the morphisms $\lambda^\bullet$. We obtain therefore a commutative diagram

$$(5.16)\quad \begin{tikzcd} \check{H}^q(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & \check{H}^q(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & \check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & \check{H}^{q+1}(\mathcal{U},\mathcal{B}) \arrow[d] \\ H^q(X,\mathcal{A}) \arrow[r] & H^q(X,\mathcal{B}) \arrow[r] & H^q(X,\mathcal{C}) \arrow[r] & H^{q+1}(X,\mathcal{A}) \arrow[r] & H^{q+1}(X,\mathcal{B}). \end{tikzcd}$$

(5.17) Theorem (Leray). *Assume that*

$$H^s(U_{\alpha_0\ldots\alpha_t},\mathcal{A}) = 0$$

for all indices $\alpha_0,\ldots,\alpha_t$ *and* $s \ge 1$. *Then* (5.14) *gives an isomorphism* $\check{H}^q(\mathcal{U},\mathcal{A}) \simeq H^q(X,\mathcal{A})$.

We say that the covering $\mathcal{U}$ is *acyclic* (with respect to $\mathcal{A}$) if the hypothesis of Th. 5.17 is satisfied. Leray's theorem asserts that the cohomology groups of $\mathcal{A}$ on X can be computed by means of an arbitrary acyclic covering (if such a covering exists), without using the direct limit procedure.

Proof. By induction on q, the result being obvious for $q = 0$. Consider the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ with $\mathcal{B} = \mathcal{A}^{[0]}$ and $\mathcal{C} = \mathcal{A}^{[0]}/\mathcal{A}$. As $\mathcal{B}$

is acyclic, the hypothesis on $\mathcal{A}$ and the long exact sequence of cohomology imply $H^s(U_{\alpha_0\ldots\alpha_t}, \mathcal{C}) = 0$ for $s \geq 1$, $t \geq 0$. Moreover $C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) = C^\bullet(\mathcal{U}, \mathcal{C})$ thanks to Cor. 3.6. The induction hypothesis in degree q and diagram (5.16) give

$$\begin{array}{ccccccc} \check{H}^q(\mathcal{U}, \mathcal{B}) & \longrightarrow & \check{H}^q(\mathcal{U}, \mathcal{C}) & \longrightarrow & \check{H}^{q+1}(\mathcal{U}, \mathcal{A}) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\ H^q(X, \mathcal{B}) & \longrightarrow & H^q(X, \mathcal{C}) & \longrightarrow & H^{q+1}(X, \mathcal{A}) & \longrightarrow & 0, \end{array}$$

hence $\check{H}^{q+1}(\mathcal{U}, \mathcal{A}) \longrightarrow H^{q+1}(X, \mathcal{A})$ is also an isomorphism. □

(5.18) Remark. The morphism $H^1(\lambda^\bullet) : \check{H}^1(\mathcal{U}, \mathcal{A}) \longrightarrow H^1(X, \mathcal{A})$ is always injective. Indeed, we have a commutative diagram

$$\begin{array}{ccccccc} \check{H}^0(\mathcal{U}, \mathcal{B}) & \longrightarrow & \check{H}^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) & \longrightarrow & \check{H}^1(\mathcal{U}, \mathcal{A}) & \longrightarrow & 0 \\ \downarrow = & & \cap \downarrow & & \downarrow & & \\ H^0(X, \mathcal{B}) & \longrightarrow & H^0(X, \mathcal{C}) & \longrightarrow & H^1(X, \mathcal{A}) & \longrightarrow & 0, \end{array}$$

where $\check{H}^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$ is the subspace of $\mathcal{C}(X) = H^0(X, \mathcal{C})$ consisting of sections which can be lifted in $\mathcal{B}$ over each U_α. As a consequence, the refinement mappings

$$H^1(\rho^\bullet) : \check{H}^1(\mathcal{U}, \mathcal{A}) \longrightarrow \check{H}^1(\mathcal{V}, \mathcal{A})$$

are also injective. □

5.C. Čech Cohomology on Paracompact Spaces

We will prove here that Čech cohomology theory coincides with the ordinary one on paracompact spaces.

(5.19) Proposition. *Assume that X is paracompact. If*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is an exact sequence of sheaves, there is an exact sequence

$$\check{H}^q(X, \mathcal{A}) \longrightarrow \check{H}^q(X, \mathcal{B}) \longrightarrow \check{H}^q(X, \mathcal{C}) \longrightarrow \check{H}^{q+1}(X, \mathcal{A}) \longrightarrow \cdots$$

which is the direct limit of the exact sequences (5.10) *over all coverings* $\mathcal{U}$.

Proof. We have to show that the natural map

$$\varinjlim \check{H}^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \longrightarrow \varinjlim \check{H}^q(\mathcal{U}, \mathcal{C})$$

is an isomorphism. This follows easily from the following lemma, which says essentially that every cochain in $\mathcal{C}$ becomes liftable in $\mathcal{B}$ after a refinement of the covering.

(5.20) Lifting lemma. *Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of X and $c \in C^q(\mathcal{U}, \mathcal{C})$. If X is paracompact, there exists a finer covering $\mathcal{V} = (V_\beta)_{\beta\in J}$ and a refinement map $\rho : J \to I$ such that $\rho^q c \in C^q_{\mathcal{B}}(\mathcal{V}, \mathcal{C})$.*

Proof. Since $\mathcal{U}$ admits a locally finite refinement, we may assume that $\mathcal{U}$ itself is locally finite. There exists an open covering $\mathcal{W} = (W_\alpha)_{\alpha\in I}$ of X such that $\overline{W}_\alpha \subset U_\alpha$. For every point $x \in X$, we can select an open neighborhood V_x of x with the following properties:

a) if $x \in W_\alpha$, then $V_x \subset W_\alpha$;

b) if $x \in U_\alpha$ or if $V_x \cap W_\alpha \neq \emptyset$, then $V_x \subset U_\alpha$;

c) if $x \in U_{\alpha_0\ldots\alpha_q}$, then $c_{\alpha_0\ldots\alpha_q} \in C^q(U_{\alpha_0\ldots\alpha_q}, \mathcal{C})$ admits a lifting in $\mathcal{B}(V_x)$.

Indeed, a) (resp. c)) can be achieved because x belongs to only finitely many sets W_α (resp. U_α), and so only finitely many sections of $\mathcal{C}$ have to be lifted in $\mathcal{B}$. b) can be achieved because x has a neighborhood V'_x that meets only finitely many sets U_α ; then we take

$$V_x \subset V'_x \cap \bigcap_{U_\alpha \ni x} U_\alpha \cap \bigcap_{U_\alpha \not\ni x} (V'_x \smallsetminus \overline{W}_\alpha).$$

Choose $\rho : X \to I$ such that $x \in W_{\rho(x)}$ for every x. Then a) implies $V_x \subset W_{\rho(x)}$, so $\mathcal{V} = (V_x)_{x\in X}$ is finer than $\mathcal{U}$, and ρ defines a refinement map. If $V_{x_0\ldots x_q} \neq \emptyset$, we have

$$V_{x_0} \cap W_{\rho(x_j)} \supset V_{x_0} \cap V_{x_j} \neq \emptyset \quad \text{for} \quad 0 \le j \le q,$$

thus $V_{x_0} \subset U_{\rho(x_0)\ldots\rho(x_q)}$ by b). Now, c) implies that the section $c_{\rho(x_0)\ldots\rho(x_q)}$ admits a lifting in $\mathcal{B}(V_{x_0})$, and in particular in $\mathcal{B}(V_{x_0\ldots x_q})$. Therefore $\rho^q c$ is liftable in $\mathcal{B}$. □

(5.21) Theorem. *If X is a paracompact space, the canonical morphism $\check{H}^q(X, \mathcal{A}) \simeq H^q(X, \mathcal{A})$ is an isomorphism.*

Proof. Argue by induction on q as in Leray's theorem, with the Čech cohomology exact sequence over $\mathcal{U}$ replaced by its direct limit in (5.16). □

In the next chapters, we will be concerned only by paracompact spaces, and most often in fact by manifolds that are either compact or countable at infinity. In these cases, we will not distinguish $H^q(X, \mathcal{A})$ and $\check{H}^q(X, \mathcal{A})$.

5.D. Alternate Čech Cochains

For explicit calculations, it is sometimes useful to consider a slightly modified Čech complex which has the advantage of producing much smaller cochain groups. If $\mathcal{A}$ is a sheaf and $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ an open covering of X, we let

$AC^q(\mathcal{U},\mathcal{A}) \subset C^q(\mathcal{U},\mathcal{A})$ be the subgroup of *alternate Čech cochains*, consisting of Čech cochains $c = (c_{\alpha_0\ldots\alpha_q})$ such that

$$(5.22)\qquad \begin{cases} c_{\alpha_0\ldots\alpha_q} = 0 & \text{if } \alpha_i = \alpha_j, \quad i \neq j, \\ c_{\alpha_{\sigma(0)}\ldots\alpha_{\sigma(q)}} = \varepsilon(\sigma)\, c_{\alpha_0\ldots\alpha_q} & \end{cases}$$

for any permutation σ of $\{1,\ldots,q\}$ of signature $\varepsilon(\sigma)$. Then the Čech differential (5.1) of an alternate cochain is still alternate, so $AC^\bullet(\mathcal{U},\mathcal{A})$ is a subcomplex of $C^\bullet(\mathcal{U},\mathcal{A})$. We are going to show that the inclusion induces an isomorphism in cohomology:

$$(5.23)\qquad H^q\big(AC^\bullet(\mathcal{U},\mathcal{A})\big) \simeq H^q\big(C^\bullet(\mathcal{U},\mathcal{A})\big) = \check{H}^q(\mathcal{U},\mathcal{A}).$$

Select a total ordering on the index set I. For each such ordering, we can define a projection $\pi^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow AC^q(\mathcal{U},\mathcal{A}) \subset C^q(\mathcal{U},\mathcal{A})$ by

$$c \longmapsto \text{alternate } \widetilde{c} \ \text{ such that } \widetilde{c}_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} \ \text{ whenever } \ \alpha_0 < \ldots < \alpha_q.$$

As $\pi^\bullet$ is a morphism of complexes, it is enough to verify that $\pi^\bullet$ is homotopic to the identity on $C^\bullet(\mathcal{U},\mathcal{A})$. For a given multi-index $\alpha = (\alpha_0,\ldots,\alpha_q)$, which may contain repeated indices, there is a unique permutation $\big(m(0),\ldots,m(q)\big)$ of $(0,\ldots,q)$ such that

$$\alpha_{m(0)} \leq \ldots \leq \alpha_{m(q)} \quad \text{and} \quad m(l) < m(l+1) \quad \text{whenever} \quad \alpha_{m(l)} = \alpha_{m(l+1)}.$$

For $p \leq q$, we let $\varepsilon(\alpha,p)$ be the sign of the permutation

$$(0,\ldots,q) \longmapsto \big(m(0),\ldots,m(p-1),0,1,\ldots,\widehat{m(0)},\ldots,\widehat{m(p-1)},\ldots,q\big)$$

if the elements $\alpha_{m(0)},\ldots,\alpha_{m(p)}$ are all distinct, and $\varepsilon(\alpha,p) = 0$ otherwise. Finally, we set $h^q = 0$ for $q \leq 0$ and

$$(h^q c)_{\alpha_0\ldots\alpha_{q-1}} = \sum_{0\leq p\leq q-1} (-1)^p \varepsilon(\alpha,p)\, c_{\alpha_{m(0)}\ldots\alpha_{m(p)}\alpha_0\alpha_1\ldots\widehat{\alpha_{m(0)}}\ldots\widehat{\alpha_{m(p-1)}}\ldots\alpha_{q-1}}$$

for $q \geq 1$; observe that the index $\alpha_{m(p)}$ is repeated twice in the right hand side. A rather tedious calculation left to the reader shows that

$$(\delta^{q-1}h^q c + h^{q+1}\delta^q c)_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} - \varepsilon(\alpha,q)\, c_{\alpha_{m(0)}\ldots\alpha_{m(q)}} = (c - \pi^q c)_{\alpha_0\ldots\alpha_q}.$$

An interesting consequence of the isomorphism (5.23) is the following:

(5.24) Proposition. *Let $\mathcal{A}$ be a sheaf on a paracompact space X. If X has arbitrarily fine open coverings or at least one acyclic open covering $\mathcal{U} = (U_\alpha)$ such that more than $n+1$ distinct sets $U_{\alpha_0},\ldots,U_{\alpha_n}$ have empty intersection, then $H^q(X,\mathcal{A}) = 0$ for $q > n$.*

Proof. In fact, we have $AC^q(\mathcal{U},\mathcal{A}) = 0$ for $q > n$. □

6. The De Rham-Weil Isomorphism Theorem

In § 3 we defined cohomology groups by means of the simplicial flabby resolution. We show here that any resolution by acyclic sheaves could have been used instead. Let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$. We assume in addition that all $\mathcal{L}^q$ are acyclic on X, i.e. $H^s(X, \mathcal{L}^q) = 0$ for all $q \geq 0$ and $s \geq 1$. Set $\mathcal{Z}^q = \ker d^q$. Then $\mathcal{Z}^0 = \mathcal{A}$ and for every $q \geq 1$ we get a short exact sequence

$$0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{L}^{q-1} \xrightarrow{d^{q-1}} \mathcal{Z}^q \longrightarrow 0.$$

Theorem 3.5 yields an exact sequence

$$(6.1)\ H^s(X, \mathcal{L}^{q-1}) \xrightarrow{d^{q-1}} H^s(X, \mathcal{Z}^q) \xrightarrow{\partial^{s,q}} H^{s+1}(X, \mathcal{Z}^{q-1}) \to H^{s+1}(X, \mathcal{L}^{q-1}) = 0.$$

If $s \geq 1$, the first group is also zero and we get an isomorphism

$$\partial^{s,q} : H^s(X, \mathcal{Z}^q) \xrightarrow{\simeq} H^{s+1}(X, \mathcal{Z}^{q-1}).$$

For $s = 0$ we have $H^0(X, \mathcal{L}^{q-1}) = \mathcal{L}^{q-1}(X)$ and $H^0(X, \mathcal{Z}^q) = \mathcal{Z}^q(X)$ is the q-cocycle group of $\mathcal{L}^\bullet(X)$, so the connecting map $\partial^{0,q}$ gives an isomorphism

$$H^q\big(\mathcal{L}^\bullet(X)\big) = \mathcal{Z}^q(X)/d^{q-1}\mathcal{L}^{q-1}(x) \xrightarrow{\widetilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}).$$

The composite map $\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q}$ therefore defines an isomorphism

$$(6.2) \qquad H^q\big(\mathcal{L}^\bullet(X)\big) \xrightarrow{\widetilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}) \xrightarrow{\partial^{1,q-1}} \cdots \xrightarrow{\partial^{q-1,1}} H^q(X, \mathcal{Z}^0) = H^q(X, \mathcal{A}).$$

This isomorphism behaves functorially with respect to morphisms of resolutions. Our assertion means that for every sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ and every morphism of resolutions $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$, there is a commutative diagram

$$(6.3) \qquad \begin{array}{ccc} H^s\big(\mathcal{L}^\bullet(X)\big) & \longrightarrow & H^s(X, \mathcal{A}) \\ \downarrow H^s(\varphi^\bullet) & & \downarrow H^s(\varphi) \\ H^s\big(\mathcal{M}^\bullet(X)\big) & \longrightarrow & H^s(X, \mathcal{B}). \end{array}$$

If $\mathcal{W}^q = \ker\big(d^q : \mathcal{M}^q \to \mathcal{M}^{q+1}\big)$, the functoriality comes from the fact that we have commutative diagrams

$$\begin{array}{ccccccccc} 0 \to & \mathcal{Z}^{q-1} & \to & \mathcal{L}^{q-1} & \to & \mathcal{Z}^q & \to 0\ , \\ & \downarrow \varphi^{q-1} & & \downarrow \varphi^{q-1} & & \downarrow \varphi^q & \\ 0 \to & \mathcal{W}^{q-1} & \to & \mathcal{M}^{q-1} & \to & \mathcal{W}^q & \to 0\ , \end{array} \qquad \begin{array}{ccc} H^s(X, \mathcal{Z}^q) & \xrightarrow{\partial^{s,q}} & H^{s+1}(X, \mathcal{Z}^{q-1}) \\ \downarrow H^s(\varphi^q) & & \downarrow H^{s+1}(\varphi^{q-1}) \\ H^s(X, \mathcal{W}^q) & \xrightarrow{\partial^{s,q}} & H^{s+1}(X, \mathcal{W}^{q-1}). \end{array}$$

(6.4) De Rham-Weil isomorphism theorem. *If $(\mathcal{L}^\bullet, d)$ is a resolution of $\mathcal{A}$ by sheaves $\mathcal{L}^q$ which are acyclic on X, there is a functorial isomorphism*

$$H^q\big(\mathcal{L}^\bullet(X)\big) \longrightarrow H^q(X, \mathcal{A}). \qquad \square$$

(6.5) Example: De Rham cohomology. Let X be a n-dimensional paracompact differential manifold. Consider the resolution

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \to \cdots \to \mathcal{E}^q \xrightarrow{d} \mathcal{E}^{q+1} \to \cdots \to \mathcal{E}^n \to 0$$

given by the exterior derivative d acting on germs of C^∞ differential q-forms (c.f. Example 2.2). The *De Rham cohomology groups* of X are precisely

$$(6.6) \quad H^q_{\mathrm{DR}}(X, \mathbb{R}) = H^q\big(\mathcal{E}^\bullet(X)\big).$$

All sheaves $\mathcal{E}^q$ are $\mathcal{E}_X$-modules, so $\mathcal{E}^q$ is acyclic by Cor. 4.19. Therefore, we get an isomorphism

$$(6.7) \quad H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} H^q(X, \mathbb{R})$$

from the De Rham cohomology onto the cohomology with values in the constant sheaf $\mathbb{R}$. Instead of using C^∞ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents:

$$0 \to \mathbb{R} \to \mathcal{D}'_n \xrightarrow{d} \mathcal{D}'_{n-1} \to \cdots \to \mathcal{D}'_{n-q} \xrightarrow{d} \mathcal{D}'_{n-q-1} \to \cdots \to \mathcal{D}'_0 \to 0.$$

The sheaves $\mathcal{D}'_q$ are also $\mathcal{E}_X$-modules, hence acyclic. Thanks to (6.3), the inclusion $\mathcal{E}^q \subset \mathcal{D}'_{n-q}$ induces an isomorphism

$$(6.8) \quad H^q\big(\mathcal{E}^\bullet(X)\big) \simeq H^q\big(\mathcal{D}'_{n-\bullet}(X)\big),$$

both groups being isomorphic to $H^q(X, \mathbb{R})$. The isomorphism between cohomology of differential forms and singular cohomology (another topological invariant) was first established by (De Rham 1931). The above proof follows essentially the method given by (Weil 1952), in a more abstract setting. As we will see, the isomorphism (6.7) can be put under a very explicit form in terms of Čech cohomology. We need a simple lemma.

(6.9) Lemma. *Let X be a paracompact differentiable manifold. There are arbitrarily fine open coverings $\mathfrak{U} = (U_\alpha)$ such that all intersections $U_{\alpha_0 \ldots \alpha_q}$ are diffeomorphic to convex sets.*

Proof. Select locally finite coverings $\Omega'_j \subset\subset \Omega_j$ of X by open sets diffeomorphic to concentric euclidean balls in $\mathbb{R}^n$. Let us denote by τ_{jk} the transition diffeomorphism from the coordinates in Ω_k to those in Ω_j. For any point $a \in \Omega'_j$, the function $x \mapsto |x - a|^2$ computed in terms of the coordinates of Ω_j becomes $|\tau_{jk}(x) - \tau_{jk}(a)|^2$ on any patch $\Omega_k \ni a$. It is clear that these functions are strictly convex at a, thus there is a euclidean ball $B(a, \varepsilon) \subset \Omega'_j$ such that all functions are strictly convex on $B(a, \varepsilon) \cap \Omega'_k \subset \Omega_k$ (only a finite number of indices k is involved). Now, choose $\mathfrak{U}$ to be a (locally finite) covering of X by such balls $U_\alpha = B(a_\alpha, \varepsilon_\alpha)$ with $U_\alpha \subset \Omega'_{\rho(\alpha)}$. Then the intersection $U_{\alpha_0 \ldots \alpha_q}$ is defined in Ω_k, $k = \rho(\alpha_0)$, by the equations

$$|\tau_{jk}(x) - \tau_{jk}(a_{\alpha_m})|^2 < \varepsilon_{\alpha_m}^2$$

where $j = \rho(\alpha_m)$, $0 \le m \le q$. Hence the intersection is convex in the open coordinate chart $\Omega_{\rho(\alpha_0)}$. □

Let Ω be an open subset of $\mathbb{R}^n$ which is starshaped with respect to the origin. Then the De Rham complex $\mathbb{R} \longrightarrow \mathcal{E}^\bullet(\Omega)$ is acyclic: indeed, Poincaré's lemma yields a homotopy operator $k^q : \mathcal{E}^q(\Omega) \longrightarrow \mathcal{E}^{q-1}(\Omega)$ such that

$$k^q f_x(\xi_1, \ldots, \xi_{q-1}) = \int_0^1 t^{q-1} f_{tx}(x, \xi_1, \ldots, \xi_{q-1})\, dt, \quad x \in \Omega, \quad \xi_j \in \mathbb{R}^n,$$
$$k^0 f = f(0) \in \mathbb{R} \quad \text{for} \quad f \in \mathcal{E}^0(\Omega).$$

Hence $H^q_{\mathrm{DR}}(\Omega, \mathbb{R}) = 0$ for $q \ge 1$. Now, consider the resolution $\mathcal{E}^\bullet$ of the constant sheaf $\mathbb{R}$ on X, and apply the proof of the De Rham-Weil isomorphism theorem to Čech cohomology groups over a covering $\mathcal{U}$ chosen as in Lemma 6.9. Since the intersections $U_{\alpha_0\ldots\alpha_s}$ are convex, all Čech cochains in $C^s(\mathcal{U}, \mathcal{Z}^q)$ are liftable in $\mathcal{E}^{q-1}$ by means of k^q. Hence for all $s = 1, \ldots, q$ we have isomorphisms $\partial^{s,q-s} : \check{H}^s(\mathcal{U}, \mathcal{Z}^{q-s}) \longrightarrow \check{H}^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ for $s \ge 1$ and we get a resulting isomorphism

$$\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q} : H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} \check{H}^q(\mathcal{U}, \mathbb{R})$$

We are going to compute the connecting homomorphisms $\partial^{s,q-s}$ and their inverses explicitly.

Let c in $C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that $\delta^s c = 0$. As $c_{\alpha_0\ldots\alpha_s}$ is d-closed, we can write $c = d(k^{q-s}c)$ where the cochain $k^{q-s}c \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ is defined as the family of sections $k^{q-s}c_{\alpha_0\ldots\alpha_s} \in \mathcal{E}^{q-s-1}(U_{\alpha_0\ldots\alpha_s})$. Then $d(\delta^s k^{q-s}c) = \delta^s(dk^{q-s}c) = \delta^s c = 0$ and

$$\partial^{s,q-s}\{c\} = \{\delta^s k^{q-s} c\} \in \check{H}^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1}).$$

The isomorphism $H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} \check{H}^q(\mathcal{U}, \mathbb{R})$ is thus defined as follows: to the cohomology class $\{f\}$ of a closed q-form $f \in \mathcal{E}^q(X)$, we associate the cocycle $(c^0_\alpha) = (f_{\restriction U_\alpha}) \in C^0(\mathcal{U}, \mathcal{Z}^q)$, then the cocycle

$$c^1_{\alpha\beta} = k^q c^0_\beta - k^q c^0_\alpha \in C^1(\mathcal{U}, \mathcal{Z}^{q-1}),$$

and by induction cocycles $(c^s_{\alpha_0\ldots\alpha_s}) \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ given by

$$\text{(6.10)} \quad c^{s+1}_{\alpha_0\ldots\alpha_{s+1}} = \sum_{0 \le j \le s+1} (-1)^j\, k^{q-s} c^s_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{s+1}} \qquad \text{on} \ \ U_{\alpha_0\ldots\alpha_{s+1}}.$$

The image of $\{f\}$ in $\check{H}^q(\mathcal{U}, \mathbb{R})$ is the class of the q-cocycle $(c^q_{\alpha_0\ldots\alpha_q})$ in $C^q(\mathcal{U}, \mathbb{R})$.

Conversely, let (ψ_α) be a C^∞ partition of unity subordinate to $\mathcal{U}$. Any Čech cocycle $c \in C^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ can be written $c = \delta^s\gamma$ with $\gamma \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ given by

$$\gamma_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} \psi_\nu \, c_{\nu\alpha_0\ldots\alpha_s},$$

(c.f. Prop. 5.11 b)), thus $\{c'\} = (\partial^{s,q-s})^{-1}\{c\}$ can be represented by the cochain $c' = d\gamma \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that

$$c'_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} d\psi_\nu \wedge c_{\nu\alpha_0\ldots\alpha_s} = (-1)^{q-s-1} \sum_{\nu\in I} c_{\nu\alpha_0\ldots\alpha_s} \wedge d\psi_\nu.$$

For a reason that will become apparent later, we shall in fact modify the sign of our isomorphism $\partial^{s,q-s}$ by the factor $(-1)^{q-s-1}$. Starting from a class $\{c\} \in \check{H}^q(\mathcal{U}, \mathbb{R})$, we obtain inductively $\{b\} \in \check{H}^0(\mathcal{U}, \mathcal{Z}^q)$ such that

$$(6.11) \quad b_{\alpha_0} = \sum_{\nu_0,\ldots,\nu_{q-1}} c_{\nu_0\ldots\nu_{q-1}\alpha_0} \, d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}} \quad \text{on } U_{\alpha_0},$$

corresponding to $\{f\} \in H^q_{\mathrm{DR}}(X, \mathbb{R})$ given by the explicit formula

$$(6.12) \quad f = \sum_{\nu_q} \psi_{\nu_q} b_{\nu_q} = \sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q} \, \psi_{\nu_q} d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}}.$$

The choice of sign corresponds to (6.2) multiplied by $(-1)^{q(q-1)/2}$.

(6.13) Example: Dolbeault cohomology groups. Let X be a $\mathbb{C}$-analytic manifold of dimension n, and let $\mathcal{E}^{p,q}$ be the sheaf of germs of C^∞ differential forms of type (p,q) with complex values. For every $p = 0, 1, \ldots, n$, the Dolbeault-Grothendieck Lemma I-2.9 shows that $(\mathcal{E}^{p,\bullet}, d'')$ is a resolution of the sheaf Ω^p_X of germs of holomorphic forms of degree p on X. The *Dolbeault cohomology groups* of X already considered in chapter 1 can be defined by

$$(6.14) \quad H^{p,q}(X, \mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big).$$

The sheaves $\mathcal{E}^{p,q}$ are acyclic, so we get the *Dolbeault isomorphism theorem*, originally proved in (Dolbeault 1953), which relates d''-cohomology and sheaf cohomology:

$$(6.15) \quad H^{p,q}(X, \mathbb{C}) \xrightarrow{\simeq} H^q(X, \Omega^p_X).$$

The case $p = 0$ is especially interesting:

$$(6.16) \quad H^{0,q}(X, \mathbb{C}) \simeq H^q(X, \mathcal{O}_X).$$

As in the case of De Rham cohomology, there is an inclusion $\mathcal{E}^{p,q} \subset \mathcal{D}'_{n-p,n-q}$ and the complex of currents $(\mathcal{D}'_{n-p,n-\bullet}, d'')$ defines also a resolution of Ω^p_X. Hence there is an isomorphism:

$$(6.17) \quad H^{p,q}(X, \mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big) \simeq H^q\big(\mathcal{D}'_{n-p,n-\bullet}(X)\big).$$

7. Cohomology with Supports

As its name indicates, cohomology with supports deals with sections of sheaves having supports in prescribed closed sets. We first introduce what is an admissible family of supports.

(7.1) Definition. *A family of supports on a topological space X is a collection Φ of closed subsets of X with the following two properties:*

a) *If F, $F' \in \Phi$, then $F \cup F' \in \Phi$;*

b) *If $F \in \Phi$ and $F' \subset F$ is closed, then $F' \in \Phi$.*

(7.2) Example. Let S be an arbitrary subset of X. Then the family of all closed subsets of X contained in S is a family of supports.

(7.3) Example. The collection of all compact (non necessarily Hausdorff) subsets of X is a family of supports, which will be denoted simply c in the sequel. □

(7.4) Definition. *For any sheaf $\mathcal{A}$ and any family of supports Φ on X, $\mathcal{A}_\Phi(X)$ will denote the set of all sections $f \in \mathcal{A}(X)$ such that* Supp $f \in \Phi$.

It is clear that $\mathcal{A}_\Phi(X)$ is a subgroup of $\mathcal{A}(X)$. We can now introduce cohomology groups with arbitrary supports.

(7.5) Definition. *The cohomology groups of $\mathcal{A}$ with supports in Φ are*

$$H^q_\Phi(X, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}_\Phi(X)\big).$$

The cohomology groups with compact supports will be denoted $H^q_c(X, \mathcal{A})$ and the cohomology groups with supports in a subset S will be denoted $H^q_S(X, \mathcal{A})$.

In particular $H^0_\Phi(X, \mathcal{A}) = \mathcal{A}_\Phi(X)$. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence, there are corresponding exact sequences

$$(7.6)\qquad \begin{array}{l} 0 \longrightarrow \mathcal{A}^{[q]}_\Phi(X) \longrightarrow \mathcal{B}^{[q]}_\Phi(X) \longrightarrow \mathcal{C}^{[q]}_\Phi(X) \longrightarrow \cdots \\ \qquad H^q_\Phi(X, \mathcal{A}) \longrightarrow H^q_\Phi(X, \mathcal{B}) \longrightarrow H^q_\Phi(X, \mathcal{C}) \longrightarrow H^{q+1}_\Phi(X, \mathcal{A}) \longrightarrow \cdots. \end{array}$$

When $\mathcal{A}$ is flabby, there is an exact sequence

$$(7.7)\qquad 0 \longrightarrow \mathcal{A}_\Phi(X) \longrightarrow \mathcal{B}_\Phi(X) \longrightarrow \mathcal{C}_\Phi(X) \longrightarrow 0$$

and every $g \in \mathcal{C}_\Phi(X)$ can be lifted to $v \in \mathcal{B}_\Phi(X)$ without enlarging the support: apply the proof of Prop. 4.3 to a maximal lifting which extends $w = 0$ on $W = \complement(\mathrm{Supp}\ g)$. It follows that a flabby sheaf $\mathcal{A}$ is Φ-acyclic, i.e. $H^q_\Phi(X, \mathcal{A}) = 0$ for all $q \geq 1$. Similarly, assume that X is paracompact and that $\mathcal{A}$ is soft, and suppose that Φ has the following additional property: every set $F \in \Phi$ has a neighborhood $G \in \Phi$. An adaptation of the proofs of Prop. 4.3 and 4.13 shows that (7.7) is again exact. Therefore every soft sheaf is also Φ-acyclic in that case.

As a consequence of (7.6), any resolution $\mathcal{L}^\bullet$ of $\mathcal{A}$ by Φ-acyclic sheaves provides a canonical De Rham-Weil isomorphism

$$(7.8) \quad H^q\big(\mathcal{L}^\bullet_\Phi(X)\big) \longrightarrow H^q_\Phi(X,\mathcal{A}).$$

(7.9) Example: De Rham cohomology with compact support. In the special case of the De Rham resolution $\mathbb{R} \longrightarrow \mathcal{E}^\bullet$ on a paracompact manifold, we get an isomorphism

$$(7.10) \quad H^q_{\mathrm{DR},c}(X,\mathbb{R}) := H^q\big((\mathcal{D}^\bullet(X)\big) \xrightarrow{\simeq} H^q_c(X,\mathbb{R}),$$

where $\mathcal{D}^q(X)$ is the space of smooth differential q-forms with compact support in X. These groups are called the *De Rham cohomology groups* of X with compact support. When X is oriented, $\dim X = n$, we can also consider the complex of compactly supported currents:

$$0 \longrightarrow \mathcal{E}'_n(X) \xrightarrow{d} \mathcal{E}'_{n-1}(X) \longrightarrow \cdots \longrightarrow \mathcal{E}'_{n-q}(X) \xrightarrow{d} \mathcal{E}'_{n-q-1}(X) \longrightarrow \cdots.$$

Note that $\mathcal{D}^\bullet(X)$ and $\mathcal{E}'_{n-\bullet}(X)$ are respectively the subgroups of compactly supported sections in $\mathcal{E}^\bullet$ and $\mathcal{D}'_{n-\bullet}$, both of which are acyclic resolutions of $\mathbb{R}$. Therefore the inclusion $\mathcal{D}^\bullet(X) \subset \mathcal{E}'_{n-\bullet}(X)$ induces an isomorphism

$$(7.11) \quad H^q\big(\mathcal{D}^\bullet(X)\big) \simeq H^q\big(\mathcal{E}'_{n-\bullet}(X)\big),$$

both groups being isomorphic to $H^q_c(X,\mathbb{R})$. □

Now, we concentrate our attention on cohomology groups with compact support. We assume until the end of this section that X is a *locally compact* space.

(7.12) Proposition. *There is an isomorphism*

$$H^q_c(X,\mathcal{A}) = \varinjlim_{U \subset\subset X} H^q(\overline{U},\mathcal{A}_U)$$

where $\mathcal{A}_U$ is the sheaf of sections of $\mathcal{A}$ vanishing on $X \smallsetminus U$ (c.f. §3).

Proof. By definition

$$H^q_c(X,\mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}_c(X)\big) = \varinjlim_{U \subset\subset X} H^q\big((\mathcal{A}^{[\bullet]})_U(\overline{U})\big)$$

since sections of $(\mathcal{A}^{[\bullet]})_U(\overline{U})$ can be extended by 0 on $X \smallsetminus U$. However, $(\mathcal{A}^{[\bullet]})_U$ is a resolution of $\mathcal{A}_U$ and $(\mathcal{A}^{[q]})_U$ is a $\mathbb{Z}^{[q]}$-module, so it is acyclic on $\overline{U}$. The De Rham-Weil isomorphism theorem implies

$$H^q\big((\mathcal{A}^{[\bullet]})_U(\overline{U})\big) = H^q(\overline{U}, \mathcal{A}_U)$$

and the proposition follows. The reader should take care of the fact that $(\mathcal{A}^{[q]})_U$ does not coincide in general with $(\mathcal{A}_U)^{[q]}$. □

The cohomology groups with compact support can also be defined by means of Čech cohomology.

(7.13) Definition. *Assume that X is a separable locally compact space. If $\mathcal{U} = (U_\alpha)$ is a locally finite covering of X by relatively compact open subsets, we let $C^q_c(\mathcal{U}, \mathcal{A})$ be the subgroups of cochains such that only finitely many coefficients $c_{\alpha_0\ldots\alpha_q}$ are non zero. The Čech cohomology groups with compact support are defined by*

$$\check{H}^q_c(\mathcal{U}, \mathcal{A}) = H^q\big(C^\bullet_c(\mathcal{U}, \mathcal{A})\big)$$
$$\check{H}^q_c(X, \mathcal{A}) = \varinjlim_{\mathcal{U}} H^q\big(C^\bullet_c(\mathcal{U}, \mathcal{A})\big)$$

For such coverings $\mathcal{U}$, Formula (5.13) yields canonical morphisms

$$(7.14)\quad H^q(\lambda^\bullet) \;:\; \check{H}^q_c(\mathcal{U}, \mathcal{A}) \longrightarrow H^q_c(X, \mathcal{A}).$$

Now, the lifting Lemma 5.20 is valid for cochains with compact supports, and the same proof as the one given in §5 gives an isomorphism

$$(7.15)\quad \check{H}^q_c(X, \mathcal{A}) \simeq H^q_c(X, \mathcal{A}).$$

8. Cup Product

Let $\mathcal{R}$ be a sheaf of commutative rings and $\mathcal{A}$, $\mathcal{B}$ sheaves of $\mathcal{R}$-modules on a space X. We denote by $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ the sheaf on X defined by

$$(8.1)\quad (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})_x = \mathcal{A}_x \otimes_{\mathcal{R}_x} \mathcal{B}_x,$$

with the weakest topology such that the range of any section given by $\mathcal{A}(U) \otimes_{\mathcal{R}(U)} \mathcal{B}(U)$ is open in $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ for any open set $U \subset X$. Given $f \in \mathcal{A}^{[p]}_x$ and $g \in \mathcal{B}^{[q]}_x$, the *cup product* $f \smile g \in (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})^{[p+q]}_x$ is defined by

$$(8.2)\quad f \smile g(x_0, \ldots, x_{p+q}) = f(x_0, \ldots, x_p)(x_{p+q}) \otimes g(x_p, \ldots, x_{p+q}).$$

A simple computation shows that

$$(8.3)\quad d^{p+q}(f \smile g) = (d^p f) \smile g + (-1)^p f \smile (d^q g).$$

In particular, $f \smile g$ is a cocycle if f, g are cocycles, and we have

$$(f + d^{p-1}f') \smile (g + d^{q-1}g') = f \smile g + d^{p+q-1}\big(f' \smile g + (-1)^p f \smile g' + f' \smile dg'\big).$$

Consequently, there is a well defined $\mathcal{R}(X)$-bilinear morphism

$$(8.4) \quad H^p(X, \mathcal{A}) \times H^q(X, \mathcal{B}) \longrightarrow H^{p+q}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$$

which maps a pair $(\{f\}, \{g\})$ to $\{f \smile g\}$.

Let $0 \to \mathcal{B} \to \mathcal{B}' \to \mathcal{B}'' \to 0$ be an exact sequence of sheaves. Assume that the sequence obtained after taking the tensor product by $\mathcal{A}$ is also exact:

$$0 \longrightarrow \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}' \longrightarrow \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}'' \longrightarrow 0.$$

Then we obtain connecting homomorphisms

$$\begin{aligned} \partial^q &: H^q(X, \mathcal{B}'') \longrightarrow H^{q+1}(X, \mathcal{B}), \\ \partial^q &: H^q(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}'') \longrightarrow H^{q+1}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}). \end{aligned}$$

For every $\alpha \in H^p(X, \mathcal{A})$, $\beta'' \in H^q(X, \mathcal{B}'')$ we have

$$(8.5) \quad \partial^{p+q}(\alpha \smile \beta'') = (-1)^p \, \alpha \smile (\partial^q \beta''),$$
$$(8.5') \quad \partial^{p+q}(\beta'' \smile \alpha) = (\partial^q \beta'') \smile \alpha,$$

where the second line corresponds to the tensor product of the exact sequence by $\mathcal{A}$ on the right side. These formulas are deduced from (8.3) applied to a representant $f \in \mathcal{A}^{[p]}(X)$ of α and to a lifting $g' \in \mathcal{B}'^{[q]}(X)$ of a representative g'' of β'' (note that $d^p f = 0$).

(8.6) Associativity and anticommutativity. *Let $i : \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B} \longrightarrow \mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}$ be the canonical isomorphism $s \otimes t \mapsto t \otimes s$. For all $\alpha \in H^p(X, \mathcal{A})$, $\beta \in H^q(X, \mathcal{B})$ we have*

$$\beta \smile \alpha = (-1)^{pq} \, i(\alpha \smile \beta).$$

If $\mathcal{C}$ is another sheaf of $\mathcal{R}$-modules and $\gamma \in H^r(X, \mathcal{C})$, then

$$(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma).$$

Proof. The associativity property is easily seen to hold already for all cochains

$$(f \smile g) \smile h = f \smile (g \smile h), \quad f \in \mathcal{A}_x^{[p]}, \quad g \in \mathcal{B}_x^{[q]}, \quad h \in \mathcal{C}_x^{[r]}.$$

The commutation property is obvious for $p = q = 0$, and can be proved in general by induction on $p+q$. Assume for example $q \geq 1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}' \longrightarrow \mathcal{B}'' \longrightarrow 0$$

where $\mathcal{B}' = \mathcal{B}^{[0]}$ and $\mathcal{B}'' = \mathcal{B}^{[0]}/\mathcal{B}$. This exact sequence splits on each stalk (but not globally, nor even locally): a left inverse $\mathcal{B}_x^{[0]} \to \mathcal{B}_x$ of the inclusion is given by $g \mapsto g(x)$. Hence the sequence remains exact after taking the tensor product with $\mathcal{A}$. Now, as $\mathcal{B}'$ is acyclic, the connecting homomorphism $H^{q-1}(X, \mathcal{B}'') \longrightarrow H^q(X, \mathcal{B})$ is onto, so there is $\beta'' \in H^{q-1}(X, \mathcal{B}'')$ such that $\beta = \partial^{q-1}\beta''$. Using (8.5′), (8.5) and the induction hypothesis, we find

$$\begin{aligned}\beta \smile \alpha &= \partial^{p+q-1}(\beta'' \smile \alpha) = \partial^{p+q-1}\big((-1)^{p(q-1)}\, i(\alpha \smile \beta'')\big)\\ &= (-1)^{p(q-1)}\, i\partial^{p+q-1}(\alpha \smile \beta'') = (-1)^{p(q-1)}(-1)^p\, i(\alpha \smile \beta). \qquad \square\end{aligned}$$

Theorem 8.6 shows in particular that $H^\bullet(X, \mathcal{R})$ is a graded associative and supercommutative algebra, i.e. $\beta \smile \alpha = (-1)^{pq}\,\alpha \smile \beta$ for all classes $\alpha \in H^p(X, \mathcal{R})$, $\beta \in H^q(X, \mathcal{R})$. If $\mathcal{A}$ is a $\mathcal{R}$-module, then $H^\bullet(X, \mathcal{A})$ is a graded $H^\bullet(X, \mathcal{R})$-module.

(8.7) Remark. The cup product can also be defined for Čech cochains. Given $c \in C^p(\mathfrak{U}, \mathcal{A})$ and $c' \in C^q(\mathfrak{U}, \mathcal{B})$, the cochain $c \smile c' \in C^{p+q}(\mathfrak{U}, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$ is defined by

$$(c \smile c')_{\alpha_0 \ldots \alpha_{p+q}} = c_{\alpha_0 \ldots \alpha_p} \otimes c'_{\alpha_p \ldots \alpha_{p+q}} \quad \text{on} \quad U_{\alpha_0 \ldots \alpha_{p+q}}.$$

Straightforward calculations show that

$$\delta^{p+q}(c \smile c') = (\delta^p c) \smile c' + (-1)^p\, c \smile (\delta^q c')$$

and that there is a commutative diagram

$$\begin{array}{ccc}\check{H}^p(\mathfrak{U}, \mathcal{A}) \times \check{H}^q(\mathfrak{U}, \mathcal{B}) & \longrightarrow & \check{H}^{p+q}(\mathfrak{U}, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})\\ \downarrow & & \downarrow\\ H^p(X, \mathcal{A}) \times H^q(X, \mathcal{B}) & \longrightarrow & H^{p+q}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}),\end{array}$$

where the vertical arrows are the canonical morphisms $H^s(\lambda^\bullet)$ of (5.14). Note that the commutativity already holds in fact on cochains.

(8.8) Remark. Let Φ and Ψ be families of supports on X. Then $\Phi \cap \Psi$ is again a family of supports, and Formula (8.2) defines a bilinear map

$$(8.9) \quad H^p_\Phi(X, \mathcal{A}) \times H^q_\Psi(X, \mathcal{B}) \longrightarrow H^{p+q}_{\Phi\cap\Psi}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$$

on cohomology groups with supports. This follows immediately from the fact that $\mathrm{Supp}(f \smile g) \subset \mathrm{Supp}\, f \cap \mathrm{Supp}\, g$.

(8.10) Remark. Assume that X is a differentiable manifold. Then the cohomology complex $H^\bullet_{\mathrm{DR}}(X, \mathbb{R})$ has a natural structure of supercommutative algebra given by the wedge product of differential forms. We shall prove the following compatibility statement:

Let $H^q(X, \mathbb{R}) \longrightarrow H^q_{\mathrm{DR}}(X, \mathbb{R})$ be the De Rham-Weil isomorphism given by Formula (6.12)*. Then the cup product $c' \smile c''$ is mapped on the wedge product $f' \wedge f''$ of the corresponding De Rham cohomology classes.*

By remark 8.7, we may suppose that c', c'' are Čech cohomology classes of respective degrees p, q. Formulas (6.11) and (6.12) give

$$f'_{\restriction U_{\nu_p}} = \sum_{\nu_0,\ldots,\nu_{p-1}} c'_{\nu_0\ldots\nu_{p-1}\nu_p}\, d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{p-1}},$$
$$f'' = \sum_{\nu_p,\ldots,\nu_{p+q}} c''_{\nu_p\ldots\nu_{p+q}}\, \psi_{\nu_{p+q}} d\psi_{\nu_p} \wedge \ldots \wedge d\psi_{\nu_{p+q-1}}.$$

We get therefore

$$f' \wedge f'' = \sum_{\nu_0,\ldots,\nu_{p+q}} c'_{\nu_0\ldots\nu_p}\, c''_{\nu_p\ldots\nu_{p+q}}\, \psi_{\nu_{p+q}} d\psi_{\nu_0} \wedge \ldots \wedge \psi_{\nu_{p+q-1}},$$

which is precisely the image of $c \smile c'$ in the De Rham cohomology. □

9. Inverse Images and Cartesian Products

9.A. Inverse Image of a Sheaf

Let $F : X \to Y$ be a continuous map between topological spaces X, Y, and let $\pi : \mathcal{A} \to Y$ be a sheaf of abelian groups. Recall that *inverse image* $F^{-1}\mathcal{A}$ is defined as the sheaf-space

$$F^{-1}\mathcal{A} = \mathcal{A} \times_Y X = \big\{(s, x)\,;\, \pi(s) = F(x)\big\}$$

with projection $\pi' = \mathrm{pr}_2 : F^{-1}\mathcal{A} \to X$. The stalks of $F^{-1}\mathcal{A}$ are given by

(9.1) $\quad (F^{-1}\mathcal{A})_x = \mathcal{A}_{F(x)},$

and the sections $\sigma \in F^{-1}\mathcal{A}(U)$ can be considered as continuous mappings $\sigma : U \to \mathcal{A}$ such that $\pi \circ \sigma = F$. In particular, any section $s \in \mathcal{A}(U)$ has a *pull-back*

(9.2) $\quad F^\star s = s \circ F \in F^{-1}\mathcal{A}\big(F^{-1}(U)\big).$

For any $v \in \mathcal{A}^{[q]}_y$, we define $F^\star v \in (F^{-1}\mathcal{A})^{[q]}_x$ by

(9.3) $\quad F^\star v(x_0, \ldots, x_q) = v\big(F(x_0), \ldots, F(x_q)\big) \in (F^{-1}\mathcal{A})_{x_q} = \mathcal{A}_{F(x_q)}$

for $x_0 \in V(x)$, $x_1 \in V(x_0), \ldots, x_q \in V(x_0, \ldots, x_{q-1})$. We get in this way a morphism of complexes $F^\star : \mathcal{A}^{[\bullet]}(Y) \longrightarrow (F^{-1}\mathcal{A})^{[\bullet]}(X)$. On cohomology groups, we thus have an induced morphism

(9.4) $\quad F^\star \;:\; H^q(Y, \mathcal{A}) \longrightarrow H^q(X, F^{-1}\mathcal{A}).$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X. Thanks to property (9.1), there is an exact sequence

$$0 \longrightarrow F^{-1}\mathcal{A} \longrightarrow F^{-1}\mathcal{B} \longrightarrow F^{-1}\mathcal{C} \longrightarrow 0.$$

It is clear on the definitions that the morphism $F^\star$ in (9.4) commutes with the associated cohomology exact sequences. Also, $F^\star$ preserves the cup product, i.e. $F^\star(\alpha \smile \beta) = F^\star\alpha \smile F^\star\beta$ whenever α, β are cohomology classes with values in sheaves $\mathcal{A}$, $\mathcal{B}$ on X. Furthermore, if $G : Y \to Z$ is a continuous map, we have

$$(9.5) \quad (G \circ F)^\star = F^\star \circ G^\star.$$

(9.6) Remark. Similar definitions can be given for Čech cohomology. If $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is an open covering of Y, then $F^{-1}\mathcal{U} = \big(F^{-1}(U_\alpha)\big)_{\alpha\in I}$ is an open covering of X. For $c \in C^q(\mathcal{U}, \mathcal{A})$, we set

$$(F^\star c)_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} \circ F \in C^q(F^{-1}\mathcal{U}, F^{-1}\mathcal{A}).$$

This definition is obviously compatible with the morphism from Čech cohomology to ordinary cohomology.

(9.7) Remark. Let Φ be a family of supports on Y. We define $F^{-1}\Psi$ to be the family of closed sets $K \subset X$ such that $F(K)$ is contained in some set $L \in \Psi$. Then (9.4) can be generalized in the form

$$(9.8) \quad F^\star \ : \ H^q_\Psi(Y, \mathcal{A}) \longrightarrow H^q_{F^{-1}\Psi}(X, F^{-1}\mathcal{A}).$$

(9.9) Remark. Assume that X and Y are paracompact differentiable manifolds and that $F : X \to Y$ is a C^∞ map. If $(\psi_\alpha)_{\alpha\in I}$ is a partition of unity subordinate to $\mathcal{U}$, then $(\psi_\alpha \circ F)_{\alpha\in I}$ is a partition of unity on X subordinate to $F^{-1}\mathcal{U}$. Let $c \in C^q(\mathcal{U}, \mathbb{R})$. The differential form associated to $F^\star c$ in the De Rham cohomology is

$$\begin{aligned} g &= \sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}(\psi_{\nu_q} \circ F) d(\psi_{\nu_0} \circ F) \wedge \ldots \wedge d(\psi_{\nu_{q-1}} \circ F) \\ &= F^\star\Big(\sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}\, \psi_{\nu_q} d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}}\Big). \end{aligned}$$

Hence we have a commutative diagram

$$\begin{array}{ccccc} H^q_{\mathrm{DR}}(Y, \mathbb{R}) & \xrightarrow{\simeq} & \check{H}^q(Y, \mathbb{R}) & \xrightarrow{\simeq} & H^q(Y, \mathbb{R}) \\ \downarrow F^\star & & \downarrow F^\star & & \downarrow F^\star \\ H^q_{\mathrm{DR}}(X, \mathbb{R}) & \xrightarrow{\simeq} & \check{H}^q(X, \mathbb{R}) & \xrightarrow{\simeq} & H^q(X, \mathbb{R}). \end{array}$$

9.B. Cohomology Groups of a Subspace

Let $\mathcal{A}$ be a sheaf on a topological space X, let S be a subspace of X and $i_S : S \hookrightarrow X$ the inclusion. Then $i_S^{-1}\mathcal{A}$ is the restriction of $\mathcal{A}$ to S, so that $H^q(S, \mathcal{A}) = H^q(S, i_S^{-1}\mathcal{A})$ by definition. For any two subspaces $S' \subset S$, the inclusion of S' in S induces a restriction morphism

$$H^q(S, \mathcal{A}) \longrightarrow H^q(S', \mathcal{A}).$$

(9.10) Theorem. *Let $\mathcal{A}$ be a sheaf on X and S a strongly paracompact subspace in X. When Ω ranges over open neighborhoods of S, we have*

$$H^q(S, \mathcal{A}) = \varinjlim_{\Omega \supset S} H^q(\Omega, \mathcal{A}).$$

Proof. When $q = 0$, the property is equivalent to Prop. 4.7. The general case follows by induction on q if we use the long cohomology exact sequences associated to the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{[0]} \longrightarrow \mathcal{A}^{[0]}/\mathcal{A} \longrightarrow 0$$

on S and on its neighborhoods Ω (note that the restriction of a flabby sheaf to S is soft by Prop. 4.7 and the fact that every closed subspace of a strongly paracompact subspace is strongly paracompact). □

9.C. Cartesian Product

We use here the formalism of inverse images to deduce the cartesian product from the cup product. Let R be a fixed commutative ring and $\mathcal{A} \to X$, $\mathcal{B} \to Y$ sheaves of R-modules. We define the *external tensor product* by

$$(9.11) \quad \mathcal{A} \boxtimes_R \mathcal{B} = \mathrm{pr}_1^{-1}\mathcal{A} \otimes_R \mathrm{pr}_2^{-1}\mathcal{B}$$

where pr_1, pr_2 are the projections of $X \times Y$ onto X, Y respectively. The sheaf $\mathcal{A} \boxtimes_R \mathcal{B}$ is thus the sheaf on $X \times Y$ whose stalks are

$$(9.12) \quad (\mathcal{A} \boxtimes_R \mathcal{B})_{(x,y)} = \mathcal{A}_x \otimes_R \mathcal{B}_y.$$

For all cohomology classes $\alpha \in H^p(X, \mathcal{A})$, $\beta \in H^q(Y, \mathcal{B})$ the *cartesian product* $\alpha \times \beta \in H^{p+q}(X \times Y, \mathcal{A} \boxtimes_R \mathcal{B})$ is defined by

$$(9.13) \quad \alpha \times \beta = (\mathrm{pr}_1^\star \alpha) \smile (\mathrm{pr}_2^\star \beta).$$

More generally, let Φ and Ψ be families of supports in X and Y respectively. If $\Phi \times \Psi$ denotes the family of all closed subsets of $X \times Y$ contained in products $K \times L$ of elements $K \in \Phi$, $L \in \Psi$, the cartesian product defines a R-bilinear map

(9.14) $H^p_\Phi(X,\mathcal{A}) \times H^q_\Psi(Y,\mathcal{B}) \longrightarrow H^{p+q}_{\Phi\times\Psi}(X\times Y,\mathcal{A}\boxtimes_R \mathcal{B})$.

If $\mathcal{A}' \to X$, $\mathcal{B}' \to Y$ are sheaves of abelian groups and if α', β' are cohomology classes of degree p', q' with values in $\mathcal{A}'$, $\mathcal{B}'$, one gets easily

(9.15) $(\alpha\times\beta)\smile(\alpha'\times\beta') = (-1)^{qp'}(\alpha\smile\alpha')\times(\beta\smile\beta')$.

Furthermore, if $F : X' \to X$ and $G : Y' \to Y$ are continuous maps, then

(9.16) $(F\times G)^\star(\alpha\times\beta) = (F^\star\alpha)\times(G^\star\beta)$.

10. Spectral Sequence of a Filtered Complex

10.A. Construction of the Spectral Sequence

The theory of spectral sequences consists essentially in computing the homology groups of a differential module (K,d) by "successive approximations", once a filtration $F_p(K)$ is given in K and the cohomology groups of the graded modules $G_p(K)$ are known. Let us first recall some standard definitions and notations concerning filtrations.

(10.1) Definition. *Let R be a commutative ring. A filtration of a R-module M is a sequence of submodules $M_p \subset M$, $p\in\mathbb{Z}$, also denoted $M_p = F_p(M)$, such that $M_{p+1}\subset M_p$ for all $p\in\mathbb{Z}$, $\bigcup M_p = M$ and $\bigcap M_p = \{0\}$. The associated graded module is*

$$G(M) = \bigoplus_{p\in\mathbb{Z}} G_p(M), \qquad G_p(M) = M_p/M_{p+1}.$$

Let (K,d) be a differential module equipped with a filtration (K_p) by differential submodules (i.e. $dK_p\subset K_p$ for every p). For any number $r\in\mathbb{N}\cup\{\infty\}$, we define Z^p_r, $B^p_r \subset G_p(K) = K_p/K_{p+1}$ by

$$Z^p_r = K_p\cap d^{-1}K_{p+r} \mod K_{p+1}, \quad Z^p_\infty = K_p\cap d^{-1}\{0\} \mod K_{p+1}, \tag{10.2}$$

$$B^p_r = K_p\cap dK_{p-r+1} \mod K_{p+1}, \quad B^p_\infty = K_p\cap dK \mod K_{p+1}. \tag{10.2$'$}$$

(10.3) Lemma. *For every p and r, there are inclusions*

$$\ldots\subset B^p_r\subset B^p_{r+1}\subset\ldots\subset B^p_\infty\subset Z^p_\infty\subset\ldots\subset Z^p_{r+1}\subset Z^p_r\subset\ldots$$

and the differential d induces an isomorphism

$$\widetilde{d} \;:\; Z^p_r/Z^p_{r+1}\longrightarrow B^{p+r}_{r+1}/B^{p+r}_r.$$

Proof. It is clear that (Z_r^p) decreases with r, that (B_r^p) increases with r, and that $B_\infty^p \subset Z_\infty^p$. By definition

$$Z_r^p = (K_p \cap d^{-1}K_{p+r})/(K_{p+1} \cap d^{-1}K_{p+r}),$$
$$B_r^p = (K_p \cap dK_{p-r+1})/(K_{p+1} \cap dK_{p-r+1}).$$

The differential d induces a morphism

$$Z_r^p \longrightarrow (dK_p \cap K_{p+r})/(dK_{p+1} \cap K_{p+r})$$

whose kernel is $(K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\}) = Z_\infty^p$, whence isomorphisms

$$\widehat{d}\ :\ Z_r^p/Z_\infty^p \longrightarrow (K_{p+r} \cap dK_p)/(K_{p+r} \cap dK_{p+1}),$$
$$\widetilde{d}\ :\ Z_r^p/Z_{r+1}^p \longrightarrow (K_{p+r} \cap dK_p)/(K_{p+r} \cap dK_{p+1} + K_{p+r+1} \cap dK_p).$$

The right hand side of the last arrow can be identified to B_{r+1}^{p+r}/B_r^{p+r}, for

$$B_r^{p+r} = (K_{p+r} \cap dK_{p+1})/(K_{p+r+1} \cap dK_{p+1}),$$
$$B_{r+1}^{p+r} = (K_{p+r} \cap dK_p)/(K_{p+r+1} \cap dK_p). \qquad \square$$

Now, for each $r \in \mathbb{N}$, we define a complex $E_r^\bullet = \bigoplus_{p\in\mathbb{Z}} E_r^p$ with a differential $d_r : E_r^p \longrightarrow E_r^{p+r}$ of degree r as follows: we set $E_r^p = Z_r^p/B_r^p$ and take

$$(10.4)\quad d_r\ :\ Z_r^p/B_r^p \longrightarrow\!\!\!\!\rightarrow Z_r^p/Z_{r+1}^p \xrightarrow{\ \widetilde{d}\ } B_{r+1}^{p+r}/B_r^{p+r} \hookrightarrow Z_r^{p+r}/B_r^{p+r}$$

where the first arrow is the obvious projection and the third arrow the obvious inclusion. Since d_r is induced by d, we actually have $d_r \circ d_r = 0$; this can also be seen directly by the fact that $B_{r+1}^{p+r} \subset Z_{r+1}^{p+r}$.

(10.5) Theorem and definition. *There is a canonical isomorphism $E_{r+1}^\bullet \simeq H^\bullet(E_r^\bullet)$. The sequence of differential complexes $(E_r^\bullet, d_r^\bullet)$ is called the spectral sequence of the filtered differential module (K, d).*

Proof. Since $\widetilde{d}$ is an isomorphism in (10.4), we have

$$\ker d_r = Z_{r+1}^p/B_r^p, \qquad \mathrm{Im}\, d_r = B_{r+1}^{p+r}/B_r^{p+r}.$$

Hence the image of $d_r : E_r^{p-r} \longrightarrow E_r^p$ is B_{r+1}^p/B_r^p and

$$H^p(E_r^\bullet) = (Z_{r+1}^p/B_r^p)/(B_{r+1}^p/B_r^p) \simeq Z_{r+1}^p/B_{r+1}^p = E_{r+1}^p. \qquad \square$$

(10.6) Theorem. *Consider the filtration of the homology module $H(K)$ defined by*

$$F_p\big(H(K)\big) = \mathrm{Im}\,\big(H(K_p) \longrightarrow H(K)\big).$$

Then there is a canonical isomorphism

$$E_\infty^p = G_p\big(H(K)\big).$$

Proof. Clearly $F_p\big(H(K)\big) = (K_p \cap d^{-1}\{0\})/(K_p \cap dK)$, whereas

$$Z_\infty^p = (K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\}), \quad B_\infty^p = (K_p \cap dK)/(K_{p+1} \cap dK),$$
$$E_\infty^p = Z_\infty^p/B_\infty^p = (K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\} + K_p \cap dK).$$

It follows that $E_\infty^p \simeq F_p\big(H(K)\big)/F_{p+1}\big(H(K)\big)$. □

In most applications, the differential module K has a natural grading compatible with the filtration. Let us consider for example the case of a cohomology complex $K^\bullet = \bigoplus_{l\in\mathbb{Z}} K^l$. The filtration $K_p^\bullet = F_p(K^\bullet)$ is said to be *compatible* with the differential complex structure if each $K_p^\bullet$ is a subcomplex of $K^\bullet$, i.e.

$$K_p^\bullet = \bigoplus_{l\in\mathbb{Z}} K_p^l$$

where (K_p^l) is a filtration of K^l. Then we define $Z_r^{p,q}$, $B_r^{p,q}$, $E_r^{p,q}$ to be the sets of elements of Z_r^p, B_r^p, E_r^p of total degree $p+q$. Therefore

$$\begin{aligned}
&(10.7) && Z_r^{p,q} = K_p^{p+q} \cap d^{-1}K_{p+r}^{p+q+1} \mod K_{p+1}^{p+q}, && Z_r^p = \bigoplus Z_r^{p,q},\\
&(10.7') && B_r^{p,q} = K_p^{p+q} \cap dK_{p-r+1}^{p+q-1} \mod K_{p+1}^{p+q}, && B_r^p = \bigoplus B_r^{p,q},\\
&(10.7'') && E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}, && E_r^p = \bigoplus E_r^{p,q},
\end{aligned}$$

and the differential d_r has bidegree $(r, -r+1)$, i.e.

$$(10.8) \quad d_r \;:\; E_r^{p,q} \longrightarrow E_r^{p+r\,,\,q-r+1}.$$

For an element of pure bidegree (p,q), p is called the *filtering degree*, q the *complementary degree* and $p+q$ the *total degree.*

(10.9) Definition. *A filtration $(K_p^\bullet)$ of a complex $K^\bullet$ is said to be regular if for each degree l there are indices $\nu(l) \le N(l)$ such that $K_p^l = K^l$ for $p < \nu(l)$ and $K_p^l = 0$ for $p > N(l)$.*

If the filtration is regular, then (10.7) and (10.7′) show that

$$\begin{aligned}
Z_r^{p,q} &= Z_{r+1}^{p,q} = \ldots = Z_\infty^{p,q} \quad \text{for } r > N(p+q+1) - p,\\
B_r^{p,q} &= B_{r+1}^{p,q} = \ldots = B_\infty^{p,q} \quad \text{for } r > p+1-\nu(p+q-1),
\end{aligned}$$

therefore $E_r^{p,q} = E_\infty^{p,q}$ for $r \ge r_0(p,q)$. We say that the spectral sequence *converges* to its limit term, and we write symbolically

$$(10.10) \quad E_r^{p,q} \Longrightarrow H^{p+q}(K^\bullet)$$

to express the following facts: there is a spectral sequence whose terms of the r-th generation are $E_r^{p,q}$, the sequence converges to a limit term $E_\infty^{p,q}$, and $E_\infty^{p,l-p}$ is the term $G_p\big(H^l(K^\bullet)\big)$ in the graded module associated to some filtration of $H^l(K^\bullet)$.

(10.11) Definition. *The spectral sequence is said to collapse in $E_r^\bullet$ if all terms $Z_k^{p,q}$, $B_k^{p,q}$, $E_k^{p,q}$ are constant for $k \geq r$, or equivalently if $d_k = 0$ in all bidegrees for $k \geq r$.*

(10.12) Special case. Assume that there exists an integer $r \geq 2$ and an index q_0 such that $E_r^{p,q} = 0$ for $q \neq q_0$. Then this property remains true for larger values of r, and we must have $d_r = 0$. It follows that the spectral sequence collapses in $E_r^\bullet$ and that

$$H^l(K^\bullet) = E_r^{l-q_0,q_0}.$$

Similarly, if $E_r^{p,q} = 0$ for $p \neq p_0$ and some $r \geq 1$ then

$$H^l(K^\bullet) = E_r^{p_0,l-p_0}. \qquad \square$$

10.B. Computation of the First Terms

Consider an arbitrary spectral sequence. For $r = 0$, we have $Z_0^p = K_p/K_{p+1}$, $B_0^p = \{0\}$, thus

$$(10.13) \quad E_0^p = K_p/K_{p+1} = G_p(K).$$

The differential d_0 is induced by d on the quotients, and

$$(10.14) \quad E_1^p = H\big(G_p(K)\big).$$

Now, there is a short exact sequence of differential modules

$$0 \longrightarrow G_{p+1}(K) \longrightarrow K_p/K_{p+2} \longrightarrow G_p(K) \longrightarrow 0.$$

We get therefore a connecting homomorphism

$$(10.15) \quad E_1^p = H\big(G_p(K)\big) \xrightarrow{\ \partial\ } H\big(G_{p+1}(K)\big) = E_1^{p+1}.$$

We claim that ∂ coincides with the differential d_1 : indeed, both are induced by d. When $K^\bullet$ is a filtered cohomology complex, d_1 is the connecting homomorphism

$$(10.16) \quad E_1^{p,q} = H^{p+q}\big(G_p(K^\bullet)\big) \xrightarrow{\ \partial\ } H^{p+q+1}\big(G_{p+1}(K^\bullet)\big) = E_1^{p+1,q}.$$

11. Spectral Sequence of a Double Complex

A double complex is a bigraded module $K^{\bullet,\bullet} = \bigoplus K^{p,q}$ together with a differential $d = d' + d''$ such that

$$(11.1) \quad d' : K^{p,q} \longrightarrow K^{p+1,q}, \quad d'' : K^{p,q+1} \longrightarrow K^{p,q+1},$$

and $d \circ d = 0$. In particular, d' and d'' satisfy the relations

$$(11.2) \quad d'^2 = d''^2 = 0, \quad d'd'' + d''d' = 0.$$

The *simple complex associated* to $K^{\bullet,\bullet}$ is defined by

$$K^l = \bigoplus_{p+q=l} K^{p,q}$$

together with the differential d. We will suppose here that both graduations of $K^{\bullet,\bullet}$ are positive, i.e. $K^{p,q} = 0$ for $p < 0$ or $q < 0$. The *first filtration* of $K^\bullet$ is defined by

$$(11.3) \quad K^l_p = \bigoplus_{i+j=l,\ i\geq p} K^{i,j} = \bigoplus_{p\leq i\leq l} K^{i,l-i}.$$

The graded module associated to this filtration is of course $G_p(K^l) = K^{p,l-p}$, and the differential induced by d on the quotient coincides with d'' because d' takes K^l_p to K^{l+1}_{p+1}. Thus we have a spectral sequence beginning by

$$(11.4) \quad E_0^{p,q} = K^{p,q}, \quad d_0 = d'', \quad E_1^{p,q} = H^q_{d''}(K^{p,\bullet}).$$

By (10.16), d_1 is the connecting homomorphism associated to the short exact sequence

$$0 \longrightarrow K^{p+1,\bullet} \longrightarrow K^{p,\bullet} \oplus K^{p+1,\bullet} \longrightarrow K^{p,\bullet} \longrightarrow 0$$

where the differential is given by (d mod $K^{p+2,\bullet}$) for the central term and by d'' for the two others. The definition of the connecting homomorphism in the proof of Th. 1.5 shows that

$$d_1 = \partial : \quad H^q_{d''}(K^{p,\bullet}) \longrightarrow H^q_{d''}(K^{p+1,\bullet})$$

is induced by d'. Consequently, we find

$$(11.5) \quad E_2^{p,q} = H^p_{d'}(E_1^{\bullet,q}) = H^p_{d'}\big(H^q_{d''}(K^{\bullet,\bullet})\big).$$

For such a spectral sequence, several interesting additional features can be pointed out. For all r and l, there is an injective homomorphism

$$E^{0,l}_{r+1} \hookrightarrow E^{0,l}_r$$

whose image can be identified with the set of d_r-cocycles in $E_r^{0,l}$; the coboundary group is zero because $E_r^{p,q} = 0$ for $q < 0$. Similarly, $E_r^{l,0}$ is equal to its cocycle submodule, and there is a surjective homomorphism

$$E_r^{l,0} \twoheadrightarrow E_{r+1}^{l,0} \simeq E_r^{l,0}/d_r E_r^{l-r,r-1}.$$

Furthermore, the filtration on $H^l(K^\bullet)$ begins at $p = 0$ and stops at $p = l$, i.e.

$$(11.6) \quad F_0\big(H^l(K^\bullet)\big) = H^l(K^\bullet), \quad F_p\big(H^l(K^\bullet)\big) = 0 \quad \text{for} \quad p > l.$$

Therefore, there are canonical maps

$$(11.7) \quad \begin{aligned} &H^l(K^\bullet) \twoheadrightarrow G_0\big(H^l(K^\bullet)\big) = E_\infty^{0,l} \hookrightarrow E_r^{0,l}, \\ &E_r^{l,0} \twoheadrightarrow E_\infty^{l,0} = G_l\big(H^l(K^\bullet)\big) \hookrightarrow H^l(K^\bullet). \end{aligned}$$

These maps are called the *edge homomorphisms* of the spectral sequence.

(11.8) Theorem. *There is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(K^\bullet)$$

where the non indicated arrows are edge homomorphisms.

Proof. By 11.6, the graded module associated to $H^1(K^\bullet)$ has only two components, and we have an exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

However $E_\infty^{1,0} = E_2^{1,0}$ because all differentials d_r starting from $E_r^{1,0}$ or abuting to $E_r^{1,0}$ must be zero for $r \geq 2$. Similarly, $E_\infty^{0,1} = E_3^{0,1}$ and $E_\infty^{2,0} = E_3^{2,0}$, thus there is an exact sequence

$$0 \longrightarrow E_\infty^{0,1} \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0.$$

A combination of the two above exact sequences yields

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0.$$

Taking into account the injection $E_\infty^{2,0} \hookrightarrow H^2(K^\bullet)$ in (11.7), we get the required exact sequence. □

(11.9) Example. Let X be a complex manifold of dimension n. Consider the double complex $K^{p,q} = C^\infty_{p,q}(X, \mathbb{C})$ together with the exterior derivative $d = d' + d''$. Then there is a spectral sequence which starts from the Dolbeault cohomology groups

$$E_1^{p,q} = H^{p,q}(X, \mathbb{C})$$

and which converges to the graded module associated to a filtration of the De Rham cohomology groups:

$$E_r^{p,q} \Longrightarrow H_{\mathrm{DR}}^{p+q}(X, \mathbb{C}).$$

This spectral sequence is called the *Hodge-Frölicher spectral sequence* (Frölicher 1955). We will study it in much more detail in chapter 6 when X is compact. □

Frequently, the spectral sequence under consideration can be obtained from two distinct double complexes and one needs to compare the final cohomology groups. The following lemma can often be applied.

(11.10) Lemma. *Let $K^{p,q} \longrightarrow L^{p,q}$ be a morphism of double complexes (i.e. a double sequence of maps commuting with d' and d''). Then there are induced morphisms*

$$_KE_r^{\bullet,\bullet} \longrightarrow {}_LE_r^{\bullet,\bullet}, \qquad r \geq 0$$

of the associated spectral sequences. If one of these morphisms is an isomorphism for some r, then $H^l(K^\bullet) \longrightarrow H^l(L^\bullet)$ is an isomorphism.

Proof. If the r-terms are isomorphic, they have the same cohomology groups, thus $_KE_{r+1}^{\bullet,\bullet} \simeq {}_LE_{r+1}^{\bullet,\bullet}$ and $_KE_\infty^{\bullet,\bullet} \simeq {}_LE_\infty^{\bullet,\bullet}$ in the limit. The lemma follows from the fact that if a morphism of graded modules $\varphi : M \longrightarrow M'$ induces an isomorphism $G_\bullet(M) \longrightarrow G_\bullet(M')$, then φ is an isomorphism. □

12. Hypercohomology Groups

Let $(\mathcal{L}^\bullet, \delta)$ be a complex of sheaves

$$0 \longrightarrow \mathcal{L}^0 \xrightarrow{\delta^0} \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{\delta^q} \cdots$$

on a topological space X. We denote by $\mathcal{H}^q = \mathcal{H}^q(\mathcal{L}^\bullet)$ the q-th sheaf of cohomology of this complex; thus $\mathcal{H}^q$ is a sheaf of abelian groups over X. Our goal is to define "generalized cohomology groups" attached to $\mathcal{L}^\bullet$ on X, in such a way that these groups only depend on the cohomology sheaves $\mathcal{H}^q$. For this, we associate to $\mathcal{L}^\bullet$ the double complex of groups

$$(12.1) \quad K_{\mathcal{L}}^{p,q} = (\mathcal{L}^q)^{[p]}(X)$$

with differential $d' = d^p$ given by (2.5), and with $d'' = (-1)^p(\delta^q)^{[p]}$. As $(\delta^q)^{[\bullet]} : (\mathcal{L}^q)^{[\bullet]} \longrightarrow (\mathcal{L}^{q+1})^{[\bullet]}$ is a morphism of complexes, we get the expected relation $d'd'' + d''d' = 0$.

(12.2) Definition. *The groups $H^q(K^\bullet_{\mathcal{L}})$ are called the hypercohomology groups of $\mathcal{L}^\bullet$ and are denoted $\mathbb{H}^q(X,\mathcal{L}^\bullet)$.*

Clearly $\mathbb{H}^0(X,\mathcal{L}^\bullet)=\mathcal{H}^0(X)$ where $\mathcal{H}^0=\ker\delta^0$ is the first cohomology sheaf of $\mathcal{L}^\bullet$. If $\varphi^\bullet:\mathcal{L}^\bullet\longrightarrow\mathcal{M}^\bullet$ is a morphism of sheaf complexes, there is an associated morphism of double complexes $\varphi^{\bullet,\bullet}:K^{\bullet,\bullet}_{\mathcal{L}}\longrightarrow K^{\bullet,\bullet}_{\mathcal{M}}$, hence a natural morphism

$$(12.3)\quad \mathbb{H}^q(\varphi^\bullet)\ :\ \mathbb{H}^q(X,\mathcal{L}^\bullet)\longrightarrow\mathbb{H}^q(X,\mathcal{M}^\bullet).$$

We first list a few immediate properties of hypercohomology groups, whose proofs are left to the reader.

(12.4) Proposition. *The following properties hold:*

a) *If $\mathcal{L}^q=0$ for $q\neq 0$, then $\mathbb{H}^q(X,\mathcal{L}^\bullet)=H^q(X,\mathcal{L}^0)$.*

b) *If $\mathcal{L}^\bullet[s]$ denotes the complex $\mathcal{L}^\bullet$ shifted of s indices to the right, i.e. $\mathcal{L}^\bullet[s]^q=\mathcal{L}^{q-s}$, then $\mathbb{H}^q(X,\mathcal{L}^\bullet[s])=\mathbb{H}^{q-s}(X,\mathcal{L}^\bullet)$.*

c) *If $0\longrightarrow\mathcal{L}^\bullet\longrightarrow\mathcal{M}^\bullet\longrightarrow\mathcal{N}^\bullet\longrightarrow 0$ is an exact sequence of sheaf complexes, there is a long exact sequence*

$$\cdots\mathbb{H}^q(X,\mathcal{L}^\bullet)\longrightarrow\mathbb{H}^q(X,\mathcal{M}^\bullet)\longrightarrow\mathbb{H}^q(X,\mathcal{N}^\bullet)\xrightarrow{\partial}\mathbb{H}^{q+1}(X,\mathcal{L}^\bullet)\cdots.\ \square$$

If $\mathcal{L}^\bullet$ is a sheaf complex, the spectral sequence associated to the first filtration of $K^\bullet_{\mathcal{L}}$ is given by

$$E_1^{p,q}=H^q_{d''}(K^{p,\bullet}_{\mathcal{L}})=H^q\big((\mathcal{L}^\bullet)^{[p]}(X)\big).$$

However by (2.9) the functor $\mathcal{A}\longmapsto\mathcal{A}^{[p]}(X)$ preserves exact sequences. Therefore, we get

$$(12.5)\quad E_1^{p,q}=\big(\mathcal{H}^q(\mathcal{L}^\bullet)\big)^{[p]}(X),$$

$$(12.5')\quad E_2^{p,q}=H^p\big(X,\mathcal{H}^q(\mathcal{L}^\bullet)\big),$$

since $E_2^{p,q}=H^p_{d'}(E_1^{\bullet,q})$. If $\varphi^\bullet:\mathcal{L}^\bullet\longrightarrow\mathcal{M}^\bullet$ is a morphism, an application of Lemma 11.10 to the E_2-term of the associated first spectral sequences of $K^{\bullet,\bullet}_{\mathcal{L}}$ and $K^{\bullet,\bullet}_{\mathcal{M}}$ yields:

(12.6) Corollary. *If $\varphi^\bullet:\mathcal{L}^\bullet\longrightarrow\mathcal{M}^\bullet$ is a quasi-isomorphism $\big($this means that $\varphi^\bullet$ induces an isomorphism $\mathcal{H}^\bullet(\mathcal{L}^\bullet)\longrightarrow\mathcal{H}^\bullet(\mathcal{M}^\bullet)$ $\big)$, then*

$$\mathbb{H}^l(\varphi^\bullet)\ :\ \mathbb{H}^l(X,\mathcal{L}^\bullet)\longrightarrow\mathbb{H}^l(X,\mathcal{M}^\bullet)$$

is an isomorphism.

Now, we may reverse the roles of the indices p,q and of the differentials d', d''. The *second filtration* $F_p(K^l_{\mathcal{L}}) = \bigoplus_{j\geq p} K^{l-j,j}_{\mathcal{L}}$ is associated to a spectral sequence such that $\widetilde{E}^{p,q}_1 = H^q_{d'}(K^{\bullet,p}_{\mathcal{L}}) = H^q_{d'}\big((\mathcal{L}^p)^{[\bullet]}(X)\big)$, hence

(12.7) $\widetilde{E}^{p,q}_1 = H^q(X, \mathcal{L}^p)$,

(12.7′) $\widetilde{E}^{p,q}_2 = H^p_\delta\big(H^q(X, \mathcal{L}^\bullet)\big)$.

These two spectral sequences converge to limit terms which are the graded modules associated to filtrations of $\mathbb{H}^\bullet(X, \mathcal{L}^\bullet)$; these filtrations are in general different. Let us mention two interesting special cases.

• Assume first that the complex $\mathcal{L}^\bullet$ is a resolution of a sheaf $\mathcal{A}$, so that $\mathcal{H}^0 = \mathcal{A}$ and $\mathcal{H}^q = 0$ for $q \geq 1$. Then we find

$$E^{p,0}_2 = H^p(X, \mathcal{A}), \qquad E^{p,q}_2 = 0 \quad \text{for} \quad q \geq 1.$$

It follows that the first spectral sequence collapses in $E^\bullet_2$, and 10.12 implies

(12.8) $\mathbb{H}^l(X, \mathcal{L}^\bullet) \simeq H^l(X, \mathcal{A})$.

• Now, assume that the sheaves $\mathcal{L}^q$ are acyclic. The second spectral sequence gives

$$\widetilde{E}^{p,0}_2 = H^p\big(\mathcal{L}^\bullet(X)\big), \qquad \widetilde{E}^{p,q}_2 = 0 \quad \text{for} \quad q \geq 1,$$

(12.9) $$\mathbb{H}^l(X, \mathcal{L}^\bullet) \simeq H^l\big(\mathcal{L}^\bullet(X)\big).$$

If both conditions hold, i.e. if $\mathcal{L}^\bullet$ is a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, then (12.8) and (12.9) can be combined to obtain a new proof of the De Rham-Weil isomorphism $H^l(X, \mathcal{A}) \simeq H^l\big(\mathcal{L}^\bullet(X)\big)$.

13. Direct Images and the Leray Spectral Sequence

13.A. Direct Images of a Sheaf

Let X, Y be topological spaces, $F : X \to Y$ a continuous map and $\mathcal{A}$ a sheaf of abelian groups on X. Recall that the *direct image* $F_\star\mathcal{A}$ is the presheaf on Y defined for any open set $U \subset Y$ by

(13.1) $(F_\star\mathcal{A})(U) = \mathcal{A}\big(F^{-1}(U)\big)$.

Axioms (II-2.4′ and (II-2.4″) are clearly satisfied, thus $F_\star\mathcal{A}$ is in fact a sheaf. The following result is obvious:

(13.2) $\mathcal{A}$ is flabby $\implies$ $F_\star\mathcal{A}$ is flabby.

Every sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ induces a corresponding morphism

$$F_\star\varphi \ : \quad F_\star\mathcal{A} \longrightarrow F_\star\mathcal{B},$$

so $F_\star$ is a functor on the category of sheaves on X to the category of sheaves on Y. This functor is exact on the left: indeed, to every exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is associated an exact sequence

$$0 \longrightarrow F_\star\mathcal{A} \longrightarrow F_\star\mathcal{B} \longrightarrow F_\star\mathcal{C},$$

but $F_\star\mathcal{B} \to F_\star\mathcal{C}$ need not be onto if $\mathcal{B} \to \mathcal{C}$ is. All this follows immediately from the considerations of §3. In particular, the simplicial flabby resolution $(\mathcal{A}^{[\bullet]}, d)$ yields a complex of sheaves

$$(13.3) \quad 0 \longrightarrow F_\star\mathcal{A}^{[0]} \longrightarrow F_\star\mathcal{A}^{[1]} \longrightarrow \cdots \longrightarrow F_\star\mathcal{A}^{[q]} \xrightarrow{F_\star d^q} F_\star\mathcal{A}^{[q+1]} \longrightarrow \cdots.$$

(13.4) Definition. *The q-th direct image of $\mathcal{A}$ by F is the q-th cohomology sheaf of the sheaf complex* (13.3). *It is denoted*

$$R^qF_\star\mathcal{A} = \mathcal{H}^q(F_\star\mathcal{A}^{[\bullet]}).$$

As $F_\star$ is exact on the left, the sequence $0 \to F_\star\mathcal{A} \to F_\star\mathcal{A}^{[0]} \to F_\star\mathcal{A}^{[1]}$ is exact, thus

$$(13.5) \quad R^0F_\star\mathcal{A} = F_\star\mathcal{A}.$$

We now compute the stalks of $R^qF_\star\mathcal{A}$. As the kernel or cokernel of a sheaf morphism is obtained stalk by stalk, we have

$$(R^qF_\star\mathcal{A})_y = H^q\big((F_\star\mathcal{A}^{[\bullet]})_y\big) = \varinjlim_{U\ni y} \ H^q\big(F_\star\mathcal{A}^{[\bullet]}(U)\big).$$

The very definition of $F_\star$ and of sheaf cohomology groups implies

$$H^q\big(F_\star\mathcal{A}^{[\bullet]}(U)\big) = H^q\big(\mathcal{A}^{[\bullet]}(F^{-1}(U))\big) = H^q\big(F^{-1}(U), \mathcal{A}\big),$$

hence we find

$$(13.6) \quad (R^qF_\star\mathcal{A})_y = \varinjlim_{U\ni y} \ H^q\big(F^{-1}(U), \mathcal{A}\big),$$

i.e. $R^qF_\star\mathcal{A}$ is the sheaf associated to the presheaf $U \mapsto H^q\big(F^{-1}(U), \mathcal{A}\big)$. We must stress here that the stronger relation $R^qF_\star\mathcal{A}(U) = H^q\big(F^{-1}(U), \mathcal{A}\big)$ need not be true in general. If the fiber $F^{-1}(y)$ is strongly paracompact in X and if the family of open sets $F^{-1}(U)$ is a fundamental family of neighborhoods of $F^{-1}(y)$ (this situation occurs for example if X and Y are locally compact spaces and F a proper map, or if $F = \mathrm{pr}_1 : X = Y \times S \longrightarrow Y$ where S is compact), Th. 9.10 implies the more natural relation

$$(13.6') \quad (R^qF_\star\mathcal{A})_y = H^q\big(F^{-1}(y), \mathcal{A}\big).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X. Apply the long exact sequence of cohomology on every open set $F^{-1}(U)$ and take the direct limit over U. We get an exact sequence of sheaves:

$$\begin{array}{c} 0 \longrightarrow F_\star\mathcal{A} \longrightarrow F_\star\mathcal{B} \longrightarrow F_\star\mathcal{C} \longrightarrow R^1F_\star\mathcal{A} \longrightarrow \cdots \\ \cdots \longrightarrow R^qF_\star\mathcal{A} \longrightarrow R^qF_\star\mathcal{B} \longrightarrow R^qF_\star\mathcal{C} \longrightarrow R^{q+1}F_\star\mathcal{A} \longrightarrow \cdots . \end{array} \tag{13.7}$$

13.B. Leray Spectral Sequence

For any continuous map $F : X \to Y$, the Leray spectral sequence relates the cohomology groups of a sheaf $\mathcal{A}$ on X and those of its direct images $R^qF_\star\mathcal{A}$ on Y. Consider the two spectral sequences $E_r^\bullet$, $\widetilde{E}_r^\bullet$ associated with the complex of sheaves $\mathcal{L}^\bullet = F_\star\mathcal{A}^{[\bullet]}$ on Y, as in § 12. By definition we have $\mathcal{H}^q(\mathcal{L}^\bullet) = R^qF_\star\mathcal{A}$. By (12.5′) the second term of the first spectral sequence is

$$E_2^{p,q} = H^p(Y, R^qF_\star\mathcal{A}),$$

and this spectral sequence converges to the graded module associated to a filtration of $\mathbb{H}^l(Y, F_\star\mathcal{A}^{[\bullet]})$. On the other hand, (13.2) implies that $F_\star\mathcal{A}^{[q]}$ is flabby. Hence, the second special case (12.9) can be applied:

$$\mathbb{H}^l(Y, F_\star\mathcal{A}^{[\bullet]}) \simeq H^l\big(F_\star\mathcal{A}^{[\bullet]}(Y)\big) = H^l\big(\mathcal{A}^{[\bullet]}(X)\big) = H^l(X, \mathcal{A}).$$

We may conclude this discussion by the following

(13.8) Theorem. *For any continuous map $F : X \to Y$ and any sheaf $\mathcal{A}$ of abelian groups on X, there exists a spectral sequence whose $E_2^\bullet$ term is*

$$E_2^{p,q} = H^p(Y, R^qF_\star\mathcal{A}),$$

which converges to a limit term $E_\infty^{p,l-p}$ equal to the graded module associated with a filtration of $H^l(X, \mathcal{A})$. The edge homomorphism

$$H^l(Y, F_\star\mathcal{A}) \longrightarrow\!\!\!\!\rightarrow E_\infty^{l,0} \hookrightarrow H^l(X, \mathcal{A})$$

coincides with the composite morphism

$$F^\# : \quad H^l(Y, F_\star\mathcal{A}) \xrightarrow{F^\star} H^l(X, F^{-1}F_\star\mathcal{A}) \xrightarrow{H^l(\mu_F)} H^l(X, \mathcal{A})$$

where $\mu_F : F^{-1}F_\star\mathcal{A} \longrightarrow \mathcal{A}$ is the canonical sheaf morphism.

Proof. Only the last statement remains to be proved. The morphism μ_F is defined as follows: every element $s \in (F^{-1}F_\star\mathcal{A})_x = (F_\star\mathcal{A})_{F(x)}$ is the germ of a section $\widetilde{s} \in F_\star\mathcal{A}(V) = \mathcal{A}\big(F^{-1}(V)\big)$ on a neighborhood V of $F(x)$. Then $F^{-1}(V)$ is a neighborhood of x and we let $\mu_F s$ be the germ of $\widetilde{s}$ at x.

Now, we observe that to any commutative diagram of topological spaces and continuous maps is associated a commutative diagram involving the direct image sheaves and their cohomology groups:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ X' & \xrightarrow{F'} & Y' \end{array} \qquad\qquad \begin{array}{ccc} H^l(X,\mathcal{A}) & \xleftarrow{F^\#} & H^l(Y,F_\star\mathcal{A}) \\ {\scriptstyle G^\#}\uparrow & & \uparrow{\scriptstyle H^\#} \\ H^l(X',G_\star\mathcal{A}) & \xleftarrow{F'^\#} & H^l(Y',F'_\star G_\star\mathcal{A}). \end{array}$$

There is a similar commutative diagram in which $F^\#$ and $F'^\#$ are replaced by the edge homomorphisms of the spectral sequences of F and F' : indeed there is a natural morphism $H^{-1}F'_\star\mathcal{B} \longrightarrow F_\star G^{-1}\mathcal{B}$ for any sheaf $\mathcal{B}$ on X', so we get a morphism of sheaf complexes

$$H^{-1}F'_\star(G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star G^{-1}(G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star(G^{-1}G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star\mathcal{A}^{[\bullet]},$$

hence also a morphism of the spectral sequences associated to both ends.

The special case $X'=Y'=Y$, $G=F$, $F'=H=\mathrm{Id}_Y$ then shows that our statement is true for F if it is true for F'. Hence we may assume that F is the identity map; in this case, we need only show that the edge homomorphism of the spectral sequence of $F_\star\mathcal{A}^{[\bullet]}=\mathcal{A}^{[\bullet]}$ is the identity map. This is an immediate consequence of the fact that we have a quasi-isomorphism

$$(\cdots \to 0 \to \mathcal{A} \to 0 \to \cdots) \longrightarrow \mathcal{A}^{[\bullet]}. \qquad \square$$

(13.9) Corollary. *If $R^qF_\star\mathcal{A}=0$ for $q\geq 1$, there is an isomorphism $H^l(Y,F_\star\mathcal{A})\simeq H^l(X,\mathcal{A})$ induced by $F^\#$.*

Proof. We are in the special case 10.12 with $E_2^{p,q}=0$ for $q\neq 0$, so

$$H^l(Y,F_\star\mathcal{A}) = E_2^{l,0} \simeq H^l(X,\mathcal{A}). \qquad \square$$

(13.10) Corollary. *Let $F: X \longrightarrow Y$ be a proper finite-to-one map. For any sheaf $\mathcal{A}$ on X, we have $R^qF_\star\mathcal{A}=0$ for $q\geq 1$ and there is an isomorphism $H^l(Y,F_\star\mathcal{A})\simeq H^l(X,\mathcal{A})$.*

Proof. By definition of higher direct images, we have

$$(R^qF_\star\mathcal{A})_y = \varinjlim_{U\ni y} H^q\big(\mathcal{A}^{[\bullet]}\big(F^{-1}(U)\big)\big).$$

If $F^{-1}(y)=\{x_1,\ldots,x_m\}$, the assumptions imply that $\big(F^{-1}(U)\big)$ is a fundamental system of neighborhoods of $\{x_1,\ldots,x_m\}$. Therefore

$$(R^qF_\star\mathcal{A})_y = \bigoplus_{1\leq j\leq m} H^q\big(\mathcal{A}^{[\bullet]}_{x_j}\big) = \begin{cases} \bigoplus \mathcal{A}_{x_j} & \text{for } q=0, \\ 0 & \text{for } q\geq 1, \end{cases}$$

and we conclude by Cor. 13.9. □

Corollary 13.10 can be applied in particular to the inclusion $j : S \to X$ of a *closed* subspace S. Then $j_\star\mathcal{A}$ coincides with the sheaf $\mathcal{A}^S$ defined in §3 and we get $H^q(S,\mathcal{A}) = H^q(X,\mathcal{A}^S)$. It is very important to observe that the property $R^q j_\star\mathcal{A} = 0$ for $q \ge 1$ need not be true if S is not closed.

13.C. Topological Dimension

As a first application of the Leray spectral sequence, we are going to derive some properties related to the concept of *topological dimension*.

(13.11) Definition. *A non empty space X is said to be of topological dimension $\le n$ if $H^q(X,\mathcal{A}) = 0$ for any $q > n$ and any sheaf $\mathcal{A}$ on X. We let* $\operatorname{topdim} X$ *be the smallest such integer n if it exists, and $+\infty$ otherwise.*

(13.12) Criterion. *For a paracompact space X, the following conditions are equivalent:*

a) $\operatorname{topdim} X \le n$;

b) *the sheaf* $\mathcal{Z}^n = \ker(\mathcal{A}^{[n]} \longrightarrow \mathcal{A}^{[n+1]})$ *is soft for every sheaf $\mathcal{A}$;*

c) *every sheaf $\mathcal{A}$ admits a resolution $0 \to \mathcal{L}^0 \to \cdots \to \mathcal{L}^n \to 0$ of length n by soft sheaves.*

Proof. b) $\Longrightarrow$ c). Take $\mathcal{L}^q = \mathcal{A}^{[q]}$ for $q < n$ and $\mathcal{L}^n = \mathcal{Z}^n$.

c) $\Longrightarrow$ a). For every sheaf $\mathcal{A}$, the De Rham-Weil isomorphism implies $H^q(X,\mathcal{A}) = H^q\big(\mathcal{L}^\bullet(X)\big) = 0$ when $q > n$.

a) $\Longrightarrow$ b). Let S be a closed set and $U = X \smallsetminus S$. As in Prop. 7.12, $(\mathcal{A}^{[\bullet]})_U$ is an acyclic resolution of $\mathcal{A}_U$. Clearly $\ker\big((\mathcal{A}^{[n]})_U \to (\mathcal{A}^{[n+1]})_U\big) = \mathcal{Z}^n_U$, so the isomorphisms (6.2) obtained in the proof of the De Rham-Weil theorem imply

$$H^1(X,\mathcal{Z}^n_U) \simeq H^{n+1}(X,\mathcal{A}_U) = 0.$$

By (3.10), the restriction map $\mathcal{Z}^n(X) \longrightarrow \mathcal{Z}^n(S)$ is onto, so $\mathcal{Z}^n$ is soft. □

(13.13) Theorem. *The following properties hold:*

a) *If X is paracompact and if every point of X has a neighborhood of topological dimension $\le n$, then* $\operatorname{topdim} X \le n$.

b) *If $S \subset X$, then* $\operatorname{topdim} S \le \operatorname{topdim} X$ *provided that S is closed or X metrizable.*

c) *If X, Y are metrizable spaces, one of them locally compact, then*

$$\operatorname{topdim}(X \times Y) \le \operatorname{topdim} X + \operatorname{topdim} Y.$$

d) *If X is metrizable and locally homeomorphic to a subspace of $\mathbb{R}^n$, then* $\operatorname{topdim} X \le n$.

Proof. a) Apply criterion 13.12 b) and the fact that softness is a local property (Prop. 4.12).

b) When S is closed in X, the property follows from Cor. 13.10. When X is metrizable, any subset S is strongly paracompact. Let $j : S \longrightarrow X$ be the injection and $\mathcal{A}$ a sheaf on S. As $\mathcal{A} = (j_\star\mathcal{A})_{\restriction S}$, we have

$$H^q(S, \mathcal{A}) = H^q(S, j_\star\mathcal{A}) = \varinjlim_{\Omega \supset S} H^q(\Omega, j_\star\mathcal{A})$$

by Th. 9.10. We may therefore assume that S is open in X. Then every point of S has a neighborhood which is closed in X, so we conclude by a) and the first case of b).

c) Thanks to a) and b), we may assume for example that X is compact. Let $\mathcal{A}$ be a sheaf on $X \times Y$ and $\pi : X \times Y \longrightarrow Y$ the second projection. Set $n_X = \operatorname{topdim} X$, $n_Y = \operatorname{topdim} Y$. In virtue of (13.6′), we have $R^q\pi_\star\mathcal{A} = 0$ for $q > n_X$. In the Leray spectral sequence, we obtain therefore

$$E_2^{p,q} = H^p(Y, R^q\pi_\star\mathcal{A}) = 0 \quad \text{for} \quad p > n_Y \quad \text{or} \quad q > n_X,$$

thus $E_\infty^{p,l-p} = 0$ when $l > n_X + n_Y$ and we infer $H^l(X \times Y, \mathcal{A}) = 0$.

d) The unit interval $[0,1] \subset \mathbb{R}$ is of topological dimension ≤ 1, because $[0,1]$ admits arbitrarily fine coverings

$$(13.14) \qquad \mathcal{U}_k = \big(\, [0,1] \cap \,](\alpha-1)/k, (\alpha+1)/k[\, \big)_{0 \le \alpha \le k}$$

for which only consecutive open sets U_α, $U_{\alpha+1}$ intersect; we may therefore apply Prop. 5.24. Hence $\mathbb{R}^n \simeq \,]0,1[^n \subset [0,1]^n$ is of topological dimension $\le n$ by b) and c). Property d) follows

14. Alexander-Spanier Cohomology

14.A. Invariance by Homotopy

Alexander-Spanier's theory can be viewed as the special case of sheaf cohomology theory with *constant coefficients*, i.e. with values in constant sheaves.

(14.1) Definition. *Let X be a topological space, R a commutative ring and M a R-module. The constant sheaf $X \times M$ is denoted M for simplicity.*

The Alexander-Spanier q-th cohomology group with values in M is the sheaf cohomology group $H^q(X, M)$.

In particular $H^0(X, M)$ is the set of locally constant functions $X \to M$, so $H^0(X, M) \simeq M^E$, where E is the set of connected components of X. We will not repeat here the properties of Alexander-Spanier cohomology groups that are formal consequences of those of general sheaf theory, but we focus our attention instead on new features, such as invariance by homotopy.

(14.2) Lemma. *Let I denote the interval $[0,1]$ of real numbers. Then $H^0(I, M) = M$ and $H^q(I, M) = 0$ for $q \neq 0$.*

Proof. Let us employ alternate Čech cochains for the coverings $\mathfrak{U}_n$ defined in (13.14). As I is paracompact, we have

$$H^q(I, M) = \varinjlim \check{H}^q(\mathfrak{U}_n, M).$$

However, the alternate Čech complex has only two non zero components and one non zero differential:

$$\begin{aligned}
&AC^0(\mathfrak{U}_n, M) = \big\{(c_\alpha)_{0 \leq \alpha \leq n}\big\} = M^{n+1},\\
&AC^1(\mathfrak{U}_n, M) = \big\{(c_{\alpha\,\alpha+1})_{0 \leq \alpha \leq n-1}\big\} = M^n,\\
&d^0 : (c_\alpha) \longmapsto (c'_{\alpha\,\alpha+1}) = (c_{\alpha+1} - c_\alpha).
\end{aligned}$$

We see that d^0 is surjective, and that $\ker d^0 = \big\{(m, m, \ldots, m)\big\} = M$. $\square$

For any continuous map $f : X \longrightarrow Y$, the inverse image of the constant sheaf M on Y is $f^{-1}M = M$. We get therefore a morphism

$$f^\star : H^q(Y, M) \longrightarrow H^q(X, M), \tag{14.3}$$

which, as already mentioned in §9, is compatible with cup product.

(14.4) Proposition. *For any space X, the projection $\pi : X \times I \longrightarrow X$ and the injections $i_t : X \longrightarrow X \times I$, $x \longmapsto (x, t)$ induce inverse isomorphisms*

$$H^q(X, M) \underset{i_t^\star}{\overset{\pi^\star}{\rightleftarrows}} H^q(X \times I, M).$$

In particular, $i_t^\star$ does not depend on t.

Proof. As $\pi \circ i_t = \mathrm{Id}$, we have $i_t^\star \circ \pi^\star = \mathrm{Id}$, so it is sufficient to check that $\pi^\star$ is an isomorphism. However $(R^q\pi_\star M)_x = H^q(I, M)$ in virtue of (13.6′), so we get

$$R^0\pi_\star M = M, \qquad R^q\pi_\star M = 0 \quad \text{for} \quad q \neq 0$$

and conclude by Cor. 13.9. □

(14.4) Theorem. *If $f, g : X \longrightarrow Y$ are homotopic maps, then*

$$f^\star = g^\star : H^q(Y, M) \longrightarrow H^q(X, M).$$

Proof. Let $H : X \times I \longrightarrow Y$ be a homotopy between f and g, with $f = H \circ i_0$ and $g = H \circ i_1$. Proposition 14.3 implies

$$f^\star = i_0^\star \circ H^\star = i_1^\star \circ H^\star = g^\star.$$ □

We denote $f \sim g$ the homotopy equivalence relation. Two spaces X, Y are said to be homotopically equivalent $(X \sim Y)$ if there exist continuous maps $u : X \longrightarrow Y$, $v : Y \longrightarrow X$ such that $v \circ u \sim \mathrm{Id}_X$ and $u \circ v \sim \mathrm{Id}_Y$. Then $H^q(X, M) \simeq H^q(Y, M)$ and $u^\star, v^\star$ are inverse isomorphisms.

(14.5) Example. A subspace $S \subset X$ is said to be a (strong) deformation retract of X if there exists a *retraction* of X onto S, i.e. a map $r : X \longrightarrow S$ such that $r \circ j = \mathrm{Id}_S$ (j = inclusion of S in X), which is a *deformation* of Id_X, i.e. there exists a homotopy $H : X \times I \longrightarrow X$ relative to S between Id_X and $j \circ r$:

$$H(x, 0) = x, \quad H(x, 1) = r(x) \ \text{ on } \ X, \qquad H(x, t) = x \ \text{ on } \ S \times I.$$

Then X and S are homotopically equivalent. In particular X is said to be *contractible* if X has a deformation retraction onto a point x_0. In this case

$$H^q(X, M) = H^q(\{x_0\}, M) = \begin{cases} M & \text{for } q = 0 \\ 0 & \text{for } q \neq 0. \end{cases}$$

(14.6) Corollary. *If X is a compact differentiable manifold, the cohomology groups $H^q(X, R)$ are finitely generated over R.*

Proof. Lemma 6.9 shows that X has a finite covering $\mathfrak{U}$ such that the intersections $U_{\alpha_0 \ldots \alpha_q}$ are contractible. Hence $\mathfrak{U}$ is acyclic, $H^q(X, R) = H^q\big(C^\bullet(\mathfrak{U}, R)\big)$ and each Čech cochain space is a finitely generated (free) module. □

(14.7) Example: Cohomology Groups of Spheres. Set

$$S^n = \big\{x \in \mathbb{R}^{n+1} \ ; \ x_0^2 + x_1^2 + \ldots + x_n^2 = 1\big\}, \quad n \geq 1.$$

We will prove by induction on n that

$$(14.8) \quad H^q(S^n, M) = \begin{cases} M & \text{for } q = 0 \text{ or } q = n \\ 0 & \text{otherwise.} \end{cases}$$

As S^n is connected, we have $H^0(S^n, M) = M$. In order to compute the higher cohomology groups, we apply the Mayer-Vietoris exact sequence 3.11 to the covering (U_1, U_2) with

$$U_1 = S^n \smallsetminus \{(-1, 0, \ldots, 0)\}, \qquad U_2 = S^n \smallsetminus \{(1, 0, \ldots, 0)\}.$$

Then $U_1, U_2 \approx \mathbb{R}^n$ are contractible, and $U_1 \cap U_2$ can be retracted by deformation on the equator $S^n \cap \{x_0 = 0\} \approx S^{n-1}$. Omitting M in the notations of cohomology groups, we get exact sequences

$$(14.9') \qquad H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(U_1 \cap U_2) \longrightarrow H^1(S^n) \longrightarrow 0,$$
$$(14.9'') \qquad 0 \longrightarrow H^{q-1}(U_1 \cap U_2) \longrightarrow H^q(S^n) \longrightarrow 0, \quad q \geq 2.$$

For $n = 1$, $U_1 \cap U_2$ consists of two open arcs, so we have

$$H^0(U_1) \oplus H^0(U_2) = H^0(U_1 \cap U_2) = M \times M,$$

and the first arrow in (14.9′) is $(m_1, m_2) \longmapsto (m_2 - m_1, m_2 - m_1)$. We infer easily that $H^1(S^1) = M$ and that

$$H^q(S^1) = H^{q-1}(U_1 \cap U_2) = 0 \quad \text{for} \quad q \geq 2.$$

For $n \geq 2$, $U_1 \cap U_2$ is connected, so the first arrow in (14.9′) is onto and $H^1(S^n) = 0$. The second sequence (14.9″) yields $H^q(S^n) \simeq H^{q-1}(S^{n-1})$. An induction concludes the proof. □

14.B. Relative Cohomology Groups and Excision Theorem

Let X be a topological space and S a subspace. We denote by $M^{[q]}(X, S)$ the subgroup of sections $u \in M^{[q]}(X)$ such that $u(x_0, \ldots, x_q) = 0$ when

$$(x_0, \ldots, x_q) \in S^q, \quad x_1 \in V(x_0), \ \ldots, \ x_q \in V(x_0, \ldots, x_{q-1}).$$

Then $M^{[\bullet]}(X, S)$ is a subcomplex of $M^{[\bullet]}(X)$ and we define the *relative cohomology group* of the pair (X, S) with values in M as

$$(14.10) \quad H^q(X, S\,; M) = H^q\big(M^{[\bullet]}(X, S)\big).$$

By definition of $M^{[q]}(X, S)$, there is an exact sequence

$$(14.11) \quad 0 \longrightarrow M^{[q]}(X, S) \longrightarrow M^{[q]}(X) \longrightarrow (M_{\restriction S})^{[q]}(S) \longrightarrow 0.$$

The reader should take care of the fact that $(M_{\restriction S})^{[q]}(S)$ does not coincide with the module of sections $M^{[q]}(S)$ of the sheaf $M^{[q]}$ on X, except if S is open. The snake lemma shows that there is an "exact sequence of the pair":

$$(14.12) \quad H^q(X, S\,; M) \to H^q(X, M) \to H^q(S, M) \to H^{q+1}(X, S\,; M) \cdots.$$

We have in particular $H^0(X,S\,;\,M)=M^E$, where E is the set of connected components of X which do not meet S. More generally, for a triple (X,S,T) with $X\supset S\supset T$, there is an "exact sequence of the triple":

$$(14.12')\qquad 0\longrightarrow M^{[q]}(X,S)\longrightarrow M^{[q]}(X,T)\longrightarrow M^{[q]}(S,T)\longrightarrow 0,$$
$$H^q(X,S\,;M)\longrightarrow H^q(X,T\,;\,M)\longrightarrow H^q(S,T\,;\,M)\longrightarrow H^{q+1}(X,S\,;\,M).$$

The definition of the cup product in (8.2) shows that $\alpha\smile\beta$ vanishes on $S\cup S'$ if α,β vanish on S, S' respectively. Therefore, we obtain a bilinear map

$$(14.13)\qquad H^p(X,S\,;\,M)\times H^q(X,S'\,;\,M')\longrightarrow H^{p+q}(X,S\cup S'\,;\,M\otimes M').$$

If $f:(X,S)\longrightarrow(Y,T)$ is a morphism of pairs, i.e. a continuous map $X\to Y$ such that $f(S)\subset T$, there is an induced pull-back morphism

$$(14.14)\qquad f^\star:\quad H^q(Y,T\,;\,M)\longrightarrow H^q(X,S\,;\,M)$$

which is compatible with the cup product. Two morphisms of pairs f,g are said to be homotopic when there is a pair homotopy $H:(X\times I,S\times I)\longrightarrow(Y,T)$. An application of the exact sequence of the pair shows that

$$\pi^\star:H^q(X,S\,;\,M)\longrightarrow H^q(X\times I,S\times I\,;\,M)$$

is an isomorphism. It follows as above that $f^\star=g^\star$ as soon as f,g are homotopic.

(14.15) Excision theorem. *For subspaces $\overline{T}\subset S^\circ$ of X, the restriction morphism $H^q(X,S\,;\,M)\longrightarrow H^q(X\smallsetminus T,S\smallsetminus T\,;\,M)$ is an isomorphism.*

Proof. Under our assumption, it is not difficult to check that the surjective restriction map $M^{[q]}(X,S)\longrightarrow M^{[q]}(X\smallsetminus T,S\smallsetminus T)$ is also injective, because the kernel consists of sections $u\in M^{[q]}(X)$ such that $u(x_0,\ldots,x_q)=0$ on $(X\smallsetminus T)^{q+1}\cup S^{q+1}$, and this set is a neighborhood of the diagonal of X^{q+1}. □

(14.16) Proposition. *If S is open or strongly paracompact in X, the relative cohomology groups can be written in terms of cohomology groups with supports in $X\smallsetminus S$:*

$$H^q(X,S\,;\,M)\simeq H^q_{X\smallsetminus S}(X,M).$$

In particular, if $X\smallsetminus S$ is relatively compact in X, we have

$$H^q(X,S\,;\,M)\simeq H^q_c(X\smallsetminus S,M).$$

Proof. We have an exact sequence

$$(14.17)\qquad 0\longrightarrow M^{[\bullet]}_{X\smallsetminus S}(X)\longrightarrow M^{[\bullet]}(X)\longrightarrow M^{[\bullet]}(S)\longrightarrow 0$$

where $M^{[\bullet]}_{X\smallsetminus S}(X)$ denotes sections with support in $X \smallsetminus S$. If S is open, then $M^{[\bullet]}(S) = (M_{\restriction S})^{[\bullet]}(S)$, hence $M^{[\bullet]}_{X\smallsetminus S}(X) = M^{[\bullet]}(X, S)$ and the result follows. If S is strongly paracompact, Prop. 4.7 and Th. 9.10 show that

$$H^q\big(M^{[\bullet]}(S)\big) = H^q\big(\varinjlim_{\Omega\supset S} M^{[\bullet]}(\Omega)\big) = \varinjlim_{\Omega\supset S} H^q(\Omega, M) = H^q(S, M_{\restriction S}).$$

If we consider the diagram

$$\begin{array}{ccccccc}
0\longrightarrow & M^{[\bullet]}_{X\smallsetminus S}(X) & \longrightarrow & M^{[\bullet]}(X) & \longrightarrow & M^{[\bullet]}(S) & \longrightarrow 0\\
 & \downarrow & & \downarrow \mathrm{Id} & & \downarrow \restriction S & \\
0\longrightarrow & M^{[\bullet]}(X, S) & \longrightarrow & M^{[\bullet]}(X) & \longrightarrow & (M_{\restriction S})^{[\bullet]}(S) & \longrightarrow 0
\end{array}$$

we see that the last two vertical arrows induce isomorphisms in cohomology. Therefore, the first one also does. □

(14.18) Corollary. *Let X, Y be locally compact spaces and $f, g : X \to Y$ proper maps. We say that f, g are properly homotopic if they are homotopic through a proper homotopy $H : X \times I \longrightarrow Y$. Then*

$$f^\star = g^\star \ : \ H^q_c(Y, M) \longrightarrow H^q_c(X, M).$$

Proof. Let $\widehat{X} = X \cup \{\infty\}$, $\widehat{Y} = Y \cup \{\infty\}$ be the Alexandrov compactifications of X, Y. Then f, g, H can be extended as continuous maps

$$\widehat{f}, \widehat{g} \ : \ \widehat{X} \longrightarrow \widehat{Y}, \qquad \widehat{H} \ : \ \widehat{X} \times I \longrightarrow \widehat{Y}$$

with $\widehat{f}(\infty) = \widehat{g}(\infty) = H(\infty, t) = \infty$, so that $\widehat{f}, \widehat{g}$ are homotopic as maps $(\widehat{X}, \infty) \longrightarrow (\widehat{Y}, \infty)$. Proposition 14.16 implies $H^q_c(X, M) = H^q(\widetilde{X}, \infty\,; M)$ and the result follows. □

15. Künneth Formula

15.A. Flat Modules and Tor Functors

The goal of this section is to investigate homological properties related to tensor products. We work in the category of modules over a commutative ring R with unit. All tensor products appearing here are tensor products over R. The starting point is the observation that tensor product with a given module is a right exact functor: if $0 \to A \to B \to C \to 0$ is an exact sequence and M a R-module, then

$$A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact, but the map $A \otimes M \longrightarrow B \otimes M$ need not be injective. A counterexample is given by the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad \text{over} \quad R = \mathbb{Z}$$

tensorized by $M = \mathbb{Z}/2\mathbb{Z}$. However, the injectivity holds if M is a free R-module. More generally, one says that M is a *flat R-module* if the tensor product by M preserves injectivity, or equivalently, if $\otimes M$ is a left exact functor.

A *flat resolution* $C_\bullet$ of a R-module A is a homology exact sequence

$$\cdots \longrightarrow C_q \longrightarrow C_{q-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

where C_q are flat R-modules and $C_q = 0$ for $q < 0$. Such a resolution always exists because every module A is a quotient of a free module C_0. Inductively, we take C_{q+1} to be a free module such that $\ker(C_q \to C_{q-1})$ is a quotient of C_{q+1}. In terms of homology groups, we have $H_0(C_\bullet) = A$ and $H_q(C_\bullet) = 0$ for $q \neq 0$. Given R-modules A, B and free resolutions $d' : C_\bullet \longrightarrow A$, $d'' : D_\bullet \longrightarrow B$, we consider the double homology complex

$$K_{p,q} = C_p \otimes D_q, \qquad d = d' \otimes \mathrm{Id} + (-1)^p \,\mathrm{Id} \otimes d''$$

and the associated first and second spectral sequences. Since C_p is free, we have

$$E^1_{p,q} = H_q(C_p \otimes D_\bullet) = \begin{cases} C_p \otimes B & \text{for} \quad q = 0, \\ 0 & \text{for} \quad q \neq 0. \end{cases}$$

Similarly, the second spectral sequence also collapses and we have

$$H_l(K_\bullet) = H_l(C_\bullet \otimes B) = H_l(A \otimes D_\bullet).$$

This implies in particular that the homology groups $H_l(K_\bullet)$ do not depend on the choice of the resolutions $C_\bullet$ or $D_\bullet$.

(15.1) Definition. *The q-th torsion module of A and B is*

$$\mathrm{Tor}_q(A, B) = H_q(K_\bullet) = H_q(C_\bullet \otimes B) = H_q(A \otimes D_\bullet).$$

Since the definition of $K_\bullet$ is symmetric with respect to A and B, we have $\mathrm{Tor}_q(A, B) \simeq \mathrm{Tor}_q(B, A)$. By the right-exactness of $\otimes B$, we find in particular $\mathrm{Tor}_0(A, B) = A \otimes B$. Moreover, if B is flat, $\otimes B$ is also left exact, thus $\mathrm{Tor}_q(A, B) = 0$ for all $q \geq 1$ and all modules A. If $0 \to A \to A' \to A'' \to 0$ is an exact sequence, there is a corresponding exact sequence of homology complexes

$$0 \longrightarrow A \otimes D_\bullet \longrightarrow A' \otimes D_\bullet \longrightarrow A'' \otimes D_\bullet \longrightarrow 0,$$

thus a long exact sequence

$$(15.2)\qquad \begin{array}{l} \longrightarrow \mathrm{Tor}_q(A,B) \longrightarrow \mathrm{Tor}_q(A',B) \longrightarrow \mathrm{Tor}_q(A'',B) \longrightarrow \mathrm{Tor}_{q-1}(A,B) \\ \cdots \longrightarrow \quad A\otimes B \quad \longrightarrow \quad A'\otimes B \quad \longrightarrow \quad A''\otimes B \quad \longrightarrow \quad 0. \end{array}$$

It follows that B is flat if and only if $\mathrm{Tor}_1(A,B)=0$ for every R-module A.

Suppose now that R is a *principal ring*. Then every module A has a free resolution $0\to C_1\to C_0\to A\to 0$ because the kernel of any surjective map $C_0\to A$ is free (every submodule of a free module is free). It follows that one always has $\mathrm{Tor}_q(A,B)=0$ for $q\geq 2$. In this case, we denote $\mathrm{Tor}_1(A,B)=A\star B$ and call it the *torsion product* of A and B. The above exact sequence (15.2) reduces to

$$(15.3)\qquad 0\to A\star B\to A'\star B\to A''\star B\to A\otimes B\to A'\otimes B\to A''\otimes B\to 0.$$

In order to compute $A\star B$, we may restrict ourselves to finitely generated modules, because every module is a direct limit of such modules and the $\star$ product commutes with direct limits. Over a principal ring R, every finitely generated module is a direct sum of a free module and of cyclic modules R/aR. It is thus sufficient to compute $R/aR\star R/bR$. The obvious free resolution $R \xrightarrow{a\times} R$ of R/aR shows that $R/aR\star R/bR$ is the kernel of the map $R/bR \xrightarrow{a\times} R/bR$. Hence

$$(15.4)\qquad R/aR\star R/bR\simeq R/(a\wedge b)R$$

where $a\wedge b$ denotes the greatest common divisor of a and b. It follows that a module B is flat if and only if it is torsion free. If R is a field, every R-module is free, thus $A\star B=0$ for all A and B.

15.B. Künneth and Universal Coefficient Formulas

The algebraic Künneth formula describes the cohomology groups of the tensor product of two differential complexes.

(15.5) Algebraic Künneth formula. *Let $(K^\bullet,d')$, $(L^\bullet,d'')$ be complexes of R-modules and $(K\otimes L)^\bullet$ the simple complex associated to the double complex $(K\otimes L)^{p,q}=K^p\otimes L^q$. If $K^\bullet$ or $L^\bullet$ is torsion free, there is a split exact sequence*

$$0\to \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet) \xrightarrow{\mu} H^l\big((K\otimes L)^\bullet\big)\to \bigoplus_{p+q=l+1} H^p(K^\bullet)\star H^q(L^\bullet) \to 0$$

where the map μ is defined by $\mu(\{k^p\}\times\{l^q\})=\{k^p\otimes l^q\}$ for all cocycles $\{k^p\}\in H^p(K^\bullet)$, $\{l^q\}\in H^q(L^\bullet)$.

(15.6) Corollary. *If R is a field, or if one of the graded modules $H^\bullet(K^\bullet)$, $H^\bullet(L^\bullet)$ is torsion free, then*

$$H^l\big((K\otimes L)^\bullet\big) \simeq \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet).$$

Proof. Assume for example that $K^\bullet$ is torsion free. There is a short exact sequence of complexes

$$0 \longrightarrow Z^\bullet \longrightarrow K^\bullet \xrightarrow{d'} B^{\bullet+1} \longrightarrow 0$$

where $Z^\bullet, B^\bullet \subset K^\bullet$ are respectively the graded modules of cocycles and coboundaries in $K^\bullet$, considered as subcomplexes with zero differential. As $B^{\bullet+1}$ is torsion free, the tensor product of the above sequence with $L^\bullet$ is still exact. The corresponding long exact sequence for the associated simple complexes yields:

$$\begin{aligned} &H^l\big((B\otimes L)^\bullet\big) \longrightarrow H^l\big((Z\otimes L)^\bullet\big) \longrightarrow H^l\big((K\otimes L)^\bullet\big) \xrightarrow{d'} H^{l+1}\big((B\otimes L)^\bullet\big)\\ &\qquad\longrightarrow H^{l+1}\big((Z\otimes L)^\bullet\big)\cdots. \end{aligned}\tag{15.7}$$

The first and last arrows are connecting homomorphisms; in this situation, they are easily seen to be induced by the inclusion $B^\bullet \subset Z^\bullet$. Since the differential of $Z^\bullet$ is zero, the simple complex $(Z\otimes L)^\bullet$ is isomorphic to the direct sum $\bigoplus_p Z^p\otimes L^{\bullet-p}$, where Z^p is torsion free. Similar properties hold for $(B\otimes L)^\bullet$, hence

$$H^l\big((Z\otimes L)^\bullet\big) = \bigoplus_{p+q=l} Z^p\otimes H^q(L^\bullet),\qquad H^l\big((B\otimes L)^\bullet\big) = \bigoplus_{p+q=l} B^p\otimes H^q(L^\bullet).$$

The exact sequence

$$0 \longrightarrow B^p \longrightarrow Z^p \longrightarrow H^p(K^\bullet) \longrightarrow 0$$

tensorized by $H^q(L^\bullet)$ yields an exact sequence of the type (15.3):

$$\begin{aligned} 0 \to H^p(K^\bullet)\star H^q(L^\bullet) \to B^p\otimes H^q(L^\bullet) &\to Z^p\otimes H^q(L^\bullet)\\ &\to H^p(K^\bullet)\otimes H^q(L^\bullet) \to 0. \end{aligned}$$

By the above equalities, we get

$$\begin{aligned} 0 \longrightarrow \bigoplus_{p+q=l} H^p(K^\bullet)\star H^q(L^\bullet) &\longrightarrow H^l\big((B\otimes L)^\bullet\big) \longrightarrow H^l\big((Z\otimes L)^\bullet\big)\\ &\longrightarrow \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet) \longrightarrow 0. \end{aligned}$$

In our initial long exact sequence (15.7), the cokernel of the first arrow is thus $\bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet)$ and the kernel of the last arrow is the torsion sum $\bigoplus_{p+q=l+1} H^p(K^\bullet)\star H^q(L^\bullet)$. This gives the exact sequence of the lemma. We leave the computation of the map μ as an exercise for the reader. The splitting assertion can be obtained by observing that there always exists a torsion free complex $\widetilde{K}^\bullet$ that splits (i.e. $\widetilde{Z}^\bullet \subset \widetilde{K}^\bullet$ splits), and a morphism $\widetilde{K}^\bullet \longrightarrow K^\bullet$ inducing an isomorphism in cohomology; then the projection $\widetilde{K}^\bullet \longrightarrow \widetilde{Z}^\bullet$ yields a projection

$$H^l\big((\widetilde{K}\otimes L)^\bullet\big) \longrightarrow H^l\big((\widetilde{Z}\otimes L)^\bullet\big) \simeq \bigoplus_{p+q=l} \widetilde{Z}^p \otimes H^q(L^\bullet)$$
$$\longrightarrow \bigoplus_{p+q=l} H^p(\widetilde{K}^\bullet)\otimes H^q(L^\bullet).$$

To construct $\widetilde{K}^\bullet$, let $\widetilde{Z}^\bullet \longrightarrow Z^\bullet$ be a surjective map with $\widetilde{Z}^\bullet$ free, $\widetilde{B}^\bullet$ the inverse image of $B^\bullet$ in $\widetilde{\mathcal{Z}}^\bullet$ and $\widetilde{K}^\bullet = \widetilde{Z}^\bullet \oplus \widetilde{B}^{\bullet+1}$, where the differential $\widetilde{K}^\bullet \longrightarrow \widetilde{K}^{\bullet+1}$ is given by $\widetilde{Z}^\bullet \longrightarrow 0$ and $\widetilde{B}^{\bullet+1} \subset \widetilde{Z}^{\bullet+1} \oplus 0$; as $\widetilde{B}^\bullet$ is free, the map $\widetilde{B}^{\bullet+1} \longrightarrow B^{\bullet+1}$ can be lifted to a map $\widetilde{B}^{\bullet+1} \longrightarrow K^\bullet$, and this lifting combined with the composite $\widetilde{Z}^\bullet \to Z^\bullet \subset K^\bullet$ yields the required complex morphism $\widetilde{K}^\bullet = \widetilde{Z}^\bullet \oplus \widetilde{B}^{\bullet+1} \longrightarrow K^\bullet$. □

(15.8) Universal coefficient formula. *Let $K^\bullet$ be a complex of R-modules and M a R-module such that either $K^\bullet$ or M is torsion free. Then there is a split exact sequence*

$$0 \longrightarrow H^p(K^\bullet)\otimes M \longrightarrow H^p(K^\bullet \otimes M) \longrightarrow H^{p+1}(K^\bullet)\star M \longrightarrow 0.$$

Indeed, this is a special case of Formula 15.5 when the complex $L^\bullet$ is reduced to one term $L^0 = M$. In general, it is interesting to observe that the spectral sequence of $K^\bullet\otimes L^\bullet$ collapses in E_2 if $K^\bullet$ is torsion free: $H^k\big((K\otimes L)^\bullet\big)$ is in fact the direct sum of the terms $E_2^{p,q} = H^p\big(K^\bullet \otimes H^q(L^\bullet)\big)$ thanks to (15.8).

15.C. Künneth Formula for Sheaf Cohomology H

ere we apply the general algebraic machinery to compute cohomology groups over a product space $X\times Y$. The main argument is a combination of the Leray spectral sequence with the universal coefficient formula for sheaf cohomology.

(15.9) Theorem. *Let $\mathcal{A}$ be a sheaf of R-modules over a topological space X and M a R-module. Assume that either $\mathcal{A}$ or M is torsion free and that either X is compact or M is finitely generated. Then there is a split exact sequence*

$$0 \longrightarrow H^p(X,\mathcal{A})\otimes M \longrightarrow H^p(X,\mathcal{A}\otimes M) \longrightarrow H^{p+1}(X,\mathcal{A})\star M \longrightarrow 0.$$

Proof. If M is finitely generated, we get $(\mathcal{A}\otimes M)^{[\bullet]}(X) = \mathcal{A}^{[\bullet]}(X)\otimes M$ easily, so the above exact sequence is a consequence of Formula 15.8. If X is compact, we may consider Čech cochains $C^q(\mathcal{U}, \mathcal{A} \otimes M)$ over finite coverings. There is an obvious morphism

$$C^q(\mathcal{U}, \mathcal{A}) \otimes M \longrightarrow C^q(\mathcal{U}, \mathcal{A} \otimes M)$$

but this morphism need not be surjective nor injective. However, since

$$(\mathcal{A} \otimes M)_x = \mathcal{A}_x \otimes M = \varinjlim_{V \ni x} \mathcal{A}(V) \otimes M,$$

the following properties are easy to verify:

a) If $c \in C^q(\mathcal{U}, \mathcal{A} \otimes M)$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ and $\rho : \mathcal{V} \longrightarrow \mathcal{U}$ such that $\rho^\star c \in C^q(\mathcal{V}, \mathcal{A} \otimes M)$ is in the image of $C^q(\mathcal{V}, \mathcal{A}) \otimes M$.

b) If a tensor $t \in C^q(\mathcal{U}, \mathcal{A}) \otimes M$ is mapped to 0 in $C^q(\mathcal{U}, \mathcal{A} \otimes M)$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\rho^\star t \in C^q(\mathcal{V}, \mathcal{A}) \otimes M$ equals 0.

From a) and b) it follows that

$$\check{H}^q(X, \mathcal{A} \otimes M) = \varinjlim_{\mathcal{U}} H^q\big(C^\bullet(\mathcal{U}, \mathcal{A} \otimes M)\big) = \varinjlim_{\mathcal{U}} H^q\big(C^\bullet(\mathcal{U}, \mathcal{A}) \otimes M\big)$$

and the desired exact sequence is the direct limit of the exact sequences of Formula 15.8 obtained for $K^\bullet = C^\bullet(\mathcal{U}, \mathcal{A})$. □

(15.10) Theorem (Künneth)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of R-modules over topological spaces X and Y. Assume that $\mathcal{A}$ is torsion free, that Y is compact and that either X is compact or the cohomology groups $H^q(Y, \mathcal{B})$ are finitely generated R-modules. There is a split exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=l} H^p(X, \mathcal{A}) \otimes H^q(Y, \mathcal{B}) \xrightarrow{\ \mu\ } H^l(X \times Y, \mathcal{A} \boxtimes \mathcal{B})$$
$$\longrightarrow \bigoplus_{p+q=l+1} H^p(X, \mathcal{A}) \star H^q(Y, \mathcal{B}) \longrightarrow 0$$

where μ is the map given by the cartesian product $\bigoplus \alpha_p \otimes \beta_q \longmapsto \sum \alpha_p \times \beta_q$.

Proof. We compute $H^l(X, \mathcal{A} \boxtimes \mathcal{B})$ by means of the Leray spectral sequence of the projection $\pi : X \times Y \longrightarrow X$. This means that we are considering the differential sheaf $\mathcal{L}^q = \pi_\star(\mathcal{A} \boxtimes \mathcal{B})^{[q]}$ and the double complex

$$K^{p,q} = (\mathcal{L}^q)^{[p]}(X).$$

By (12.5′) we have ${}_KE_2^{p,q} = H^p\big(X, \mathcal{H}^q(\mathcal{L}^\bullet)\big)$. As Y is compact, the cohomology sheaves $\mathcal{H}^q(\mathcal{L}^\bullet) = R^q\pi_\star(\mathcal{A} \boxtimes \mathcal{B})$ are given by

$$R^q\pi_\star(\mathcal{A} \boxtimes \mathcal{B})_x = H^q(\{x\} \times Y, \mathcal{A} \boxtimes \mathcal{B}_{\restriction \{x\} \times Y}) = H^q(Y, \mathcal{A}_x \otimes \mathcal{B}) = \mathcal{A}_x \otimes H^q(Y, \mathcal{B})$$

thanks to the compact case of Th. 15.9 where $M = \mathcal{A}_x$ is torsion free. We obtain therefore

$$R^q\pi_\star(\mathcal{A}\boxtimes\mathcal{B}) = \mathcal{A}\otimes H^q(Y,\mathcal{B}),$$
$$_K E_2^{p,q} = H^p\big(X,\mathcal{A}\otimes H^q(Y,\mathcal{B})\big).$$

Theorem 15.9 shows that the E_2-term is actually given by the desired exact sequence, but it is not a priori clear that the spectral sequence collapses in E_2. In order to check this, we consider the double complex

$$C^{p,q} = \mathcal{A}^{[p]}(X)\otimes\mathcal{B}^{[q]}(Y)$$

and construct a natural morphism $C^{\bullet,\bullet}\longrightarrow K^{\bullet,\bullet}$. We may consider $K^{p,q} = \big(\pi_\star(\mathcal{A}\boxtimes\mathcal{B})^{[q]}\big)^{[p]}(X)$ as the set of equivalence classes of functions

$$h\big(\xi_0,\ldots,\xi_p\big)\in\pi_\star(\mathcal{A}\boxtimes\mathcal{B})^{[q]}_{\xi_p} = \varinjlim\ (\mathcal{A}\boxtimes\mathcal{B})^{[q]}\big(\pi^{-1}\big(V(\xi_p)\big)\big)$$

or more precisely

$$\begin{aligned}
&h\big(\xi_0,\ldots,\xi_p\,;\,(x_0,y_0),\ldots,(x_q,y_q)\big)\in\mathcal{A}_{x_q}\otimes\mathcal{B}_{y_q}\qquad\text{with}\\
&\xi_0\in X,\quad \xi_j\in V(\xi_0,\ldots,\xi_{j-1}),\quad 1\le j\le p,\\
&(x_0,y_0)\in V(\xi_0,\ldots,\xi_p)\times Y,\\
&(x_j,y_j)\in V\big(\xi_0,\ldots,\xi_p\,;\,(x_0,y_0),\ldots,(x_{j-1},y_{j-1})\big),\quad 1\le j\le q.
\end{aligned}$$

Then $f\otimes g\in C^{p,q}$ is mapped to $h\in K^{p,q}$ by the formula

$$h\big(\xi_0,\ldots,\xi_p\,;\,(x_0,y_0),\ldots,(x_q,y_q)\big) = f(\xi_0,\ldots,\xi_p)(x_q)\otimes g(y_0,\ldots,y_q).$$

As $\mathcal{A}^{[p]}(X)$ is torsion free, we find

$$_C E_1^{p,q} = \mathcal{A}^{[p]}(X)\otimes H^q(Y,\mathcal{B}).$$

Since either X is compact or $H^q(Y,\mathcal{B})$ finitely generated, Th. 15.9 yields

$$_C E_2^{p,q} = H^p\big(X,\mathcal{A}\otimes H^q(Y,\mathcal{B})\big)\simeq {}_K E_2^{p,q}$$

hence $H^l(K^\bullet)\simeq H^l(C^\bullet)$ and the algebraic Künneth formula 15.5 concludes the proof. □

(15.11) Remark. The exact sequences of Th. 15.9 and of Künneth's theorem also hold for cohomology groups with compact support, provided that X and Y are locally compact and $\mathcal{A}$ (or $\mathcal{B}$) is torsion free. This is an immediate consequence of Prop. 7.12 on direct limits of cohomology groups over compact subsets.

(15.12) Corollary. *When $\mathcal{A}$ and $\mathcal{B}$ are torsion free constant sheaves, e.g. $\mathcal{A}=\mathcal{B}=\mathbb{Z}$ or $\mathbb{R}$, the Künneth formula holds as soon as X or Y has the same homotopy type as a finite cell complex.*

Proof. If Y satisfies the assumption, we may suppose in fact that Y is a finite cell complex by the homotopy invariance. Then Y is compact and $H^\bullet(Y,\mathcal{B})$ is finitely generated, so Th. 15.10 can be applied. □

16. Poincaré duality

16.A. Injective Modules and Ext Functors

The study of duality requires some algebraic preliminaries on the Hom functor and its derived functors Ext^q. Let R be a commutative ring with unit, M a R-module and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

an exact sequence of R-modules. Then we have exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(M,A) &\longrightarrow \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C),\\ \mathrm{Hom}_R(A,M) &\longleftarrow \mathrm{Hom}_R(B,M) \longleftarrow \mathrm{Hom}_R(C,M) \longleftarrow 0,\end{aligned}$$

i.e. $\mathrm{Hom}(M,\bullet)$ is a covariant left exact functor and $\mathrm{Hom}(\bullet,M)$ a contravariant right exact functor. The module M is said to be *projective* if $\mathrm{Hom}(M,\bullet)$ is also right exact, and *injective* if $\mathrm{Hom}(\bullet,M)$ is also left exact. Every free R-module is projective. Conversely, if M is projective, any surjective morphism $F \longrightarrow M$ from a free module F onto M must split $\big(\mathrm{Id}_M$ has a preimage in $\mathrm{Hom}(M,F)\big)$; if R is a principal ring, "projective" is therefore equivalent to "free".

(16.1) Proposition. *Over a principal ring R, a module M is injective if and only if it is divisible, i.e. if for every $x \in M$ and $\lambda \in R \smallsetminus \{0\}$, there exists $y \in M$ such that $\lambda y = x$.*

Proof. If M is injective, the exact sequence $0 \longrightarrow R \xrightarrow{\lambda\times} R \longrightarrow R/\lambda R \longrightarrow 0$ shows that

$$M = \mathrm{Hom}(R,M) \xrightarrow{\lambda\times} \mathrm{Hom}(R,M) = M$$

must be surjective, thus M is divisible.

Conversely, assume that R is divisible. Let $f : A \longrightarrow M$ be a morphism and $B \supset A$. Zorn's lemma implies that there is a maximal extension $\widetilde{f} : \widetilde{A} \longrightarrow M$ of f where $A \subset \widetilde{A} \subset B$. If $\widetilde{A} \neq B$, select $x \in B \smallsetminus \widetilde{A}$ and consider the ideal I of elements $\lambda \in R$ such that $\lambda x \in \widetilde{A}$. As R is principal we have $I = \lambda_0 R$ for some λ_0. If $\lambda_0 \neq 0$, select $y \in M$ such that $\lambda_0 y = \widetilde{f}(\lambda_0 x)$ and if $\lambda_0 = 0$ take y arbitrary. Then $\widetilde{f}$ can be extended to $\widetilde{A} + Rx$ by letting $\widetilde{f}(x) = y$. This is a contradiction, so we must have $\widetilde{A} = B$. □

(16.2) Proposition. *Every module M can be embedded in an injective module $\widetilde{M}$.*

Proof. Assume first $R = \mathbb{Z}$. Then set

$$M' = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \qquad M'' = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \subset \mathbb{Q}/\mathbb{Z}^{M'}.$$

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}^{M'}$ are injective. It is therefore sufficient to show that the canonical morphism $M \longrightarrow M''$ is injective. In fact, for any $x \in M \smallsetminus \{0\}$, the subgroup $\mathbb{Z}x$ is cyclic (finite or infinite), so there is a non trivial morphism $\mathbb{Z}x \longrightarrow \mathbb{Q}/\mathbb{Z}$, and we can extend this morphism into a morphism $u : M \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then $u \in M'$ and $u(x) \neq 0$, so $M \longrightarrow M''$ is injective.

Now, for an arbitrary ring R, we set $\widetilde{M} = \mathrm{Hom}_{\mathbb{Z}}\big(R, \mathbb{Q}/\mathbb{Z}^{M'}\big)$. There are R-linear embeddings

$$M = \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(R, \mathbb{Q}/\mathbb{Z}^{M'}\big) = \widetilde{M}$$

and since $\mathrm{Hom}_R(\bullet, \widetilde{M}) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(\bullet, \mathbb{Q}/\mathbb{Z}^{M'}\big)$, it is clear that $\widetilde{M}$ is injective over the ring R. □

As a consequence, any module has a (cohomological) resolution by injective modules. Let A, B be given R-modules, let $d' : B \to D^\bullet$ be an injective resolution of B and let $d'' : C_\bullet \to A$ be a free (or projective) resolution of A. We consider the cohomology double complex

$$K^{p,q} = \mathrm{Hom}(C_q, D^p), \qquad d = d' + (-1)^p (d'')^\dagger$$

($\dagger$ means transposition) and the associated first and second spectral sequences. Since $\mathrm{Hom}(\bullet, D^p)$ and $\mathrm{Hom}(C_q, \bullet)$ are exact, we get

$$\begin{aligned} &E_1^{p,0} = \mathrm{Hom}(A, D^p), \qquad \widetilde{E}_1^{p,0} = \mathrm{Hom}(C_p, B), \\ &E_1^{p,q} = \widetilde{E}_1^{p,q} = 0 \quad \text{ for } \ q \neq 0. \end{aligned}$$

Therefore, both spectral sequences collapse in E_1 and we get

$$H^l(K^\bullet) = H^l\big(\mathrm{Hom}(A, D^\bullet)\big) = H^l\big(\mathrm{Hom}(C_\bullet, B)\big) \ ;$$

in particular, the cohomology groups $H^l(K^\bullet)$ do not depend on the choice of the resolutions $C_\bullet$ or $D^\bullet$.

(16.3) Definition. *The q-th extension module of A, B is*

$$\mathrm{Ext}^q_R(A, B) = H^q(K^\bullet) = H^q\big(\mathrm{Hom}(A, D^\bullet)\big) = H^q\big(\mathrm{Hom}(C_\bullet, B)\big).$$

By the left exactness of $\mathrm{Hom}(A, \bullet)$, we get in particular $\mathrm{Ext}^0(A, B) = \mathrm{Hom}(A, B)$. If A is projective or B injective, then clearly $\mathrm{Ext}^q(A, B) = 0$ for all $q \geq 1$. Any exact sequence $0 \to A \to A' \to A'' \to 0$ is converted into an exact sequence by $\mathrm{Hom}(\bullet, D^\bullet)$, thus we get a long exact sequence

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}(A'', B) \longrightarrow \mathrm{Hom}(A', B) \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Ext}^1(A'', B) \cdots \\ &\quad \longrightarrow \mathrm{Ext}^q(A'', B) \longrightarrow \mathrm{Ext}^q(A', B) \longrightarrow \mathrm{Ext}^q(A, B) \longrightarrow \mathrm{Ext}^{q+1}(A'', B) \cdots \end{aligned}$$

Similarly, any exact sequence $0 \to B \to B' \to B'' \to 0$ yields

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B') \longrightarrow \mathrm{Hom}(A,B'') \longrightarrow \mathrm{Ext}^1(A,B) \cdots \\ &\quad \longrightarrow \mathrm{Ext}^q(A,B) \longrightarrow \mathrm{Ext}^q(A,B') \longrightarrow \mathrm{Ext}^q(A,B'') \longrightarrow \mathrm{Ext}^{q+1}(A,B) \cdots \end{aligned}$$

Suppose now that R is a principal ring. Then the resolutions $C_\bullet$ or $D^\bullet$ can be taken of length 1 (any quotient of a divisible module is divisible), thus $\mathrm{Ext}^q(A,B)$ is always 0 for $q \geq 2$. In this case, we simply denote $\mathrm{Ext}^1(A,B) = \mathrm{Ext}(A,B)$. When A is finitely generated, the computation of $\mathrm{Ext}(A,B)$ can be reduced to the cyclic case, since $\mathrm{Ext}(A,B) = 0$ when A is free. For $A = R/aR$, the obvious free resolution $R \xrightarrow{a\times} R$ gives

(16.4) $\quad \mathrm{Ext}_R(R/aR, B) = B/aB.$

(16.5) Lemma. *Let $K_\bullet$ be a homology complex and let $M \to M^\bullet$ be an injective resolution of a R-module M. Let $L^\bullet$ be the simple complex associated to $L^{p,q} = \mathrm{Hom}_R(K_q, M^p)$. There is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}\big(H_{q-1}(K_\bullet), M\big) \longrightarrow H^q(L^\bullet) \longrightarrow \mathrm{Hom}\,\big(H_q(K_\bullet), M\big) \longrightarrow 0.$$

Proof. As the functor $\mathrm{Hom}_R(\bullet, M^p)$ is exact, we get

$$\begin{aligned} {}_LE_1^{p,q} &= \mathrm{Hom}\,\big(H_q(K_\bullet), M^p\big), \\ {}_LE_2^{p,q} &= \begin{cases} \mathrm{Hom}\,\big(H_q(K_\bullet), M\big) & \text{for } \ p = 0, \\ \mathrm{Ext}\big(H_q(K_\bullet), M\big) & \text{for } \ p = 1, \\ 0 & \text{for } \ p \geq 2. \end{cases} \end{aligned}$$

The spectral sequence collapses in E_2, therefore we get

$$\begin{aligned} G_0\big(H^q(L^\bullet)\big) &= \mathrm{Hom}\,\big(H_q(K_\bullet), M\big), \\ G_1\big(H^q(L^\bullet)\big) &= \mathrm{Ext}\big(H_{q-1}(K_\bullet), M\big) \end{aligned}$$

and the expected exact sequence follows. By the same arguments as at the end of the proof of Formula 15.5, we may assume that $K_\bullet$ is split, so that there is a projection $K_q \longrightarrow Z_q$. Then the composite morphism

$$\begin{aligned} \mathrm{Hom}\,\big(H_q(K_\bullet), M\big) = \mathrm{Hom}(Z_q/B_q, M) &\longrightarrow \mathrm{Hom}(K_q/B_q, M) \\ &\subset Z^q(L^\bullet) \longrightarrow H^q(L^\bullet) \end{aligned}$$

defines a splitting of the exact sequence. $\square$

16.B. Poincaré Duality for Sheaves

Let $\mathcal{A}$ be a sheaf of abelian groups on a locally compact topological space X of finite topological dimension $n = \operatorname{topdim} X$. By 13.12 c), $\mathcal{A}$ admits a soft resolution $\mathcal{L}^\bullet$ of length n. If $M \to M^0 \to M^1 \to 0$ is an injective resolution of a R-module M, we introduce the double complex of presheaves $\mathcal{F}^{p,q}_{\mathcal{A},M}$ defined by

$$(16.6)\qquad \mathcal{F}^{p,q}_{\mathcal{A},M}(U) = \operatorname{Hom}_R\big(\mathcal{L}^{n-q}_c(U), M^p\big),$$

where the restriction map $\mathcal{F}^{p,q}_{\mathcal{A},M}(U) \longrightarrow \mathcal{F}^{p,q}_{\mathcal{A},M}(V)$ is the adjoint of the inclusion $\mathcal{L}^{n-q}_c(V) \longrightarrow \mathcal{L}^{n-q}_c(U)$ when $V \subset U$. As $\mathcal{L}^{n-q}$ is soft, any $f \in \mathcal{L}^{n-q}_c(U)$ can be written as $f = \sum f_\alpha$ with (f_α) subordinate to any open covering (U_α) of U ; it follows easily that $\mathcal{F}^{p,q}_{\mathcal{A},M}$ satisfy axioms (II-2.4) of sheaves. The injectivity of M^p implies that $\mathcal{F}^{p,q}_{\mathcal{A},M}$ is a flabby sheaf. By Lemma 16.5, we get a split exact sequence

$$(16.7)\qquad \begin{aligned} 0 \longrightarrow \operatorname{Ext}\big(H^{n-q+1}_c(X,\mathcal{A}), M\big) \longrightarrow H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big) \\ \longrightarrow \operatorname{Hom}\big(H^{n-q}_c(X,\mathcal{A}), M\big) \longrightarrow 0. \end{aligned}$$

This can be seen as an abstract Poincaré duality formula, relating the cohomology groups of a differential sheaf $\mathcal{F}^\bullet_{\mathcal{A},M}$ "dual" of $\mathcal{A}$ to the dual of the cohomology with compact support of $\mathcal{A}$. In concrete applications, it still remains to compute $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big)$. This can be done easily when X is a manifold and $\mathcal{A}$ is a constant or locally constant sheaf.

16.C. Poincaré Duality on Topological Manifolds

Here, X denotes a paracompact topological manifold of dimension n.

(16.8) Definition. *Let L be a R-module. A locally constant sheaf of stalk L on X is a sheaf $\mathcal{A}$ such that every point has a neighborhood Ω on which $\mathcal{A}_{\restriction\Omega}$ is R-isomorphic to the constant sheaf L.*

Thus, a locally constant sheaf $\mathcal{A}$ can be seen as a discrete fiber bundle over X whose fibers are R-modules and whose transition automorphisms are R-linear. If X is locally contractible, a locally constant sheaf of stalk L is given, up to isomorphism, by a representation $\rho : \pi_1(X) \longrightarrow \operatorname{Aut}_R(L)$ of the fundamental group of X, up to conjugation; denoting by $\widetilde{X}$ the universal covering of X, the sheaf $\mathcal{A}$ associated to ρ can be viewed as the quotient of $\widetilde{X} \times L$ by the diagonal action of $\pi_1(X)$. We leave the reader check himself the details of these assertions: in fact similar arguments will be explained in full details in §V-6 when properties of flat vector bundles are discussed.

Let $\mathcal{A}$ be a locally constant sheaf of stalk L, let $\mathcal{L}^\bullet$ be a soft resolution of $\mathcal{A}$ and $\mathcal{F}^{p,q}_{\mathcal{A},M}$ the associated flabby sheaves. For an arbitrary open set $U \subset X$, Formula (16.7) gives a (non canonical) isomorphism

$$H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) \simeq \mathrm{Hom}\,\big(H^{n-q}_c(U,\mathcal{A}),M\big) \oplus \mathrm{Ext}\big(H^{n-q+1}_c(U,\mathcal{A}),M\big)$$

and in the special case $q=0$ a canonical isomorphism

$$(16.9)\quad H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) = \mathrm{Hom}\,\big(H^n_c(U,\mathcal{A}),M\big).$$

For an open subset Ω homeomorphic to $\mathbb{R}^n$, we have $\mathcal{A}_{\restriction\Omega} \simeq L$. Proposition 14.16 and the exact sequence of the pair yield

$$H^q_c(\Omega,L) \simeq H^q(S^n,\{\infty\}\,;\,L) = \begin{cases} L & \text{for } q=n,\\ 0 & \text{for } q\neq n.\end{cases}$$

If $\Omega \simeq \mathbb{R}^n$, we find

$$H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \simeq \mathrm{Hom}(L,M), \qquad H^1\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \simeq \mathrm{Ext}(L,M)$$

and $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) = 0$ for $q \neq 0,1$. Consider open sets $V \subset \Omega$ where V is a deformation retract of Ω. Then the restriction map $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \longrightarrow H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(V)\big)$ is an isomorphism. Taking the direct limit over all such neighborhoods V of a given point $x \in \Omega$, we see that $\mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M})$ and $\mathcal{H}^1(\mathcal{F}^\bullet_{\mathcal{A},M})$ are locally constant sheaves of stalks $\mathrm{Hom}(L,M)$ and $\mathrm{Ext}(L,M)$, and that $\mathcal{H}^q(\mathcal{F}^\bullet_{\mathcal{A},M}) = 0$ for $q \neq 0,1$. When $\mathrm{Ext}(L,M)=0$, the complex $\mathcal{F}^\bullet_{\mathcal{A},M}$ is thus a flabby resolution of $\mathcal{H}^0 = \mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M})$ and we get isomorphisms

$$(16.10)\ H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big) = H^q(X,\mathcal{H}^0),$$
$$(16.11)\ \mathcal{H}^0(U) = H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) = \mathrm{Hom}\,\big(H^n_c(U,\mathcal{A}),M\big).$$

(16.12) Definition. *The locally constant sheaf* $\tau_X = \mathcal{H}^0(\mathcal{F}^\bullet_{\mathbb{Z},\mathbb{Z}})$ *of stalk* $\mathbb{Z}$ *defined by*

$$\tau_X(U) = \mathrm{Hom}_{\mathbb{Z}}\,\big(H^n_c(U,\mathbb{Z}),\mathbb{Z}\big)$$

is called the orientation sheaf (or dualizing sheaf) of X.

This sheaf is given by a homomorphism $\pi_1(X) \longrightarrow \{1,-1\}$; it is not difficult to check that τ_X coincides with the trivial sheaf $\mathbb{Z}$ if and only if X is orientable (cf. exercice 18.?). In general, $H^n_c(U,\mathcal{A}) = H^n_c(U,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{A}(U)$ for any small open set U on which $\mathcal{A}$ is trivial, thus

$$\mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M}) = \tau_X \otimes_{\mathbb{Z}} \mathrm{Hom}(\mathcal{A},M).$$

A combination of (16.7) and (16.10) then gives:

(16.13) Poincaré duality theorem. *Let* X *be a topological manifold, let* $\mathcal{A}$ *be a locally constant sheaf over* X *of stalk* L *and let* M *be a* R*-module such that* $\mathrm{Ext}(L,M)=0$. *There is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}\big(H_c^{n-q+1}(X,\mathcal{A}),M\big) \longrightarrow H^q\big(X,\tau_X\otimes \mathrm{Hom}(\mathcal{A},M)\big) \\ \longrightarrow \mathrm{Hom}\,\big(H_c^{n-q}(X,\mathcal{A}),M\big) \longrightarrow 0.$$

In particular, if either X is orientable or R has characteristic 2, then

$$0 \longrightarrow \mathrm{Ext}\big(H_c^{n-q+1}(X,R),R\big) \longrightarrow H^q(X,R) \longrightarrow \mathrm{Hom}\,\big(H_c^{n-q}(X,R),R\big) \\ \longrightarrow 0. \qquad \square$$

(16.14) Corollary. *Let X be a connected topological manifold, $n=\dim X$. Then for any R-module L*

a) $H_c^n(X,\tau_X\otimes L)\simeq L$;

b) $H_c^n(X,L)\simeq L/2L$ *if X is not orientable.*

Proof. First assume that L is free. For $q=0$ and $\mathcal{A}=\tau_X\otimes L$, the Poincaré duality formula gives an isomorphism

$$\mathrm{Hom}\,\big(H_c^n(X,\tau_X\otimes L),M\big)\simeq \mathrm{Hom}(L,M)$$

and the isomorphism is functorial with respect to morphisms $M\longrightarrow M'$. Taking $M=L$ or $M=H_c^n(X,\tau_X\otimes L)$, we easily obtain the existence of inverse morphisms $H_c^n(X,\tau_X\otimes L)\longrightarrow L$ and $L\longrightarrow H_c^n(X,\tau_X\otimes L)$, hence equality a). Similarly, for $\mathcal{A}=L$ we get

$$\mathrm{Hom}\,\big(H_c^n(X,L),M\big)\simeq H^0\big(X,\tau_X\otimes \mathrm{Hom}(L,M)\big).$$

If X is non orientable, then τ_X is non trivial and the global sections of the sheaf $\tau_X\otimes\mathrm{Hom}(L,M)$ consist of 2-torsion elements of $\mathrm{Hom}(L,M)$, that is

$$\mathrm{Hom}\,\big(H_c^n(X,L),M\big)\simeq \mathrm{Hom}(L/2L,M).$$

Formula b) follows. If L is not free, the result can be extended by using a free resolution $0\to L_1\to L_0\to L\to 0$ and the associated long exact sequence. $\square$

(16.15) Remark. If X is a connected non compact n-dimensional manifold, it can be proved (exercise 18.?) that $H^n(X,\mathcal{A})=0$ for every locally constant sheaf $\mathcal{A}$ on X. $\square$

Assume from now on that X is oriented. Replacing M by $L\otimes M$ and using the obvious morphism $M\longrightarrow \mathrm{Hom}(L,L\otimes M)$, the Poincaré duality theorem yields a morphism

$$H^q(X,M)\longrightarrow \mathrm{Hom}\,\big(H_c^{n-q}(X,L),L\otimes M\big), \tag{16.16}$$

in other words, a bilinear pairing

$$(16.16') \quad H_c^{n-q}(X,L)\times H^q(X,M)\longrightarrow L\otimes M.$$

(16.17) Proposition. *Up to the sign, the above pairing is given by the cup product, modulo the identification $H_c^n(X,L\otimes M)\simeq L\otimes M$.*

Proof. By functoriality in L, we may assume $L=R$. Then we make the following special choices of resolutions:

$$\mathcal{L}^q = R^{[q]} \quad \text{for} \quad q<n, \qquad \mathcal{L}^n=\ker(R^{[q]}\longrightarrow R^{[q+1]}),$$
$$M^0 = \text{an injective module containing} \;\; M_c^{[n]}(X)/d^{n-1}M_c^{[n-1]}(X).$$

We embed M in M^0 by $\lambda\mapsto u\otimes_{\mathbb{Z}}\lambda$ where $u\in\mathbb{Z}^{[n]}(X)$ is a representative of a generator of $H_c^n(X,\mathbb{Z})$, and we set $M^1=M^0/M$. The projection map $M^0\longrightarrow M^1$ can be seen as an extension of

$$\widetilde{d^n} \;:\; M_c^{[n]}(X)/d^{n-1}M_c^{[n-1]}(X)\longrightarrow d^nM_c^{[n]}(X),$$

since $\operatorname{Ker}\widetilde{d^n}\simeq H_c^n(X,M)=M$. The inclusion $d^nM_c^{[n]}(X)\subset M^1$ can be extended into a map $\pi: M_c^{[n+1]}(X)\longrightarrow M^1$. The cup product bilinear map

$$M^{[q]}(U)\times R_c^{[n-q]}(U)\longrightarrow M_c^{[n]}(X)\longrightarrow M^0$$

gives rise to a morphism $M^{[q]}(U)\longrightarrow \mathcal{F}^q_{R,M}(U)$ defined by

$$(16.18)\qquad \begin{array}{rcccc} M^{[q]}(U)\longrightarrow & \mathrm{Hom}\big(\mathcal{L}_c^{n-q}(U),M^0\big) & \oplus & \mathrm{Hom}\big(\mathcal{L}_c^{n-q+1}(U),M^1\big) \\ f\longmapsto & (g\longmapsto f\smile g) & \oplus & \big(h\longmapsto \pi(f\smile h)\big). \end{array}$$

This morphism is easily seen to give a morphism of differential sheaves $M^{[\bullet]}\longrightarrow\mathcal{F}^\bullet_{R,M}$, when $M^{[\bullet]}$ is truncated in degree n, i.e. when $M^{[n]}$ is replaced by $\operatorname{Ker}d^n$. The induced morphism

$$M=\mathcal{H}^0(M^{[\bullet]})\longrightarrow\mathcal{H}^0(\mathcal{F}^\bullet_{R,M})$$

is then the identity map, hence the cup product morphism (16.18) actually induces the Poincaré duality map (16.16). □

(16.19) Remark. If X is an oriented differentiable manifold, the natural isomorphism $H_c^n(X,\mathbb{R})\simeq\mathbb{R}$ given by 16.14 a) corresponds in De Rham cohomology to the integration morphism $f\longmapsto\int_X f$, $f\in\mathcal{D}_n(X)$. Indeed, by a partition of unity, we may assume that $\operatorname{Supp} f\subset\Omega\simeq\mathbb{R}^n$. The proof is thus reduced to the case $X=\mathbb{R}^n$, which itself reduces to $X=\mathbb{R}$ since the cup product is compatible with the wedge product of forms. Let us consider the covering $\mathcal{U}=(]k-1,k+1[)_{k\in\mathbb{Z}}$ and a partition of unity (ψ_k) subordinate to $\mathcal{U}$. The Čech differential

$$\begin{aligned} AC^0(\mathcal{U},\mathbb{Z})&\longrightarrow AC^1(\mathcal{U},\mathbb{Z})\\ (c_k)&\longmapsto(c_{k\,k+1})=(c_{k+1}-c_k)\end{aligned}$$

shows immediately that the generators of $H^1_c(\mathbb{R},\mathbb{Z})$ are the 1-cocycles c such that $c_{01} = \pm 1$ and $c_{k\,k+1} = 0$ for $k \neq 0$. By Formula (6.12), the associated closed differential form is

$$f = c_{01}\psi_1 d\psi_0 + c_{10}\psi_0 d\psi_1,$$

hence $f = \pm\mathbf{1}_{[0,1]}d\psi_0$ and f does satisfy $\int_{\mathbb{R}} f = \pm 1$.

(16.20) Corollary. *If X is an oriented C^∞ manifold, the bilinear map*

$$H^{n-q}_c(X,\mathbb{R}) \times H^q(X,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (\{f\},\{g\}) \longmapsto \int_X f \wedge g$$

is well defined and identifies $H^q(X,\mathbb{R})$ to the dual of $H^{n-q}_c(X,\mathbb{R})$.