Chapter V
Hermitian Vector Bundles

This chapter introduces the basic definitions concerning vector bundles and connections. In the first sections, the concepts of connection, curvature form, first Chern class are considered in the framework of differentiable manifolds. Although we are mainly interested in complex manifolds, the ideas which will be developed in the next chapters also involve real analysis and real geometry as essential tools. In the second part, the special features of connections over complex manifolds are investigated in detail: Chern connections, first Chern class of type $(1,1)$, induced curvature forms on sub- and quotient bundles, ... These general concepts are then illustrated by the example of universal vector bundles over $\mathbb{P}^n$ and over Grassmannians.

1. Definition of Vector Bundles

Let M be a C^∞ differentiable manifold of dimension m and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field. A (real, complex) *vector bundle* of rank r over M is a C^∞ manifold E together with

i) a C^∞ map $\pi : E \longrightarrow M$ called the projection,

ii) a $\mathbb{K}$-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$

such that the vector space structure is *locally trivial.* This means that there exists an open covering $(V_\alpha)_{\alpha \in I}$ of M and C^∞ diffeomorphisms called *trivializations*

$$\theta_\alpha : E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{K}^r, \qquad \text{where} \quad E_{\restriction V_\alpha} = \pi^{-1}(V_\alpha),$$

such that for every $x \in V_\alpha$ the map

$$E_x \xrightarrow{\theta_\alpha} \{x\} \times \mathbb{K}^r \longrightarrow \mathbb{K}^r$$

is a linear isomorphism. For each $\alpha, \beta \in I$, the map

$$\theta_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{K}^r \longrightarrow (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

acts as a linear automorphism on each fiber $\{x\} \times \mathbb{K}^r$. It can thus be written

$$\theta_{\alpha\beta}(x, \xi) = (x, g_{\alpha\beta}(x) \cdot \xi), \qquad (x, \xi) \in (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

where $(g_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ is a collection of invertible matrices with coefficients in $C^\infty(V_\alpha \cap V_\beta, \mathbb{K})$, satisfying the cocycle relation

(1.1) $\quad g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on} \quad V_\alpha \cap V_\beta \cap V_\gamma.$

The collection $(g_{\alpha\beta})$ is called a *system of transition matrices*. Conversely, any collection of invertible matrices satisfying (1.1) defines a vector bundle E, obtained by gluing the charts $V_\alpha \times \mathbb{K}^r$ via the identifications $\theta_{\alpha\beta}$.

(1.2) Example. The product manifold $E = M \times \mathbb{K}^r$ is a vector bundle over M, and is called the *trivial vector bundle* of rank r over M. We shall often simply denote it $\mathbb{K}^r$ for brevity.

(1.3) Example. A much more interesting example of real vector bundle is the *tangent bundle* TM ; if $\tau_\alpha : V_\alpha \longrightarrow \mathbb{R}^n$ is a collection of coordinate charts on M, then $\theta_\alpha = \pi \times d\tau_\alpha : TM_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{R}^m$ define trivializations of TM and the transition matrices are given by $g_{\alpha\beta}(x) = d\tau_{\alpha\beta}(x^\beta)$ where $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ and $x^\beta = \tau_\beta(x)$. The dual $T^\star M$ of TM is called the *cotangent bundle* and the p-th exterior power $\Lambda^p T^\star M$ is called the bundle of differential forms of degree p on M.

(1.4) Definition. *If $\Omega \subset M$ is an open subset and k a positive integer or $+\infty$, we let $C^k(\Omega, E)$ denote the space of C^k sections of $E_{\restriction\Omega}$, i.e. the space of C^k maps $s : \Omega \longrightarrow E$ such that $s(x) \in E_x$ for all $x \in \Omega$ (that is $\pi \circ s = \mathrm{Id}_\Omega$).*

Let $\theta : E_{\restriction V} \longrightarrow V \times \mathbb{K}^r$ be a trivialization of E. To θ, we associate the C^∞ *frame* $(e_1, \ldots, e_r)$ of $E_{\restriction V}$ defined by

$$e_\lambda(x) = \theta^{-1}(x, \varepsilon_\lambda), \qquad x \in V,$$

where (ε_λ) is the standard basis of $\mathbb{K}^r$. A section $s \in C^k(V, E)$ can then be represented in terms of its components $\theta(s) = \sigma = (\sigma_1, \ldots, \sigma_r)$ by

$$s = \sum_{1\le\lambda\le r} \sigma_\lambda e_\lambda \quad \text{on} \quad V, \qquad \sigma_\lambda \in C^k(V, \mathbb{K}).$$

Let (θ_α) be a family of trivializations relative to a covering (V_α) of M. Given a global section $s \in C^k(M, E)$, the components $\theta_\alpha(s) = \sigma^\alpha = (\sigma^\alpha_1, \ldots, \sigma^\alpha_r)$ satisfy the *transition relations*

(1.5) $\quad \sigma^\alpha = g_{\alpha\beta}\, \sigma^\beta \quad \text{on} \quad V_\alpha \cap V_\beta.$

Conversely, any collection of vector valued functions $\sigma^\alpha : V_\alpha \longrightarrow \mathbb{K}^r$ satisfying the transition relations defines a global section s of E.

More generally, we shall also consider differential forms on M with values in E. Such forms are nothing else than sections of the tensor product bundle $\Lambda^p T^\star M \otimes_{\mathbb{R}} E$. We shall write

(1.6) $\quad C_p^k(\Omega, E) = C^k(\Omega, \Lambda^p T^\star M \otimes_{\mathbb{R}} E)$

(1.7) $\quad C_\bullet^k(\Omega, E) = \bigoplus_{0 \le p \le m} C_p^k(\Omega, E).$

2. Linear Connections

A (linear) connection D on the bundle E is a linear differential operator of order 1 acting on $C_\bullet^\infty(M, E)$ and satisfying the following properties:

(2.1) $\quad D : C_q^\infty(M, E) \longrightarrow C_{q+1}^\infty(M, E),$

(2.1′) $\quad D(f \wedge s) = df \wedge s + (-1)^p f \wedge Ds$

for any $f \in C_p^\infty(M, \mathbb{K})$ and $s \in C_q^\infty(M, E)$, where df stands for the usual exterior derivative of f.

Assume that $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$ is a trivialization of $E_{\restriction\Omega}$, and let $(e_1, \ldots, e_r)$ be the corresponding frame of $E_{\restriction\Omega}$. Then any $s \in C_q^\infty(\Omega, E)$ can be written in a unique way

$$s = \sum_{1 \le \lambda \le r} \sigma_\lambda \otimes e_\lambda, \qquad \sigma_\lambda \in C_q^\infty(\Omega, \mathbb{K}).$$

By axiom (2.1′) we get

$$Ds = \sum_{1 \le \lambda \le r} \big(d\sigma_\lambda \otimes e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda\big).$$

If we write $De_\mu = \sum_{1 \le \lambda \le r} a_{\lambda\mu} \otimes e_\lambda$ where $a_{\lambda\mu} \in C_1^\infty(\Omega, \mathbb{K})$, we thus have

$$Ds = \sum_\lambda \big(d\sigma_\lambda + \sum_\mu a_{\lambda\mu} \wedge \sigma_\mu\big) \otimes e_\lambda.$$

Identify $E_{\restriction\Omega}$ with $\Omega \times \mathbb{K}^r$ via θ and denote by d the trivial connection $d\sigma = (d\sigma_\lambda)$ on $\Omega \times \mathbb{K}^r$. Then the operator D can be written

(2.2) $\quad Ds \simeq_\theta \; d\sigma + A \wedge \sigma$

where $A = (a_{\lambda\mu}) \in C_1^\infty(\Omega, \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r))$. Conversely, it is clear that any operator D defined in such a way is a connection on $E_{\restriction\Omega}$. The matrix 1-form A will be called the *connection form* of D associated to the trivialization θ. If $\widetilde{\theta} : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$ is another trivialization and if we set

$$g = \widetilde{\theta} \circ \theta^{-1} \in C^\infty(\Omega, \mathrm{Gl}(\mathbb{K}^r))$$

then the new components $\widetilde{\sigma} = (\widetilde{\sigma}_\lambda)$ are related to the old ones by $\widetilde{\sigma} = g\sigma$. Let $\widetilde{A}$ be the connection form of D with respect to $\widetilde{\theta}$. Then

$$
\begin{aligned}
Ds &\simeq_{\widetilde{\theta}} \; d\widetilde{\sigma} + \widetilde{A} \wedge \widetilde{\sigma} \\
Ds &\simeq_{\theta} \; g^{-1}(d\widetilde{\sigma} + \widetilde{A} \wedge \widetilde{\sigma}) = g^{-1}(d(g\sigma) + \widetilde{A} \wedge g\sigma) \\
&= \; d\sigma + (g^{-1}\widetilde{A}g + g^{-1}dg) \wedge \sigma.
\end{aligned}
$$

Therefore we obtain the *gauge transformation law*:

(2.3) $\quad A = g^{-1}\widetilde{A}g + g^{-1}dg.$

3. Curvature Tensor

Let us compute $D^2 : C^\infty_q(M, E) \to C^\infty_{q+2}(M, E)$ with respect to the trivialization $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$. We obtain

$$
\begin{aligned}
D^2 s &\simeq_\theta \; d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma) \\
&= d^2\sigma + (dA \wedge \sigma - A \wedge d\sigma) + (A \wedge d\sigma + A \wedge A \wedge \sigma) \\
&= (dA + A \wedge A) \wedge \sigma.
\end{aligned}
$$

It follows that there exists a global 2-form $\Theta(D) \in C^\infty_2(M, \mathrm{Hom}(E, E))$ called *the curvature tensor* of D, such that

$$D^2 s = \Theta(D) \wedge s,$$

given with respect to any trivialization θ by

(3.1) $\quad \Theta(D) \simeq_\theta \; dA + A \wedge A.$

(3.2) Remark. If E is of rank $r = 1$, then $A \in C^\infty_1(M, \mathbb{K})$ and $\mathrm{Hom}(E, E)$ is canonically isomorphic to the trivial bundle $M \times \mathbb{K}$, because the endomorphisms of each fiber E_x are homotheties. With the identification $\mathrm{Hom}(E, E) = \mathbb{K}$, the curvature tensor $\Theta(D)$ can be considered as a closed 2-form with values in $\mathbb{K}$:

(3.3) $\quad \Theta(D) = dA.$

In this case, the gauge transformation law can be written

(3.4) $\quad A = \widetilde{A} + g^{-1}dg, \qquad g = \widetilde{\theta} \circ \theta^{-1} \in C^\infty(\Omega, \mathbb{K}^\star).$

It is then immediately clear that $dA = d\widetilde{A}$, and this equality shows again that dA does not depend on θ. □

Now, we show that the curvature tensor is closely related to commutation properties of covariant derivatives.

(3.5) Definition. *If ξ is a C^∞ vector field with values in TM, the covariant derivative of a section $s \in C^\infty(M,E)$ in the direction ξ is the section $\xi_D \cdot s \in C^\infty(M,E)$ defined by $\xi_D \cdot s = Ds \cdot \xi$.*

(3.6) Proposition. *For all sections $s \in C^\infty(M,E)$ and all vector fields $\xi, \eta \in C^\infty(M,TM)$, we have*

$$\xi_D \cdot (\eta_D \cdot s) - \eta_D \cdot (\xi_D \cdot s) = [\xi,\eta]_D \cdot s + \Theta(D)(\xi,\eta) \cdot s$$

where $[\xi,\eta] \in C^\infty(M,TM)$ is the Lie bracket of ξ, η.

Proof. Let $(x_1,\ldots,x_m)$ be local coordinates on an open set $\Omega \subset M$. Let $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{K}^r$ be a trivialization of E and let A be the corresponding connection form. If $\xi = \sum \xi_j\, \partial/\partial x_j$ and $A = \sum A_j\, dx_j$, we find

$$(3.7) \quad \xi_D s \simeq_\theta (d\sigma + A\sigma)\cdot\xi = \sum_j \xi_j \Big(\frac{\partial\sigma}{\partial x_j} + A_j \cdot \sigma\Big).$$

Now, we compute the above commutator $[\xi_D, \eta_D]$ at a given point $z_0 \in \Omega$. Without loss of generality, we may assume $A(z_0) = 0$; in fact, one can always find a gauge transformation g near z_0 such that $g(z_0) = \mathrm{Id}$ and $dg(z_0) = A(z_0)$; then (2.3) yields $\widetilde{A}(z_0) = 0$. If $\eta = \sum \eta_k\, \partial/\partial x_k$, we find $\eta_D \cdot s \simeq_\theta \sum \eta_k\, \partial\sigma/\partial x_k$ at z_0, hence

$$\begin{aligned}
&\eta_D \cdot (\xi_D \cdot s) \simeq_\theta \sum_k \eta_k \frac{\partial}{\partial x_k} \sum_j \xi_j \Big(\frac{\partial\sigma}{\partial x_j} + A_j \cdot \sigma\Big),\\
&\xi_D \cdot (\eta_D \cdot s) - \eta_D \cdot (\xi_D \cdot s) \simeq_\theta\\
&\qquad\qquad \simeq_\theta \sum_{j,k} \Big(\xi_k \frac{\partial\eta_j}{\partial x_k} - \eta_k \frac{\partial\xi_j}{\partial x_k}\Big)\frac{\partial\sigma}{\partial x_j} + \sum_{j,k} \frac{\partial A_j}{\partial x_k}(\xi_j\eta_k - \eta_j\xi_k)\cdot\sigma\\
&\qquad\qquad = d\sigma([\xi,\eta]) + dA(\xi,\eta)\cdot\sigma,
\end{aligned}$$

whereas $\Theta(D) \simeq_\theta dA$ and $[\xi,\eta]_D s \simeq_\theta d\sigma([\xi,\eta])$ at point z_0. □

4. Operations on Vector Bundles

Let E, F be vector bundles of rank r_1, r_2 over M. Given any functorial operation on vector spaces, a corresponding operation can be defined on bundles by applying the operation on each fiber. For example $E^\star$, $E \oplus F$, $\mathrm{Hom}(E,F)$ are defined by

$$(E^\star)_x = (E_x)^\star, \qquad (E \oplus F)_x = E_x \oplus F_x, \qquad \mathrm{Hom}(E,F)_x = \mathrm{Hom}(E_x, F_x).$$

The bundles E and F can be trivialized over the same covering V_α of M (otherwise take a common refinement). If $(g_{\alpha\beta})$ and $(\gamma_{\alpha\beta})$ are the transition matrices of E and F, then for example $E \otimes F$, $\Lambda^k E$, $E^\star$ are the bundles defined by the transition matrices $g_{\alpha\beta} \otimes \gamma_{\alpha\beta}$, $\Lambda^k g_{\alpha\beta}$, $(g^\dagger_{\alpha\beta})^{-1}$ where $\dagger$ denotes transposition.

Suppose now that E, F are equipped with connections D_E, D_F. Then natural connections can be associated to all derived bundles. Let us mention a few cases. First, we let

$$(4.1)\qquad D_{E\oplus F} = D_E \oplus D_F.$$

It follows immediately that

$$(4.1')\qquad \Theta(D_{E\oplus F}) = \Theta(D_E) \oplus \Theta(D_F).$$

$D_{E\otimes F}$ will be defined in such a way that the usual formula for the differentiation of a product remains valid. For every $s \in C^\infty_\bullet(M, E)$, $t \in C^\infty_\bullet(M, F)$, the wedge product $s \wedge t$ can be combined with the bilinear map $E \times F \longrightarrow E \otimes F$ in order to obtain a section $s \wedge t \in C^\infty(M, E \otimes F)$ of degree $\deg s + \deg t$. Then there exists a unique connection $D_{E\otimes F}$ such that

$$(4.2)\qquad D_{E\otimes F}(s \wedge t) = D_E s \wedge t + (-1)^{\deg\, s} s \wedge D_F t.$$

As the products $s \wedge t$ generate $C^\infty_\bullet(M, E \otimes F)$, the uniqueness is clear. If E, F are trivial on an open set $\Omega \subset M$ and if A_E, A_F, are their connection 1-forms, the induced connection $D_{E\otimes F}$ is given by the connection form $A_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes A_F$. The existence follows. An easy computation shows that $D^2_{E\otimes F}(s \wedge t) = D^2_E s \wedge t + s \wedge D^2_F t$, thus

$$(4.2')\qquad \Theta(D_{E\otimes F}) = \Theta(D_E) \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Theta(D_F).$$

Similarly, there are unique connections $D_{E^\star}$ and $D_{\mathrm{Hom}(E,F)}$ such that

$$(4.3)\qquad (D_{E^\star} u) \cdot s = d(u \cdot s) - (-1)^{\deg\, u} u \cdot D_E s,$$

$$(4.4)\qquad (D_{\mathrm{Hom}(E,F)} v) \cdot s = D_F(v \cdot s) - (-1)^{\deg\, v} v \cdot D_E s$$

whenever $s \in C^\infty_\bullet(M, E)$, $u \in C^\infty_\bullet(M, E^\star)$, $v \in C^\infty_\bullet\big(\mathrm{Hom}(E, F)\big)$. It follows that

$$0 = d^2(u \cdot s) = \big(\Theta(D_{E^\star}) \cdot u\big) \cdot s + u \cdot \big(\Theta(D_E) \cdot s\big).$$

If $\dagger$ denotes the transposition operator $\mathrm{Hom}(E, E) \to \mathrm{Hom}(E^\star, E^\star)$, we thus get

$$(4.3')\qquad \Theta(D_{E^\star}) = -\Theta(D_E)^\dagger.$$

With the identification $\mathrm{Hom}(E, F) = E^\star \otimes F$, Formula (4.2′) implies

$$(4.4')\qquad \Theta(D_{\mathrm{Hom}(E,F)}) = \mathrm{Id}_{E^\star} \otimes \Theta(D_F) - \Theta(D_E)^\dagger \otimes \mathrm{Id}_F\,.$$

Finally, $\Lambda^k E$ carries a natural connection $D_{\Lambda^k E}$. For every $s_1, \ldots, s_k$ in $C^\infty_\bullet(M, E)$ of respective degrees $p_1, \ldots, p_k$, this connection satisfies

$$(4.5)\qquad D_{\Lambda^k E}(s_1 \wedge \ldots \wedge s_k) = \sum_{1\le j\le k} (-1)^{p_1+\ldots+p_{j-1}} s_1 \wedge \ldots D_E s_j \ldots \wedge s_k,$$

$$(4.5')\qquad \Theta(D_{\Lambda^k E}) \cdot (s_1 \wedge \ldots \wedge s_k) = \sum_{1\le j\le k} s_1 \wedge \ldots \wedge \Theta(D_E) \cdot s_j \wedge \ldots \wedge s_k.$$

In particular, the *determinant bundle*, defined by $\det E = \Lambda^r E$ where r is the rank of E, has a curvature form given by

$$(4.6)\qquad \Theta(D_{\det E}) = \mathcal{T}_E\big(\Theta(D_E)\big)$$

where $\mathcal{T}_E : \mathrm{Hom}(E, E) \longrightarrow \mathbb{K}$ is the trace operator. As a conclusion of this paragraph, we mention the following simple identity.

(4.7) Bianchi identity. $D_{\mathrm{Hom}(E,E)}\big(\Theta(D_E)\big) = 0.$

Proof. By definition of $D_{\mathrm{Hom}(E,E)}$, we find for any $s \in C^\infty(M, E)$

$$\begin{aligned} D_{\mathrm{Hom}(E,E)}\big(\Theta(D_E)\big) \cdot s &= D_E\big(\Theta(D_E) \cdot s\big) - \Theta(D_E) \cdot (D_E s) \\ &= D_E^3 s - D_E^3 s = 0. \end{aligned}$$

□

5. Pull-Back of a Vector Bundle

Let $\widetilde{M}$, M be C^∞ manifolds and $\psi : \widetilde{M} \to M$ a smooth map. If E is a vector bundle on M, one can define in a natural way a C^∞ vector bundle $\widetilde{\pi} : \widetilde{E} \to \widetilde{M}$ and a C^∞ linear morphism $\Psi : \widetilde{E} \to E$ such that the diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\Psi} & E \\ \downarrow\widetilde{\pi} & & \downarrow\pi \\ \widetilde{M} & \xrightarrow{\psi} & M \end{array}$$

commutes and such that $\Psi : \widetilde{E}_x \longrightarrow E_{\psi(x)}$ is an isomorphism for every $x \in M$. The bundle $\widetilde{E}$ can be defined by

$$(5.1)\qquad \widetilde{E} = \{(\widetilde{x}, \xi) \in \widetilde{M} \times E\ ;\ \psi(\widetilde{x}) = \pi(\xi)\}$$

and the maps $\widetilde{\pi}$ and Ψ are then the restrictions to $\widetilde{E}$ of the projections of $\widetilde{M} \times E$ on $\widetilde{M}$ and E respectively.

If $\theta_\alpha : E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{K}^r$ are trivializations of E, the maps

$$\widetilde{\theta}_\alpha = \theta_\alpha \circ \Psi : \widetilde{E}_{\restriction \psi^{-1}(V_\alpha)} \longrightarrow \psi^{-1}(V_\alpha) \times \mathbb{K}^r$$

define trivializations of $\widetilde{E}$ with respect to the covering $\widetilde{V}_\alpha = \psi^{-1}(V_\alpha)$ of $\widetilde{M}$. The corresponding system of transition matrices is given by

(5.2) $\widetilde{g}_{\alpha\beta} = g_{\alpha\beta} \circ \psi \quad \text{on} \quad \widetilde{V}_\alpha \cap \widetilde{V}_\beta.$

(5.3) Definition. *$\widetilde{E}$ is termed the pull-back of E under the map ψ and is denoted $\widetilde{E} = \psi^\star E$.*

Let D be a connection on E. If (A_α) is the collection of connection forms of D with respect to the θ_α's, one can define a connection $\widetilde{D}$ on $\widetilde{E}$ by the collection of connection forms $\widetilde{A}_\alpha = \psi^\star A_\alpha \in C^\infty_1\big(\widetilde{V}_\alpha, \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r)\big)$, i.e. for every $\widetilde{s} \in C^\infty_p(\widetilde{V}_\alpha, \widetilde{E})$

$$\widetilde{D}\widetilde{s} \simeq_{\widetilde{\theta}_\alpha} d\widetilde{\sigma} + \psi^\star A_\alpha \wedge \widetilde{\sigma}.$$

Given any section $s \in C^\infty_p(M, E)$, one defines a pull back $\psi^\star s$ which is a section in $C^\infty_p(\widetilde{M}, \widetilde{E})$: for $s = f \otimes u$, $f \in C^\infty_p(M, \mathbb{K})$, $u \in C^\infty(M, E)$, set $\psi^\star s = \psi^\star f \otimes (u \circ \psi)$. Then we have the formula

(5.4) $\widetilde{D}(\psi^\star s) = \psi^\star(Ds).$

Using (5.4), a simple computation yields

(5.5) $\Theta(\widetilde{D}) = \psi^\star(\Theta(D)).$

6. Parallel Translation and Flat Vector Bundles

Let $\gamma : [0,1] \longrightarrow M$ be a smooth curve and $s : [0,1] \to E$ a C^∞ section of E along γ, i.e. a C^∞ map s such that $s(t) \in E_{\gamma(t)}$ for all $t \in [0,1]$. Then s can be viewed as a section of $\widetilde{E} = \gamma^\star E$ over $[0,1]$. The *covariant derivative* of s is the section of E along γ defined by

(6.1) $$\frac{Ds}{dt} = \widetilde{D}s(t) \cdot \frac{d}{dt} \in E_{\gamma(t)},$$

where $\widetilde{D}$ is the induced connection on $\widetilde{E}$. If A is a connection form of D with respect to a trivialization $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{K}^r$, we have $\widetilde{D}s \simeq_\theta d\sigma + \gamma^\star A \cdot \sigma$, i.e.

(6.2) $$\frac{Ds}{dt} \simeq_\theta \frac{d\sigma}{dt} + \big(A(\gamma(t)) \cdot \gamma'(t)\big) \cdot \sigma(t) \quad \text{for} \quad \gamma(t) \in \Omega.$$

For $v \in E_{\gamma(0)}$ given, the Cauchy uniqueness and existence theorem for ordinary linear differential equations implies that there exists a unique section s of $\widetilde{E}$ such that $s(0) = v$ and $Ds/dt = 0$.

(6.3) Definition. *The linear map*

$$T_\gamma : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}, \qquad v = s(0) \longmapsto s(1)$$

is called parallel translation along γ.

If $\gamma = \gamma_2\gamma_1$ is the composite of two paths γ_1, γ_2 such that $\gamma_2(0) = \gamma_1(1)$, it is clear that $T_\gamma = T_{\gamma_2} \circ T_{\gamma_1}$, and the inverse path $\gamma^{-1} : t \mapsto \gamma(1-t)$ is such that $T_{\gamma^{-1}} = T_\gamma^{-1}$. It follows that T_γ is a linear isomorphism from $E_{\gamma(0)}$ onto $E_{\gamma(1)}$.

More generally, if $h : W \longrightarrow M$ is a C^∞ map from a domain $W \subset \mathbb{R}^p$ into M and if s is a section of $h^\star E$, we define covariant derivatives $Ds/\partial t_j$, $1 \le j \le p$, by $\widetilde{D} = h^\star D$ and

$$(6.4) \qquad \frac{Ds}{\partial t_j} = \widetilde{D}s \cdot \frac{\partial}{\partial t_j}.$$

Since $\partial/\partial t_j$, $\partial/\partial t_k$ commute and since $\Theta(\widetilde{D}) = h^\star\,\Theta(D)$, Prop. 3.6 implies

$$(6.5) \qquad \frac{D}{\partial t_j}\frac{Ds}{\partial t_k} - \frac{D}{\partial t_k}\frac{Ds}{\partial t_j} = \Theta(\widetilde{D})\Big(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\Big) \cdot s = \Theta(D)_{h(t)}\Big(\frac{\partial h}{\partial t_j}, \frac{\partial h}{\partial t_k}\Big) \cdot s(t).$$

(6.6) Definition. *The connection D is said to be flat if $\Theta(D) = 0$.*

Assume from now on that D is flat. We then show that T_γ only depends on the homotopy class of γ. Let $h : [0,1] \times [0,1] \longrightarrow M$ be a smooth homotopy $h(t,u) = \gamma_u(t)$ from γ_0 to γ_1 with fixed end points $a = \gamma_u(0)$, $b = \gamma_u(1)$. Let $v \in E_a$ be given and let $s(t,u)$ be such that $s(0,u) = v$ and $Ds/\partial t = 0$ for all $u \in [0,1]$. Then s is C^∞ in both variables (t,u) by standard theorems on the dependence of parameters. Moreover (6.5) implies that the covariant derivatives $D/\partial t$, $D/\partial u$ commute. Therefore, if we set $s' = Ds/\partial u$, we find $Ds'/\partial t = 0$ with initial condition $s'(0,u) = 0$ (recall that $s(0,u)$ is a constant). The uniqueness of solutions of differential equations implies that s' is identically zero on $[0,1] \times [0,1]$, in particular $T_{\gamma_u}(v) = s(1,u)$ must be constant, as desired.

(6.7) Proposition. *Assume that D is flat. If Ω is a simply connected open subset of M, then $E_{\restriction\Omega}$ admits a C^∞ parallel frame $(e_1, \ldots, e_r)$, in the sense that $De_\lambda = 0$ on Ω, $1 \le \lambda \le r$. For any two simply connected open subsets Ω, Ω' the transition automorphism between the corresponding parallel frames (e_λ) and (e'_λ) is locally constant.*

The converse statement "*E has parallel frames near every point implies that $\Theta(D) = 0$* " can be immediately verified from the equality $\Theta(D) = D^2$.

Proof. Choose a base point $a \in \Omega$ and define a linear isomorphism $\Phi : \Omega \times E_a \longrightarrow E_{\restriction\Omega}$ by sending (x,v) on $T_\gamma(v) \in E_x$, where γ is any path from a to x in Ω (two such paths are always homotopic by hypothesis). Now, for any path γ from a to x, we have by construction $(D/dt)\Phi(\gamma(t), v) = 0$. Set $e_v(x) = \Phi(x,v)$. As γ may reach any point $x \in \Omega$ with an arbitrary tangent vector $\xi = \gamma'(1) \in T_xM$, we get $De_v(x) \cdot \xi = (D/dt)\Phi(\gamma(t), v)_{\restriction t=1} = 0$. Hence De_v is parallel for any fixed vector $v \in E_a$; Prop. 6.7 follows. □

Assume that M is connected. Let a be a base point and $\widetilde{M} \longrightarrow M$ the universal covering of M. The manifold $\widetilde{M}$ can be considered as the set of pairs $(x, [\gamma])$, where $[\gamma]$ is a homotopy class of paths from a to x. Let $\pi_1(M)$ be the fundamental group of M with base point a, acting on $\widetilde{M}$ on the left by $[\kappa] \cdot (x, [\gamma]) = (x, [\gamma\kappa^{-1}])$. If D is flat, $\pi_1(M)$ acts also on E_a by $([\kappa], v) \mapsto T_\kappa(v)$, $[\kappa] \in \pi_1(M)$, $v \in E_a$, and we have a well defined map

$$\Psi : \widetilde{M} \times E_a \longrightarrow E, \qquad \Psi(x, [\gamma]) = T_\gamma(v).$$

Then Ψ is invariant under the left action of $\pi_1(M)$ on $\widetilde{M} \times E_a$ defined by

$$[\kappa] \cdot \big((x, [\gamma]), v\big) = \big((x, [\gamma\kappa^{-1}]), T_\kappa(v)\big),$$

therefore we have an isomorphism $E \simeq (\widetilde{M} \times E_a)/\pi_1(M)$.

Conversely, let S be a $\mathbb{K}$-vector space of dimension r together with a left action of $\pi_1(M)$. The quotient $E = (\widetilde{M} \times S)/\pi_1(M)$ is a vector bundle over M with locally constant transition automorphisms $(g_{\alpha\beta})$ relatively to any covering (V_α) of M by simply connected open sets. The relation $\sigma^\alpha = g_{\alpha\beta}\,\sigma^\beta$ implies $d\sigma^\alpha = g_{\alpha\beta}\,d\sigma^\beta$ on $V_\alpha \cap V_\beta$. We may therefore define a connection D on E by letting $Ds \simeq_{\theta_\alpha} d\sigma^\alpha$ on each V_α. Then clearly $\Theta(D) = 0$.

7. Hermitian Vector Bundles and Connections

A complex vector bundle E is said to be *hermitian* if a positive definite hermitian form $|\ \ |^2$ is given on each fiber E_x in such a way that the map $E \to \mathbb{R}_+,\ \xi \mapsto |\xi|^2$ is smooth. The notion of a euclidean (real) vector bundle is similar, so we leave the reader adapt our notations to that case.

Let $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{C}^r$ be a trivialization and let $(e_1, \ldots, e_r)$ be the corresponding frame of $E_{\restriction\Omega}$. The associated inner product of E is given by a positive definite hermitian matrix $(h_{\lambda\mu})$ with C^∞ coefficients on Ω, such that

$$\langle e_\lambda(x), e_\mu(x)\rangle = h_{\lambda\mu}(x), \qquad \forall x \in \Omega.$$

When E is hermitian, one can define a natural sesquilinear map

$$\begin{aligned} C^\infty_p(M,E) \times C^\infty_q(M,E) &\longrightarrow C^\infty_{p+q}(M,\mathbb{C}) \\ (s,t) &\longmapsto \{s,t\} \end{aligned} \tag{7.1}$$

combining the wedge product of forms with the hermitian metric on E ; if $s = \sum \sigma_\lambda \otimes e_\lambda$, $t = \sum \tau_\mu \otimes e_\mu$, we let

$$\{s,t\} = \sum_{1\le\lambda,\mu\le r} \sigma_\lambda \wedge \overline{\tau}_\mu \, \langle e_\lambda, e_\mu\rangle.$$

A connection D is said to be compatible with the hermitian structure of E, or briefly *hermitian*, if for every $s \in C^\infty_p(M,E)$, $t \in C^\infty_q(M,E)$ we have

$$(7.2)\quad d\{s,t\} = \{Ds,t\} + (-1)^p\{s,Dt\}.$$

Let $(e_1,\ldots,e_r)$ be an *orthonormal frame* of $E_{\restriction\Omega}$. Denote $\theta(s) = \sigma = (\sigma_\lambda)$ and $\theta(t) = \tau = (\tau_\lambda)$. Then

$$\{s,t\} = \{\sigma,\tau\} = \sum_{1\le\lambda\le r} \sigma_\lambda \wedge \overline{\tau}_\lambda,$$
$$d\{s,t\} = \{d\sigma,\tau\} + (-1)^p\{\sigma,d\tau\}.$$

Therefore $D_{\restriction\Omega}$ is hermitian if and only if its connection form A satisfies

$$\{A\sigma,\tau\} + (-1)^p\{\sigma,A\tau\} = \{(A+A^\star)\wedge\sigma,\tau\} = 0$$

for all σ,τ, i.e.

$$(7.3)\quad A^\star = -A \quad \text{or} \quad (\overline{a_{\mu\lambda}}) = -(a_{\lambda\mu}).$$

This means that $\mathrm{i}A$ is a 1-form with values in the space $\mathrm{Herm}(\mathbb{C}^r,\mathbb{C}^r)$ of hermitian matrices. The identity $d^2\{s,t\} = 0$ implies $\{D^2s,t\}+\{s,D^2t\} = 0$, i.e. $\{\Theta(D)\wedge s,t\} + \{s,\Theta(D)\wedge t\} = 0$. Therefore $\Theta(D)^\star = -\Theta(D)$ and the curvature tensor $\Theta(D)$ is such that

$$\mathrm{i}\,\Theta(D) \in C^\infty_2(M,\mathrm{Herm}(E,E)).$$

(7.4) Special case. If E is a hermitian line bundle ($r=1$), $D_{\restriction\Omega}$ is a hermitian connection if and only if its connection form A associated to any given orthonormal frame of $E_{\restriction\Omega}$ is a 1-form with purely imaginary values.

If $\theta,\widetilde{\theta} : E_{\restriction\Omega} \to \Omega$ are two such trivializations on a simply connected open subset $\Omega \subset M$, then $g = \widetilde{\theta}\circ\theta^{-1} = e^{\mathrm{i}\varphi}$ for some real phase function $\varphi \in C^\infty(\Omega,\mathbb{R})$. The gauge transformation law can be written

$$A = \widetilde{A} + \mathrm{i}\,d\varphi.$$

In this case, we see that $\mathrm{i}\,\Theta(D) \in C^\infty_2(M,\mathbb{R})$.

(7.5) Remark. If $s,s' \in C^\infty(M,E)$ are two sections of E along a smooth curve $\gamma : [0,1] \longrightarrow M$, one can easily verify the formula

$$\frac{d}{dt}\langle s(t),s'(t)\rangle = \langle\frac{Ds}{dt},s'\rangle + \langle s,\frac{Ds'}{dt}\rangle.$$

In particular, if $(e_1, \ldots, e_r)$ is a parallel frame of E along γ such that $\big(e_1(0), \ldots, e_r(0)\big)$ is orthonormal, then $\big(e_1(t), \ldots, e_r(t)\big)$ is orthonormal for all t. All parallel translation operators T_γ defined in §6 are thus isometries of the fibers. It follows that E has a flat hermitian connection D if and only if E can be defined by means of locally constant unitary transition automorphisms $g_{\alpha\beta}$, or equivalently if E is isomorphic to the hermitian bundle $(\widetilde{M}\times S)/\pi_1(M)$ defined by a unitary representation of $\pi_1(M)$ in a hermitian vector space S. Such a bundle E is said to be *hermitian flat.*

8. Vector Bundles and Locally Free Sheaves

We denote here by $\mathcal{E}$ the sheaf of germs of C^∞ complex functions on M. Let $F \longrightarrow M$ be a C^∞ complex vector bundle of rank r. We let $\mathcal{F}$ be the sheaf of germs of C^∞ sections of F, i.e. the sheaf whose space of sections on an open subset $U \subset M$ is $\mathcal{F}(U) = C^\infty(U, F)$. It is clear that $\mathcal{F}$ is a $\mathcal{E}$-module. Furthermore, if $F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}^r$ is trivial, the sheaf $\mathcal{F}_{\restriction\Omega}$ is isomorphic to $\mathcal{E}^r_{\restriction\Omega}$ as a $\mathcal{E}_{\restriction\Omega}$-module.

(8.1) Definition. *A sheaf $\mathcal{S}$ of modules over a sheaf of rings $\mathcal{R}$ is said to be locally free of rank k if every point in the base has a neighborhood Ω such that $\mathcal{S}_{\restriction\Omega}$ is $\mathcal{R}$-isomorphic to $\mathcal{R}^k_{\restriction\Omega}$.*

Suppose that $\mathcal{S}$ is a locally free $\mathcal{E}$-module of rank r. There exists a covering $(V_\alpha)_{\alpha\in I}$ of M and sheaf isomorphisms

$$\theta_\alpha : \mathcal{S}_{\restriction V_\alpha} \longrightarrow \mathcal{E}^r_{\restriction V_\alpha}.$$

Then we have transition isomorphisms $g_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : \mathcal{E}^r \to \mathcal{E}^r$ defined on $V_\alpha \cap V_\beta$, and such an isomorphism is the multiplication by an invertible matrix with C^∞ coefficients on $V_\alpha \cap V_\beta$. The concepts of vector bundle and of locally free $\mathcal{E}$-module are thus completely equivalent.

Assume now that $F \longrightarrow M$ is a line bundle ($r = 1$). Then every collection of transition automorphisms $g = (g_{\alpha\beta})$ defines a Čech 1-cocycle with values in the multiplicative sheaf $\mathcal{E}^\star$ of invertible C^∞ functions on M. In fact the definition of the Čech differential (cf. (IV-5.1)) gives $(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta}$, and we have $\delta g = 1$ in view of (1.1). Let θ'_α be another family of trivializations and $(g'_{\alpha\beta})$ the associated cocycle (it is no loss of generality to assume that both are defined on the same covering since we may otherwise take a refinement). Then we have

$$\theta'_\alpha \circ \theta_\alpha^{-1} : V_\alpha \times \mathbb{C} \longrightarrow V_\alpha \times \mathbb{C}, \qquad (x, \xi) \longmapsto (x, u_\alpha(x)\xi), \qquad u_\alpha \in \mathcal{E}^\star(V_\alpha).$$

It follows that $g_{\alpha\beta} = g'_{\alpha\beta} u_\alpha^{-1} u_\beta$, i.e. that the Čech 1-cocycles g, g' differ only by the Čech 1-coboundary δu. Therefore, there is a well defined map which associates to every line bundle F over M the Čech cohomology class $\{g\} \in H^1(M, \mathcal{E}^\star)$ of its cocycle of transition automorphisms. It is easy to verify that the cohomology classes associated to two line bundles F, F' are equal if and only if these bundles are isomorphic: if $g = g' \cdot \delta u$, then the collection of maps

$$\begin{aligned} F_{\restriction V_\alpha} \xrightarrow{\theta_\alpha} V_\alpha \times \mathbb{C} \longrightarrow V_\alpha \times \mathbb{C} &\xrightarrow{\theta'^{-1}_\alpha} F'_{\restriction V_\alpha} \\ (x, \xi) \longmapsto (x, u_\alpha(x)\xi) & \end{aligned}$$

defines a global isomorphism $F \to F'$. It is clear that the multiplicative group structure on $H^1(M, \mathcal{E}^\star)$ corresponds to the tensor product of line bundles (the inverse of a line bundle being given by its dual). We may summarize this discussion by the following:

(8.2) Theorem. *The group of isomorphism classes of complex C^∞ line bundles is in one-to-one correspondence with the Čech cohomology group $H^1(M, \mathcal{E}^\star)$.*

9. First Chern Class

Throughout this section, we assume that E is a complex line bundle (that is, $\operatorname{rk} E = r = 1$). Let D be a connection on E. By (3.3), $\Theta(D)$ is a closed 2-form on M. Moreover, if D' is another connection on E, then (2.2) shows that $D' = D + \Gamma \wedge \bullet$ where $\Gamma \in C_1^\infty(M, \mathbb{C})$. By (3.3), we get

(9.1) $\Theta(D') = \Theta(D) + d\Gamma.$

This formula shows that the De Rham class $\{\Theta(D)\} \in H^2_{DR}(M, \mathbb{C})$ does not depend on the particular choice of D. If D is chosen to be hermitian with respect to a given hermitian metric on E (such a connection can always be constructed by means of a partition of unity) then $\mathrm{i}\,\Theta(D)$ is a real 2-form, thus $\{\mathrm{i}\,\Theta(D)\} \in H^2_{DR}(M, \mathbb{R})$. Consider now the one-to-one correspondence given by Th. 8.2:

$$\begin{aligned} \{\text{isomorphism classes of line bundles}\} &\longrightarrow H^1(M, \mathcal{E}^\star) \\ \text{class } \{E\} \text{ defined by the cocycle } (g_{\alpha\beta}) &\longmapsto \text{class of } (g_{\alpha\beta}). \end{aligned}$$

Using the exponential exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} &\longrightarrow \mathcal{E}^\star \longrightarrow 1 \\ f &\longmapsto e^{2\pi \mathrm{i} f} \end{aligned}$$

and the fact that $H^1(M, \mathcal{E}) = H^2(M, \mathcal{E}) = 0$, we obtain:

(9.2) Theorem and Definition. *The coboundary morphism*

$$H^1(M, \mathcal{E}^\star) \xrightarrow{\partial} H^2(M, \mathbb{Z})$$

is an isomorphism. The first Chern class of a line bundle E is the image $c_1(E)$ in $H^2(M, \mathbb{Z})$ of the Čech cohomology class of the 1-cocycle $(g_{\alpha\beta})$ associated to E :

$$(9.3) \quad c_1(E) = \partial\{(g_{\alpha\beta})\}.$$

Consider the natural morphism

$$(9.4) \quad H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R}) \simeq H^2_{DR}(M, \mathbb{R})$$

where the isomorphism $\simeq$ is that given by the De Rham-Weil isomorphism theorem and the sign convention of Formula (IV-6.11).

(9.5) Theorem. *The image of $c_1(E)$ in $H^2_{DR}(M, \mathbb{R})$ under* (9.4) *coincides with the De Rham cohomology class $\{\frac{\mathrm{i}}{2\pi}\Theta(D)\}$ associated to any (hermitian) connection D on E.*

Proof. Choose an open covering $(V_\alpha)_{\alpha \in I}$ of M such that E is trivial on each V_α, and such that all intersections $V_\alpha \cap V_\beta$ are simply connected (as in §IV-6, choose the V_α to be small balls relative to a given locally finite covering of M by coordinate patches). Denote by A_α the connection forms of D with respect to a family of isometric trivializations

$$\theta_\alpha : \ E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{C}^r.$$

Let $g_{\alpha\beta} \in \mathcal{E}^\star(V_\alpha \cap V_\beta)$ be the corresponding transition automorphisms. Then $|g_{\alpha\beta}| = 1$, and as $V_\alpha \cap V_\beta$ is simply connected, we may choose real functions $u_{\alpha\beta} \in \mathcal{E}(V_\alpha \cap V_\beta)$ such that

$$g_{\alpha\beta} = \exp(2\pi\mathrm{i}\, u_{\alpha\beta}).$$

By definition, the first Chern class $c_1(E)$ is the Čech 2-cocycle

$$\begin{aligned} c_1(E) =& \partial\{(g_{\alpha\beta})\} = \{(\delta u)_{\alpha\beta\gamma})\} \in H^2(M, \mathbb{Z}) \quad \text{where} \\ (\delta u)_{\alpha\beta\gamma} :=& u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}. \end{aligned}$$

Now, if $\mathcal{E}^q$ (resp. $\mathcal{Z}^q$) denotes the sheaf of real (resp. real d-closed) q-forms on M, the short exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{Z}^2 \longrightarrow 0 \\ 0 &\longrightarrow \mathbb{R} \ \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{Z}^1 \longrightarrow 0 \end{aligned}$$

yield isomorphisms (with the sign convention of (IV-6.11))

$$(9.6) \quad H^2_{DR}(M, \mathbb{R}) := H^0(M, \mathcal{Z}^2)/dH^0(M, \mathcal{E}^1) \xrightarrow{-\partial} H^1(M, \mathcal{Z}^1),$$

$$(9.7) \quad H^1(M, \mathcal{Z}^1) \xrightarrow{\partial} H^2(M, \mathbb{R}).$$

Formula 3.4 gives $A_\beta = A_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$. Since $\Theta(D) = dA_\alpha$ on V_α, the image of $\{\frac{\mathrm{i}}{2\pi}\Theta(D)\}$ under (9.6) is the Čech 1-cocycle with values in $\mathcal{Z}^1$

$$\Big\{ -\frac{\mathrm{i}}{2\pi}(A_\beta - A_\alpha)\Big\} = \Big\{\frac{1}{2\pi\mathrm{i}} g_{\alpha\beta}^{-1} dg_{\alpha\beta}\Big\} = \{du_{\alpha\beta}\}$$

and the image of this cocycle under (9.7) is the Čech 2-cocycle $\{\delta u\}$ in $H^2(M,\mathbb{R})$. But $\{\delta u\}$ is precisely the image of $c_1(E) \in H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$. □

Let us assume now that M is oriented and that $s \in C^\infty(M,E)$ is transverse to the zero section of E, i.e. that $Ds \in \mathrm{Hom}(TM,E)$ is surjective at every point of the zero set $Z := s^{-1}(0)$. Then Z is an oriented 2-codimensional submanifold of M (the orientation of Z is uniquely defined by those of M and E). We denote by $[Z]$ the current of integration over Z and by $\{[Z]\} \in H^2_{DR}(M,\mathbb{R})$ its cohomology class.

(9.8) Theorem. *We have* $\{[Z]\} = c_1(E)_{\mathbb{R}}$.

Proof. Consider the differential 1-form

$$u = s^{-1} \otimes Ds \in C^\infty_1(M \smallsetminus Z, \mathbb{C}).$$

Relatively to any trivialization θ of $E_{\restriction\Omega}$, one has $D_{\restriction\Omega} \simeq_\theta d + A \wedge \bullet$, thus

$$u_{\restriction\Omega} = \frac{d\sigma}{\sigma} + A \quad \text{where} \quad \sigma = \theta(s).$$

It follows that u has locally integrable coefficients on M. If $d\sigma/\sigma$ is considered as a current on Ω, then

$$d\Big(\frac{d\sigma}{\sigma}\Big) = d\Big(\sigma^\star \frac{dz}{z}\Big) = \sigma^\star d\Big(\frac{dz}{z}\Big) = \sigma^\star(2\pi\mathrm{i}\delta_0) = 2\pi\mathrm{i}[Z]$$

because of the Cauchy residue formula (cf. Lemma I-2.10) and because σ is a submersion in a neighborhood of Z (cf. (I-1.19)). Now, we have $dA = \Theta(D)$ and Th. 9.8 follows from the resulting equality:

$$du = 2\pi\mathrm{i}\,[Z] + \Theta(E). \tag{9.9}$$ □

10. Connections of Type (1,0) and (0,1) over Complex Manifolds

Let X be a complex manifold, $\dim_{\mathbb{C}} X = n$ and E a C^∞ vector bundle of rank r over X ; here, E is not assumed to be holomorphic. We denote by $C^\infty_{p,q}(X,E)$ the space of C^∞ sections of the bundle $\Lambda^{p,q}T^\star X \otimes E$. We have therefore a direct sum decomposition

$$C_l^\infty(X,E) = \bigoplus_{p+q=l} C_{p,q}^\infty(X,E).$$

Connections of type $(1,0)$ or $(0,1)$ are operators acting on vector valued forms, which imitate the usual operators d', d'' acting on $C_{p,q}^\infty(X,\mathbb{C})$. More precisely, a connection of type (1,0) on E is a differential operator D' of order 1 acting on $C_{\bullet,\bullet}^\infty(X,E)$ and satisfying the following two properties:

$$(10.1)\quad D' : C_{p,q}^\infty(X,E) \longrightarrow C_{p+1,q}^\infty(X,E),$$
$$(10.1')\quad D'(f\wedge s) = d'f\wedge s + (-1)^{\deg f} f\wedge D's$$

for any $f \in C_{p_1,q_1}^\infty(X,\mathbb{C})$, $s \in C_{p_2,q_2}^\infty(X,E)$. The definition of a connection D'' of type (0,1) is similar. If $\theta : E_{\restriction\Omega} \to \Omega\times\mathbb{C}^r$ is a C^∞ trivialization of $E_{\restriction\Omega}$ and if $\sigma = (\sigma_\lambda) = \theta(s)$, then all such connections D' and D'' can be written

$$(10.2')\quad D's \simeq_\theta d'\sigma + A'\wedge\sigma,$$
$$(10.2'')\quad D''s \simeq_\theta d''\sigma + A''\wedge\sigma$$

where $A' \in C_{1,0}^\infty\big(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)\big)$, $A'' \in C_{0,1}^\infty\big(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)\big)$ are arbitrary forms with matrix coefficients.

It is clear that $D = D' + D''$ is then a connection in the sense of §2 ; conversely any connection D admits a unique decomposition $D = D' + D''$ in terms of a (1,0)-connection and a (0,1)-connection.

Assume now that E has a hermitian structure and that θ is an *isometry*. The connection D is hermitian if and only if the connection form $A = A' + A''$ satisfies $A^\star = -A$, and this condition is equivalent to $A' = -(A'')^\star$. From this observation, we get immediately:

(10.3) Proposition. *Let D_0'' be a given $(0,1)$-connection on a hermitian bundle $\pi : E \to X$. Then there exists a unique hermitian connection $D = D' + D''$ such that $D'' = D_0''$.*

11. Holomorphic Vector Bundles

From now on, the vector bundles E in which we are interested are supposed to have a *holomorphic structure*:

(11.1) Definition. *A vector bundle $\pi : E \to X$ is said to be holomorphic if E is a complex manifold, if the projection map π is holomorphic and if there exists a covering $(V_\alpha)_{\alpha\in I}$ of X and a family of holomorphic trivializations $\theta_\alpha : E_{\restriction V_\alpha} \to V_\alpha\times\mathbb{C}^r$.*

It follows that the transition matrices $g_{\alpha\beta}$ are holomorphic on $V_\alpha\cap V_\beta$. In complete analogy with the discussion of §8, we see that the concept of holomorphic vector bundle is equivalent to the concept of locally free sheaf of modules over the ring $\mathcal{O}$ of germs of holomorphic functions on X. We shall denote by $\mathcal{O}(E)$ the associated sheaf of germs of holomorphic sections of E. In the case $r = 1$, there is a one-to-one correspondence between the isomorphism classes of holomorphic line bundles and the Čech cohomology group $H^1(X, \mathcal{O}^\star)$.

(11.2) Definition. *The group $H^1(X, \mathcal{O}^\star)$ of isomorphism classes of holomorphic line bundles is called the Picard group of X.*

If $s \in C^\infty_{p,q}(X, E)$, the components $\sigma^\alpha = (\sigma^\alpha_\lambda)_{1\le\lambda\le r} = \theta_\alpha(s)$ of s under θ_α are related by

$$\sigma^\alpha = g_{\alpha\beta} \cdot \sigma^\beta \quad \text{on} \quad V_\alpha \cap V_\beta.$$

Since $d'' g_{\alpha\beta} = 0$, it follows that

$$d''\sigma^\alpha = g_{\alpha\beta} \cdot d''\sigma^\beta \quad \text{on} \quad V_\alpha \cap V_\beta.$$

The collection of forms $(d''\sigma^\alpha)$ therefore corresponds to a unique global $(p, q+1)$-form $d''s$ such that $\theta_\alpha(d''s) = d''\sigma^\alpha$, and the operator d'' defined in this way is a $(0,1)$-connection on E.

(11.3) Definition. *The operator d'' is called the canonical $(0,1)$-connection of the holomorphic bundle E.*

It is clear that $d''^2 = 0$. Therefore, for any integer $p = 0, 1, \ldots, n$, we get a complex

$$C^\infty_{p,0}(X, E) \xrightarrow{d''} \cdots \longrightarrow C^\infty_{p,q}(X, E) \xrightarrow{d''} C^\infty_{p,q+1}(X, E) \longrightarrow \cdots$$

known as the *Dolbeault complex* of $(p, \bullet)$-forms with values in E.

(11.4) Notation. *The q-th cohomology group of the Dolbeault complex is denoted $H^{p,q}(X, E)$ and is called the (p, q) Dolbeault cohomology group with values in E.*

The Dolbeault-Grothendieck lemma I-2.11 shows that the complex of sheaves $d'' : \mathcal{C}^\infty_{0,\bullet}(X, E)$ is a soft resolution of the sheaf $\mathcal{O}(E)$. By the De Rham-Weil isomorphism theorem IV-6.4, we get:

(11.5) Proposition. $H^{0,q}(X, E) \simeq H^q\big(X, \mathcal{O}(E)\big)$.

Most often, we will identify the locally free sheaf $\mathcal{O}(E)$ and the bundle E itself ; the above sheaf cohomology group will therefore be simply denoted $H^q(X, E)$. Another standard notation in analytic or algebraic geometry is:

(11.6) Notation. *If X is a complex manifold, Ω_X^p denotes the vector bundle $\Lambda^p T^\star X$ or its sheaf of sections.*

It is clear that the complex $C^\infty_{p,\bullet}(X,E)$ is identical to the complex $C^\infty_{0,\bullet}(X,\Omega_X^p\otimes E)$, therefore we obtain a canonical isomorphism:

(11.7) Dolbeault isomorphism. $H^{p,q}(X,E)\simeq H^q(X,\Omega_X^p\otimes E)$.

In particular, $H^{p,0}(X,E)$ is the space of global holomorphic sections of the bundle $\Omega_X^p\otimes E$.

12. Chern Connection

Let $\pi: E\to X$ be a *hermitian holomorphic* vector bundle of rank r. By Prop. 10.3, there exists a unique hermitian connection D such that $D''=d''$.

(12.1) Definition. *The unique hermitian connection D such that $D''=d''$ is called the Chern connection of E. The curvature tensor of this connection will be denoted by $\Theta(E)$ and is called the Chern curvature tensor of E.*

Let us compute D with respect to an arbitrary *holomorphic trivialization* $\theta: E_{\restriction\Omega}\to\Omega\times\mathbb{C}^r$. Let $H=(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ denote the hermitian matrix with C^∞ coefficients representing the metric along the fibers of $E_{\restriction\Omega}$. For any $s,t\in C^\infty_{\bullet,\bullet}(X,E)$ and $\sigma=\theta(s)$, $\tau=\theta(t)$ one can write

$$\{s,t\}=\sum_{\lambda,\mu}h_{\lambda\mu}\sigma_\lambda\wedge\overline{\tau}_\mu=\sigma^\dagger\wedge H\overline{\tau},$$

where $\sigma^\dagger$ is the transposed matrix of σ. It follows that

$$\begin{aligned}\{Ds,t\}+(-1)^{\deg s}\{s,Dt\}&=d\{s,t\}\\ &=(d\sigma)^\dagger\wedge H\overline{\tau}+(-1)^{\deg\sigma}\sigma^\dagger\wedge(dH\wedge\overline{\tau}+H\overline{d\tau})\\ &=\big(d\sigma+\overline{H}^{-1}d'\overline{H}\wedge\sigma\big)^\dagger\wedge H\overline{\tau}+(-1)^{\deg\sigma}\sigma^\dagger\wedge\overline{(d\tau+\overline{H}^{-1}d'\overline{H}\wedge\tau)}\end{aligned}$$

using the fact that $dH=d'H+\overline{d'\overline{H}}$ and $\overline{H}^\dagger=H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$\text{(12.2)}\quad Ds\simeq_\theta d\sigma+\overline{H}^{-1}d'\overline{H}\wedge\sigma,$$

$$\text{(12.3)}\quad D'\simeq_\theta d'+\overline{H}^{-1}d'\overline{H}\wedge\bullet=\overline{H}^{-1}d'(\overline{H}\bullet),\qquad D''=d''.$$

It is clear from this relations that $D'^2=D''^2=0$. Consequently D^2 is given by to $D^2=D'D''+D''D'$, and the curvature tensor $\Theta(E)$ is of type $(1,1)$. Since $d'd''+d''d'=0$, we get

$$(D'D''+D''D')s \simeq_\theta \overline{H}^{-1} d'\overline{H} \wedge d''\sigma + d''(\overline{H}^{-1} d'\overline{H} \wedge \sigma) = d''(\overline{H}^{-1} d'\overline{H}) \wedge \sigma.$$

(12.4) Theorem. *The Chern curvature tensor is such that*

$$\mathrm{i}\,\Theta(E) \in C^\infty_{1,1}(X, \mathrm{Herm}(E,E)).$$

If $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $E_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta(E) = \mathrm{i}\, d''(\overline{H}^{-1} d'\overline{H}) \quad \text{on} \quad \Omega.$$

Let $(e_1, \ldots, e_r)$ be a C^∞ orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1, \ldots, z_n)$. On Ω the Chern curvature tensor can be written

$$(12.5) \quad \mathrm{i}\Theta(E) = \mathrm{i} \sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu$$

for some coefficients $c_{jk\lambda\mu} \in \mathbb{C}$. The hermitian property of $\mathrm{i}\Theta(E)$ means that $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$.

(12.6) Special case. When $r = \operatorname{rank} E = 1$, the hermitian matrix H is a positive function which we write $H = e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. By the above formulas we get

$$(12.7) \quad D' \simeq_\theta d' - d'\varphi \wedge \bullet = e^\varphi d'(e^{-\varphi} \bullet),$$

$$(12.8) \quad \mathrm{i}\Theta(E) = \mathrm{i} d'd''\varphi \quad \text{on} \quad \Omega.$$

Especially, we see that $\mathrm{i}\,\Theta(E)$ is a *closed* real (1,1)-form on X.

(12.9) Remark. In general, it is not possible to find local frames $(e_1, \ldots, e_r)$ of $E_{\restriction\Omega}$ that are simultaneously *holomorphic* and *orthonormal*. In fact, we have in this case $H = (\delta_{\lambda\mu})$, so a necessary condition for the existence of such a frame is that $\Theta(E) = 0$ on Ω. Conversely, if $\Theta(E) = 0$, Prop. 6.7 and Rem. 7.5 show that E possesses local orthonormal parallel frames (e_λ) ; we have in particular $D''e_\lambda = 0$, so (e_λ) is holomorphic; such a bundle E arising from a unitary representation of $\pi_1(X)$ is said to be *hermitian flat.* The next proposition shows in a more local way that the Chern curvature tensor is the obstruction to the existence of orthonormal holomorphic frames: a holomorphic frame can be made "almost orthonormal" only up to curvature terms of order 2 in a neighborhood of any point.

(12.10) Proposition. *For every point $x_0 \in X$ and every coordinate system $(z_j)_{1\le j\le n}$ at x_0, there exists a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ in a neighborhood of x_0 such that*

$$\langle e_\lambda(z), e_\mu(z)\rangle = \delta_{\lambda\mu} - \sum_{1\le j,k\le n} c_{jk\lambda\mu}\, z_j \overline{z}_k + O(|z|^3)$$

where $(c_{jk\lambda\mu})$ *are the coefficients of the Chern curvature tensor* $\Theta(E)_{x_0}$. *Such a frame* (e_λ) *is called a normal coordinate frame at* x_0.

Proof. Let (h_λ) be a holomorphic frame of E. After replacing (h_λ) by suitable linear combinations with constant coefficients, we may assume that $\big(h_\lambda(x_0)\big)$ is an orthonormal basis of E_{x_0}. Then the inner products $\langle h_\lambda, h_\mu\rangle$ have an expansion

$$\langle h_\lambda(z), h_\mu(z)\rangle = \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu}\, z_j + a'_{j\lambda\mu}\, \overline{z}_j) + O(|z|^2)$$

for some complex coefficients $a_{j\lambda\mu}$, $a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu} = \overline{a}_{j\mu\lambda}$. Set first

$$g_\lambda(z) = h_\lambda(z) - \sum_{j,\mu} a_{j\lambda\mu}\, z_j\, h_\mu(z).$$

Then there are coefficients $a_{jk\lambda\mu}$, $a'_{jk\lambda\mu}$, $a''_{jk\lambda\mu}$ such that

$$\begin{aligned}\langle g_\lambda(z), g_\mu(z)\rangle &= \delta_{\lambda\mu} + O(|z|^2)\\ &= \delta_{\lambda\mu} + \sum_{j,k}\big(a_{jk\lambda\mu}\, z_j\overline{z}_k + a'_{jk\lambda\mu}\, z_j z_k + a''_{jk\lambda\mu}\overline{z}_j\overline{z}_k\big) + O(|z|^3).\end{aligned}$$

The holomorphic frame (e_λ) we are looking for is

$$e_\lambda(z) = g_\lambda(z) - \sum_{j,k,\mu} a'_{jk\lambda\mu}\, z_j z_k\, g_\mu(z).$$

Since $a''_{jk\lambda\mu} = \overline{a}'_{jk\mu\lambda}$, we easily find

$$\begin{aligned}\langle e_\lambda(z), e_\mu(z)\rangle &= \delta_{\lambda\mu} + \sum_{j,k} a_{jk\lambda\mu}\, z_j\overline{z}_k + O(|z|^3),\\ d'\langle e_\lambda, e_\mu\rangle = \{D'e_\lambda, e_\mu\} &= \sum_{j,k} a_{jk\lambda\mu}\,\overline{z}_k\, dz_j + O(|z|^2),\\ \Theta(E)\cdot e_\lambda = D''(D'e_\lambda) &= \sum_{j,k,\mu} a_{jk\lambda\mu}\, d\overline{z}_k\wedge dz_j\otimes e_\mu + O(|z|),\end{aligned}$$

therefore $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$. □

13. Lelong-Poincaré Equation and First Chern Class

Our goal here is to extend the Lelong-Poincaré equation III-2.15 to any meromorphic section of a holomorphic line bundle.

(13.1) Definition. *A meromorphic section of a bundle $E \to X$ is a section s defined on an open dense subset of X, such that for every trivialization $\theta_\alpha : E_{\restriction V_\alpha} \to V_\alpha \times \mathbb{C}^r$ the components of $\sigma^\alpha = \theta_\alpha(s)$ are meromorphic functions on V_α.*

Let E be a hermitian line bundle, s a meromorphic section which does not vanish on any component of X and $\sigma = \theta(s)$ the corresponding meromorphic function in a trivialization $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{C}$. The divisor of s is the current on X defined by $\operatorname{div} s_{\restriction\Omega} = \operatorname{div}\sigma$ for all trivializing open sets Ω. One can write $\operatorname{div} s = \sum m_j Z_j$, where the sets Z_j are the irreducible components of the sets of zeroes and poles of s (cf. § II-5). The Lelong-Poincaré equation (II-5.32) gives

$$\frac{\mathrm{i}}{\pi} d'd'' \log|\sigma| = \sum m_j [Z_j],$$

and from the equalities $|s|^2 = |\sigma|^2 e^{-\varphi}$ and $d'd''\varphi = \Theta(E)$ we get

$$(13.2) \quad id'd'' \log|s|^2 = 2\pi \sum m_j [Z_j] - \mathrm{i}\,\Theta(E).$$

This equality can be viewed as a complex analogue of (9.9) (except that here the hypersurfaces Z_j are not necessarily smooth). In particular, if s is a *non vanishing holomorphic* section of $E_{\restriction\Omega}$, we have

$$(13.3) \quad \mathrm{i}\,\Theta(E) = -\mathrm{i}d'd'' \log|s|^2 \quad \text{on} \quad \Omega.$$

(13.4) Theorem. *Let $E \to X$ be a line bundle and let s be a meromorphic section of E which does not vanish identically on any component of X. If $\sum m_j Z_j$ is the divisor of s, then*

$$c_1(E)_{\mathbb{R}} = \Big\{ \sum m_j [Z_j] \Big\} \in H^2(X, \mathbb{R}).$$

Proof. Apply Formula (13.2) and Th. 9.5, and observe that the bidimension $(1,1)$-current $\mathrm{i}d'd'' \log|s|^2 = d\big(\mathrm{i}d'' \log|s|^2\big)$ has zero cohomology class. □

(13.5) Example. If $\Delta = \sum m_j Z_j$ is an arbitrary divisor on X, we associate to Δ the sheaf $\mathcal{O}(\Delta)$ of germs of meromorphic functions f such that $\operatorname{div}(f) + \Delta \geq 0$. Let (V_α) be a covering of X and u_α a meromorphic function on V_α such that $\operatorname{div}(u_\alpha) = \Delta$ on V_α. Then $\mathcal{O}(\Delta)_{\restriction V_\alpha} = u_\alpha^{-1}\mathcal{O}$, thus $\mathcal{O}(\Delta)$ is a locally free $\mathcal{O}$-module of rank 1. This sheaf can be identified to the line bundle E over X defined by the cocycle $g_{\alpha\beta} := u_\alpha/u_\beta \in \mathcal{O}^\star(V_\alpha \cap V_\beta)$. In fact, there is a sheaf isomorphism $\mathcal{O}(\Delta) \longrightarrow \mathcal{O}(E)$ defined by

$$\mathcal{O}(\Delta)(\Omega) \ni f \longmapsto s \in \mathcal{O}(E)(\Omega) \quad \text{with} \quad \theta_\alpha(s) = f u_\alpha \quad \text{on } \Omega \cap V_\alpha.$$

The constant meromorphic function $f = 1$ induces a meromorphic section s of E such that $\operatorname{div} s = \operatorname{div} u_\alpha = \Delta$; in the special case when $\Delta \geq 0$, the section s is holomorphic and its zero set $s^{-1}(0)$ is the support of Δ. By Th. 13.4, we have

$$(13.6) \quad c_1\big(\mathcal{O}(\Delta)\big)_{\mathbb{R}} = \{[\Delta]\}.$$

Let us consider the exact sequence $1 \to \mathcal{O}^\star \to \mathcal{M}^\star \to \operatorname{Div} \to 0$ already described in (II-5.36). There is a corresponding cohomology exact sequence

$$(13.7) \quad \mathcal{M}^\star(X) \longrightarrow \operatorname{Div}(X) \xrightarrow{\partial^0} H^1(X, \mathcal{O}^\star).$$

The connecting homomorphism ∂^0 is equal to the map

$$\Delta \longmapsto \text{isomorphism class of } \mathcal{O}(\Delta)$$

defined above. The kernel of this map consists of divisors which are divisors of global meromorphic functions in $\mathcal{M}^\star(X)$. In particular, two divisors Δ_1 and Δ_2 give rise to isomorphic line bundles $\mathcal{O}(\Delta_1) \simeq \mathcal{O}(\Delta_2)$ if and only if $\Delta_2 - \Delta_1 = \operatorname{div}(f)$ for some global meromorphic function $f \in \mathcal{M}^\star(X)$; such divisors are called *linearly equivalent*. The image of ∂^0 consists of classes of line bundles E such that E has a global meromorphic section which does not vanish on any component of X. Indeed, if s is such a section and $\Delta = \operatorname{div} s$, there is an isomorphism

$$(13.8) \qquad \mathcal{O}(\Delta) \xrightarrow{\simeq} \mathcal{O}(E), \qquad f \longmapsto fs. \qquad \square$$

The last result of this section is a characterization of 2-forms on X which can be written as the curvature form of a hermitian holomorphic line bundle.

(13.9) Theorem. *Let X be an arbitrary complex manifold.*

a) *For any hermitian line bundle E over M, the Chern curvature form $\frac{i}{2\pi}\Theta(E)$ is a closed real $(1,1)$-form whose De Rham cohomology class is the image of an integral class.*

b) *Conversely, let ω be a C^∞ closed real $(1,1)$-form such that the class $\{\omega\} \in H^2_{DR}(X,\mathbb{R})$ is the image of an integral class. Then there exists a hermitian line bundle $E \to X$ such that $\frac{i}{2\pi}\Theta(E) = \omega$.*

Proof. a) is an immediate consequence of Formula (12.9) and Th. 9.5, so we have only to prove the converse part b). By Prop. III-1.20, there exist an open covering (V_α) of X and functions $\varphi_\alpha \in C^\infty(V_\alpha, \mathbb{R})$ such that $\frac{i}{2\pi}d'd''\varphi_\alpha = \omega$

on V_α. It follows that the function $\varphi_\beta - \varphi_\alpha$ is pluriharmonic on $V_\alpha \cap V_\beta$. If (V_α) is chosen such that the intersections $V_\alpha \cap V_\beta$ are simply connected, then Th. I-3.35 yields holomorphic functions $f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ such that

$$2\,\mathrm{Re}\, f_{\alpha\beta} = \varphi_\beta - \varphi_\alpha \quad \text{on} \quad V_\alpha \cap V_\beta.$$

Now, our aim is to prove (roughly speaking) that $\big(\exp(-f_{\alpha\beta})\big)$ is a cocycle in $\mathcal{O}^\star$ that defines the line bundle E we are looking for. The Čech differential $(\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$ takes values in the constant sheaf $\mathrm{i}\mathbb{R}$ because

$$2\,\mathrm{Re}\,(\delta f)_{\alpha\beta\gamma} = (\varphi_\gamma - \varphi_\beta) - (\varphi_\gamma - \varphi_\alpha) + (\varphi_\beta - \varphi_\alpha) = 0.$$

Consider the real 1-forms $A_\alpha = \frac{\mathrm{i}}{4\pi}(d''\varphi_\alpha - d'\varphi_\alpha)$. As $d'(\varphi_\beta - \varphi_\alpha)$ is equal to $d'(f_{\alpha\beta} + \overline{f}_{\alpha\beta}) = df_{\alpha\beta}$, we get

$$(\delta A)_{\alpha\beta} = A_\beta - A_\alpha = \frac{\mathrm{i}}{4\pi} d(\overline{f}_{\alpha\beta} - f_{\alpha\beta}) = \frac{1}{2\pi} d\,\mathrm{Im}\, f_{\alpha\beta}.$$

Since $\omega = dA_\alpha$, it follows by (9.6) and (9.7) that the Čech cohomology class $\{\delta(\frac{1}{2\pi}\,\mathrm{Im}\, f_{\alpha\beta})\}$ is equal to $\{\omega\} \in H^2(X, \mathbb{R})$, which is by hypothesis the image of a 2-cocycle $(n_{\alpha\beta\gamma}) \in H^2(X, \mathbb{Z})$. Thus we can write

$$\delta\Big(\frac{1}{2\pi}\,\mathrm{Im}\, f_{\alpha\beta}\Big) = (n_{\alpha\beta\gamma}) + \delta(c_{\alpha\beta})$$

for some 1-chain $(c_{\alpha\beta})$ with values in $\mathbb{R}$. If we replace $f_{\alpha\beta}$ by $f_{\alpha\beta} - 2\pi\mathrm{i}c_{\alpha\beta}$, then we can achieve $c_{\alpha\beta} = 0$, so $\delta(f_{\alpha\beta}) \in 2\pi\mathrm{i}\mathbb{Z}$ and $g_{\alpha\beta} := \exp(-f_{\alpha\beta})$ will be a cocycle with values in $\mathcal{O}^\star$. Since

$$\varphi_\beta - \varphi_\alpha = 2\,\mathrm{Re}\, f_{\alpha\beta} = -\log|g_{\alpha\beta}|^2,$$

the line bundle E associated to this cocycle admits a global hermitian metric defined in every trivialization by the matrix $H_\alpha = (\exp(-\varphi_\alpha))$ and therefore

$$\frac{\mathrm{i}}{2\pi}\Theta(E) = \frac{\mathrm{i}}{2\pi} d'd''\varphi_\alpha = \omega \quad \text{on} \quad V_\alpha. \qquad \square$$

14. Exact Sequences of Hermitian Vector Bundles

Let us consider an exact sequence of holomorphic vector bundles over X :

$$(14.1) \quad 0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0.$$

Then E is said to be an *extension of S by Q*. A (holomorphic, resp. C^∞) splitting of the exact sequence is a (holomorphic, resp. C^∞) homomorphism $h : Q \longrightarrow E$ which is a right inverse of the projection $E \longrightarrow Q$, i.e. such that $g \circ h = \mathrm{Id}_Q$.

Assume that a C^∞ hermitian metric on E is given. Then S and Q can be endowed with the induced and quotient metrics respectively. Let us denote by D_E, D_S, D_Q the corresponding Chern connections. The adjoint homomorphisms

$$j^\star : E \longrightarrow S, \qquad g^\star : Q \longrightarrow E$$

are C^∞ and can be described respectively as the orthogonal projection of E onto S and as the orthogonal splitting of the exact sequence (14.1). They yield a C^∞ (in general non analytic) isomorphism

$$(14.2) \quad j^\star \oplus g : E \xrightarrow{\simeq} S \oplus Q.$$

(14.3) Theorem. *According to the C^∞ isomorphism $j^\star \oplus g$, D_E can be written*

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix}$$

where $\beta \in C^\infty_{1,0}\big(X, \mathrm{Hom}(S,Q)\big)$ is called the second fundamental of S in E and where $\beta^\star \in C^\infty_{0,1}\big(X, \mathrm{Hom}(Q,S)\big)$ is the adjoint of β. Furthermore, the following identities hold:

a) $D'_{\mathrm{Hom}(S,E)} j = g^\star \circ \beta, \quad d''j = 0$;
b) $D'_{\mathrm{Hom}(E,Q)} g = -\beta \circ j^\star, \quad d''g = 0$;
c) $D'_{\mathrm{Hom}(E,S)} j^\star = 0, \quad d''j^\star = \beta^\star \circ g$;
d) $D'_{\mathrm{Hom}(Q,E)} g^\star = 0, \quad d''g^\star = -j \circ \beta^\star$.

Proof. If we define $\nabla_E \simeq D_S \oplus D_Q$ via (14.2), then ∇_E is a hermitian connection on E. By (7.3), we have therefore $D_E = \nabla_E + \Gamma \wedge \bullet$, where $\Gamma \in C^\infty_1(X, \mathrm{Hom}(E,E))$ and $\Gamma^\star = -\Gamma$. Let us write

$$\Gamma = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \qquad \alpha^\star = -\alpha,\ \delta^\star = -\delta,\ \gamma = -\beta^\star,$$

$$(14.4) \quad D_E = \begin{pmatrix} D_S + \alpha & \gamma \\ \beta & D_Q + \delta \end{pmatrix}.$$

For any section $u \in C^\infty_{\bullet,\bullet}(X, E)$ we have

$$\begin{aligned} D_E u &= D_E(jj^\star u + g^\star g u) \\ &= jD_S(j^\star u) + g^\star D_Q(gu) + (D_{\mathrm{Hom}(S,E)} j) \wedge j^\star u + (D_{\mathrm{Hom}(E,Q)} g^\star) \wedge gu. \end{aligned}$$

A comparison with (14.4) yields

$$\begin{aligned} D_{\mathrm{Hom}(S,E)} j &= j \circ \alpha + g^\star \circ \beta, \\ D_{\mathrm{Hom}(E,Q)} g^\star &= j \circ \gamma + g^\star \circ \delta, \end{aligned}$$

Since j is holomorphic, we have $d''j = j \circ \alpha^{0,1} + g^\star \circ \beta^{0,1} = 0$, thus $\alpha^{0,1} = \beta^{0,1} = 0$. But $\alpha^\star = -\alpha$, hence $\alpha = 0$ and $\beta \in C^\infty_{1,0}(\mathrm{Hom}(S,Q))$; identity a) follows. Similarly, we get

$$\begin{aligned} D_S(j^\star u) &= j^\star D_E u + (D_{\mathrm{Hom}(E,S)} j^\star) \wedge u, \\ D_Q(gu) &= g D_E u + (D_{\mathrm{Hom}(E,Q)} g) \wedge u, \end{aligned}$$

and comparison with (14.4) yields

$$\begin{aligned} D_{\mathrm{Hom}(E,S)} j^\star &= -\alpha \circ j^\star - \gamma \circ g = \beta^\star \circ g, \\ D_{\mathrm{Hom}(E,Q)} g &= -\beta \circ j^\star - \delta \circ g. \end{aligned}$$

Since $d''g = 0$, we get $\delta^{0,1} = 0$, hence $\delta = 0$. Identities b), c), d) follow from the above computations. □

(14.5) Theorem. *We have $d''(\beta^\star) = 0$, and the Chern curvature of E is*

$$\Theta(E) = \begin{pmatrix} \Theta(S) - \beta^\star \wedge \beta & D'_{\mathrm{Hom}(Q,S)} \beta^\star \\ d''\beta & \Theta(Q) - \beta \wedge \beta^\star \end{pmatrix}.$$

Proof. A computation of D_E^2 yields

$$D_E^2 = \begin{pmatrix} D_S^2 - \beta^\star \wedge \beta & -(D_S \circ \beta^\star + \beta^\star \circ D_Q) \\ \beta \circ D_S + D_Q \circ \beta & D_Q^2 - \beta \wedge \beta^\star \end{pmatrix}.$$

Formula (13.4) implies

$$\begin{aligned} D_{\mathrm{Hom}(S,Q)} \beta &= \beta \circ D_S + D_Q \circ \beta, \\ D_{\mathrm{Hom}(Q,S)} \beta^\star &= D_S \circ \beta^\star + \beta^\star \circ D_Q. \end{aligned}$$

Since D_E^2 is of type (1,1), it follows that $d''\beta^\star = D''_{\mathrm{Hom}(Q,S)} \beta^\star = 0$. The proof is achieved. □

A consequence of Th. 14.5 is that $\Theta(S)$ and $\Theta(Q)$ are given in terms of $\Theta(E)$ by the following formulas, where $\Theta(E)_{\restriction S}$, $\Theta(E)_{\restriction Q}$ denote the blocks in the matrix of $\Theta(E)$ corresponding to $\mathrm{Hom}(S,S)$ and $\mathrm{Hom}(Q,Q)$:

$$\Theta(S) = \Theta(E)_{\restriction S} + \beta^\star \wedge \beta, \tag{14.6}$$

$$\Theta(Q) = \Theta(E)_{\restriction Q} + \beta \wedge \beta^\star. \tag{14.7}$$

By 14.3 c) the second fundamental form β vanishes identically if and only if the orthogonal splitting $E \simeq S \oplus Q$ is holomorphic ; then we have $\Theta(E) = \Theta(S) \oplus \Theta(Q)$.

Next, we show that the d''-cohomology class $\{\beta^\star\} \in H^{0,1}\big(X, \mathrm{Hom}(Q,S)\big)$ characterizes the isomorphism class of E among all extensions of S by Q.

Two extensions E and F are said to be isomorphic if there is a commutative diagram of holomorphic maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & S & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \end{array} \tag{14.8}$$

in which the rows are exact sequences. The central vertical arrow is then necessarily an isomorphism. It is easily seen that $0 \to S \to E \to Q \to 0$ has a holomorphic splitting if and only if E is isomorphic to the trivial extension $S \oplus Q$.

(14.9) Proposition. *The correspondence*

$$\{E\} \longmapsto \{\beta^\star\}$$

induces a bijection from the set of isomorphism classes of extensions of S by Q onto the cohomology group $H^1\big(X, \mathrm{Hom}(Q,S)\big)$. In particular $\{\beta^\star\}$ vanishes if and only if the exact sequence

$$0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0$$

splits holomorphically.

Proof. a) The map is well defined, i.e. $\{\beta^\star\}$ does not depend on the choice of the hermitian metric on E. Indeed, a new hermitian metric produces a new C^∞ splitting $\widehat{g}^\star$ and a new form $\widehat{\beta}^\star$ such that $d''\widehat{g}^\star = -j \circ \widehat{\beta}^\star$. Then $gg^\star = g\widehat{g}^\star = \mathrm{Id}_Q$, thus $\widehat{g} - g = j \circ v$ for some section $v \in C^\infty\big(X, \mathrm{Hom}(Q,S)\big)$. It follows that $\widehat{\beta}^\star - \beta^\star = -d''v$. Moreover, it is clear that an isomorphic extension F has the same associated form $\beta^\star$ if F is endowed with the image of the hermitian metric of E.

b) The map is injective. Let E and F be extensions of S by Q. Select C^∞ splittings $E, F \simeq S \oplus Q$. We endow S, Q with arbitrary hermitian metrics and E, F with the direct sum metric. Then we have corresponding $(0,1)$-connections

$$D''_E = \begin{pmatrix} D''_S & -\beta^\star \\ 0 & D''_Q \end{pmatrix}, \qquad D''_F = \begin{pmatrix} D''_S & -\widetilde{\beta}^\star \\ 0 & D''_Q \end{pmatrix}.$$

Assume that $\widetilde{\beta}^\star = \beta^\star + d''v$ for some $v \in C^\infty\big(X, \mathrm{Hom}(Q,S)\big)$. The isomorphism $\Psi : E \longrightarrow F$ of class C^∞ defined by the matrix

$$\begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix}.$$

is then holomorphic, because the relation $D''_S \circ v - v \circ D''_Q = d''v = \widetilde{\beta}^\star - \beta^\star$ implies

$$\begin{aligned} D''_{\mathrm{Hom}(E,F)}\Psi &= D''_F \circ \Psi - \Psi \circ D''_E \\ &= \begin{pmatrix} D''_S & -\widetilde{\beta}^\star \\ 0 & D''_Q \end{pmatrix} \begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix} - \begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix} \begin{pmatrix} D''_S & -\beta^\star \\ 0 & D''_Q \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\widetilde{\beta}^\star + \beta^\star + (D''_S \circ v - v \circ D''_Q) \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Hence the extensions E and F are isomorphic.

c) The map is surjective. Let γ be an arbitrary d''-closed $(0,1)$-form on X with values in $\mathrm{Hom}(Q,S)$. We define E as the C^∞ hermitian vector bundle $S \oplus Q$ endowed with the $(0,1)$-connection

$$D''_E = \begin{pmatrix} D''_S & \gamma \\ 0 & D''_Q \end{pmatrix}.$$

We only have to show that this connection is induced by a holomorphic structure on E ; then we will have $\beta^\star = -\gamma$. However, the Dolbeault-Grothendieck lemma implies that there is a covering of X by open sets U_α on which $\gamma = d''v_\alpha$ for some $v_\alpha \in C^\infty\big(U_\alpha, \mathrm{Hom}(Q,S)\big)$. Part b) above shows that the matrix

$$\begin{pmatrix} \mathrm{Id}_S & v_\alpha \\ 0 & \mathrm{Id}_Q \end{pmatrix}$$

defines an isomorphism ψ_α from $E_{\restriction U_\alpha}$ onto the trivial extension $(S \oplus Q)_{\restriction U_\alpha}$ such that $D''_{\mathrm{Hom}(E,S\oplus Q)}\psi_\alpha = 0$. The required holomorphic structure on $E_{\restriction U_\alpha}$ is the inverse image of the holomorphic structure of $(S \oplus Q)_{\restriction U_\alpha}$ by ψ_α ; it is independent of α because $v_\alpha - v_\beta$ and $\psi_\alpha \circ \psi_\beta^{-1}$ are holomorphic on $U_\alpha \cap U_\beta$. □

(14.10) Remark. If E and F are extensions of S by Q such that the corresponding forms $\beta^\star$ and $\widetilde{\beta}^\star = u \circ \beta^\star \circ v^{-1}$ differ by $u \in H^0\big(X, \mathrm{Aut}(S)\big)$, $v \in H^0\big(X, \mathrm{Aut}(Q)\big)$, it is easy to see that the bundles E and F are isomorphic. To see this, we need only replace the vertical arrows representing the identity maps of S and Q in (14.8) by u and v respectively. Thus, if we want to classify isomorphism classes of bundles E which are extensions of S by Q rather than the extensions themselves, the set of classes is the quotient of $H^1\big(X, \mathrm{Hom}(Q,S)\big)$ by the action of $H^0\big(X, \mathrm{Aut}(S)\big) \times H^0\big(X, \mathrm{Aut}(Q)\big)$. In particular, if S, Q are line bundles and if X is compact connected, then $H^0\big(X, \mathrm{Aut}(S)\big)$, $H^0\big(X, \mathrm{Aut}(Q)\big)$ are equal to $\mathbb{C}^\star$ and the set of classes is the projective space $P\big(H^1(X, \mathrm{Hom}(Q,S))\big)$.

15. Line Bundles $\mathcal{O}(k)$ over $\mathbb{P}^n$

15.A. Algebraic properties of $\mathcal{O}(k)$

Let V be a complex vector space of dimension $n+1$, $n \geq 1$. The quotient topological space $P(V) = (V \setminus \{0\})/\mathbb{C}^\star$ is called the *projective space of* V, and can be considered as the set of lines in V if $\{0\}$ is added to each class $\mathbb{C}^\star \cdot x$. Let

$$\begin{aligned} \pi :\ V \setminus \{0\} &\longrightarrow P(V) \\ x &\longmapsto [x] = \mathbb{C}^\star \cdot x \end{aligned}$$

be the canonical projection. When $V = \mathbb{C}^{n+1}$, we simply denote $P(V) = \mathbb{P}^n$. The space $\mathbb{P}^n$ is the quotient S^{2n+1}/S^1 of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the multiplicative action of the unit circle $S^1 \subset \mathbb{C}$, so $\mathbb{P}^n$ is compact. Let $(e_0, \ldots, e_n)$ be a basis of V, and let $(x_0, \ldots, x_n)$ be the coordinates of a vector $x \in V \setminus \{0\}$. Then $(x_0, \ldots, x_n)$ are called the *homogeneous coordinates* of $[x] \in P(V)$. The space $P(V)$ can be covered by the open sets Ω_j defined by $\Omega_j = \{[x] \in P(V)\,;\ x_j \neq 0\}$ and there are homeomorphisms

$$\begin{aligned} \tau_j \ :\ \ \Omega_j &\longrightarrow \mathbb{C}^n \\ [x] &\longmapsto (z_0, \ldots, \widehat{z_j}, \ldots, z_n), \qquad z_l = x_l/x_j \ \text{ for } \ l \neq j. \end{aligned}$$

The collection (τ_j) defines a holomorphic atlas on $P(V)$, thus $P(V) = \mathbb{P}^n$ is a compact n-dimensional complex analytic manifold.

Let $\underline{V}$ be the trivial bundle $P(V) \times V$. We denote by $\mathcal{O}(-1) \subset \underline{V}$ the *tautological line subbundle*

$$(15.1) \quad \mathcal{O}(-1) = \big\{([x], \xi) \in P(V) \times V\ ;\ \xi \in \mathbb{C} \cdot x\big\}$$

such that $\mathcal{O}(-1)_{[x]} = \mathbb{C} \cdot x \subset V$, $x \in V \setminus \{0\}$. Then $\mathcal{O}(-1)_{\restriction \Omega_j}$ admits a non vanishing holomorphic section

$$[x] \longrightarrow \varepsilon_j([x]) = x/x_j = z_0 e_0 + \ldots + e_j + z_{j+1} e_{j+1} + \ldots + z_n e_n,$$

and this shows in particular that $\mathcal{O}(-1)$ is a holomorphic line bundle.

(15.2) Definition. *For every $k \in \mathbb{Z}$, the line bundle $\mathcal{O}(k)$ is defined by*

$$\begin{aligned} \mathcal{O}(1) &= \mathcal{O}(-1)^\star, \qquad \mathcal{O}(0) = P(V) \times \mathbb{C}, \\ \mathcal{O}(k) &= \mathcal{O}(1)^{\otimes k} = \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1) \quad \text{for} \quad k \geq 1, \\ \mathcal{O}(-k) &= \mathcal{O}(-1)^{\otimes k} \quad \text{for} \quad k \geq 1 \end{aligned}$$

We also introduce the quotient vector bundle $H = \underline{V}/\mathcal{O}(-1)$ of rank n. Therefore we have canonical exact sequences of vector bundles over $P(V)$:

$$(15.3)\quad 0 \to \mathcal{O}(-1) \to \underline{V} \to H \to 0, \qquad 0 \to H^\star \to \underline{V}^\star \to \mathcal{O}(1) \to 0.$$

The total manifold of the line bundle $\mathcal{O}(-1)$ gives rise to the so called *monoidal transformation*, or *Hopf σ-process*:

(15.4) Lemma. *The holomorphic map $\mu : \mathcal{O}(-1) \to V$ defined by*

$$\mu : \mathcal{O}(-1) \hookrightarrow \underline{V} = P(V) \times V \xrightarrow{pr_2} V$$

sends the zero section $P(V) \times \{0\}$ of $\mathcal{O}(-1)$ to the point $\{0\}$ and induces a biholomorphism of $\mathcal{O}(-1) \smallsetminus \big(P(V) \times \{0\}\big)$ onto $V \smallsetminus \{0\}$.

Proof. The inverse map $\mu^{-1} : V \smallsetminus \{0\} \longrightarrow \mathcal{O}(-1)$ is clearly defined by

$$\mu^{-1} : x \longmapsto \big([x], x\big). \qquad \square$$

The space $H^0(\mathbb{P}^n, \mathcal{O}(k))$ of global holomorphic sections of $\mathcal{O}(k)$ can be easily computed by means of the above map μ.

(15.5) Theorem. *$H^0\big(P(V), \mathcal{O}(k)\big) = 0$ for $k < 0$, and there is a canonical isomorphism*

$$H^0\big(P(V), \mathcal{O}(k)\big) \simeq S^k V^\star, \qquad k \geq 0,$$

where $S^k V^\star$ denotes the k-th symmetric power of $V^\star$.

(15.6) Corollary. *We have $\dim H^0\big(\mathbb{P}^n, \mathcal{O}(k)\big) = \binom{n+k}{n}$ for $k \geq 0$, and this group is 0 for $k < 0$.*

Proof. Assume first that $k \geq 0$. There exists a canonical morphism

$$\Phi : S^k V^\star \longrightarrow H^0\big(P(V), \mathcal{O}(k)\big)\,;$$

indeed, any element $a \in S^k V^\star$ defines a homogeneous polynomial of degree k on V and thus by restriction to $\mathcal{O}(-1) \subset \underline{V}$ a section $\Phi(a) = \widetilde{a}$ of $(\mathcal{O}(-1)^\star)^{\otimes k} = \mathcal{O}(k)$; in other words Φ is induced by the k-th symmetric power $S^k \underline{V}^\star \to \mathcal{O}(k)$ of the canonical morphism $\underline{V}^\star \to \mathcal{O}(1)$ in (15.3).

Assume now that $k \in \mathbb{Z}$ is arbitrary and that s is a holomorphic section of $\mathcal{O}(k)$. For every $x \in V \smallsetminus \{0\}$ we have $s([x]) \in \mathcal{O}(k)_{[x]}$ and $\mu^{-1}(x) \in \mathcal{O}(-1)_{[x]}$. We can therefore associate to s a holomorphic function on $V \smallsetminus \{0\}$ defined by

$$f(x) = s([x]) \cdot \mu^{-1}(x)^k, \qquad x \in V \smallsetminus \{0\}.$$

Since $\dim V = n + 1 \geq 2$, f can be extended to a holomorphic function on V and f is clearly homogeneous of degree k (μ and μ^{-1} are homogeneous of degree 1). It follows that $f = 0$, $s = 0$ if $k < 0$ and that f is a homogeneous polynomial of degree k on V if $k \geq 0$. Thus, there exists a unique element $a \in S^k V^\star$ such that

$$f(x) = a \cdot x^k = \widetilde{a}([x]) \cdot \mu^{-1}(x)^k.$$

Therefore Φ is an isomorphism. □

The tangent bundle on $\mathbb{P}^n$ is closely related to the bundles H and $\mathcal{O}(1)$ as shown by the following proposition.

(15.7) Proposition. *There is a canonical isomorphism of bundles*

$$TP(V) \simeq H \otimes \mathcal{O}(1).$$

Proof. The differential $d\pi_x$ of the projection $\pi : V \smallsetminus \{0\} \to P(V)$ may be considered as a map

$$d\pi_x : V \to T_{[x]}P(V).$$

As $d\pi_x(x) = 0$, $d\pi_x$ can be factorized through $V/\mathbb{C} \cdot x = V/\mathcal{O}(-1)_{[x]} = H_{[x]}$. Hence we get an isomorphism

$$d\widetilde{\pi}_x : H_{[x]} \longrightarrow T_{[x]}P(V),$$

but this isomorphism depends on x and not only on the base point $[x]$ in $P(V)$. The formula $\pi(\lambda x + \xi) = \pi(x + \lambda^{-1}\xi)$, $\lambda \in \mathbb{C}^\star$, $\xi \in V$, shows that $d\pi_{\lambda x} = \lambda^{-1} d\pi_x$, hence the map

$$d\widetilde{\pi}_x \otimes \mu^{-1}(x) \ : \quad H_{[x]} \longrightarrow \big(TP(V) \otimes \mathcal{O}(-1)\big)_{[x]}$$

depends only on $[x]$. Therefore $H \simeq TP(V) \otimes \mathcal{O}(-1)$. □

15.B. Curvature of the Tautological Line Bundle

Assume now that V is a *hermitian vector space*. Then (15.3) yields exact sequences of hermitian vector bundles. We shall compute the curvature of $\mathcal{O}(1)$ and H.

Let $a \in P(V)$ be fixed. Choose an orthonormal basis $(e_0, e_1, \ldots, e_n)$ of V such that $a = [e_0]$. Consider the embedding

$$\mathbb{C}^n \hookrightarrow P(V), \qquad 0 \longmapsto a$$

which sends $z = (z_1, \ldots, z_n)$ to $[e_0 + z_1 e_1 + \cdots + z_n e_n]$. Then

$$\varepsilon(z) = e_0 + z_1 e_1 + \cdots + z_n e_n$$

defines a non-zero holomorphic section of $\mathcal{O}(-1)_{\restriction\mathbb{C}^n}$ and Formula (13.3) for $\Theta\big(\mathcal{O}(1)\big) = -\Theta\big(\mathcal{O}(-1)\big)$ implies

$$\Theta\big(\mathcal{O}(1)\big) = d'd'' \log|\varepsilon(z)|^2 = d'd'' \log(1+|z|^2) \quad \text{on } \mathbb{C}^n, \tag{15.8}$$

$$\Theta\big(\mathcal{O}(1)\big)_a = \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j. \tag{15.8$'$}$$

On the other hand, Th. 14.3 and (14.7) imply

$$d''g^\star = -j \circ \beta^\star, \qquad \Theta(H) = \beta \wedge \beta^\star,$$

where $j : \mathcal{O}(-1) \longrightarrow \underline{V}$ is the inclusion, $g^\star : H \longrightarrow \underline{V}$ the orthogonal splitting and $\beta^\star \in C^\infty_{0,1}\big(P(V), \mathrm{Hom}(H, \mathcal{O}(-1))\big)$. The images $(\widetilde{e}_1, \ldots, \widetilde{e}_n)$ of $e_1, \ldots, e_n$ in $H = \underline{V}/\mathcal{O}(-1)$ define a holomorphic frame of $H_{\restriction\mathbb{C}^n}$ and we have

$$g^\star \cdot \widetilde{e}_j = e_j - \frac{\langle e_j, \varepsilon\rangle}{|\varepsilon|^2} = e_j - \frac{\overline{z}_j}{1+|z|^2}\varepsilon, \qquad d''g^\star_a \cdot \widetilde{e}_j = -d\overline{z}_j \otimes \varepsilon,$$

$$\beta^\star_a = \sum_{1\le j\le n} d\overline{z}_j \otimes \widetilde{e}^\star_j \otimes \varepsilon, \qquad \beta_a = \sum_{1\le j\le n} dz_j \otimes \varepsilon^\star \otimes \widetilde{e}_j,$$

$$\Theta(H)_a = \sum_{1\le j,k\le n} dz_j \wedge d\overline{z}_k \otimes \widetilde{e}^\star_k \otimes \widetilde{e}_j. \tag{15.9}$$

(15.10) Theorem. *The cohomology algebra $H^\bullet(\mathbb{P}^n, \mathbb{Z})$ is isomorphic to the quotient ring $\mathbb{Z}[h]/(h^{n+1})$ where the generator h is given by $h = c_1(\mathcal{O}(1))$ in $H^2(\mathbb{P}^n, \mathbb{Z})$.*

Proof. Consider the inclusion $\mathbb{P}^{n-1} = P(\mathbb{C}^n \times \{0\}) \subset \mathbb{P}^n$. Topologically, $\mathbb{P}^n$ is obtained from $\mathbb{P}^{n-1}$ by attaching a $2n$-cell B_{2n} to $\mathbb{P}^{n-1}$, via the map

$$\begin{aligned} f \;:\; & B_{2n} \longrightarrow \mathbb{P}^n \\ & z \longmapsto [z, 1-|z|^2], \qquad z \in \mathbb{C}^n, \quad |z| \le 1 \end{aligned}$$

which sends $S^{2n-1} = \{|z| = 1\}$ onto $\mathbb{P}^{n-1}$. That is, $\mathbb{P}^n$ is homeomorphic to the quotient space of $B_{2n} \amalg \mathbb{P}^{n-1}$, where every point $z \in S^{2n-1}$ is identified with its image $f(z) \in \mathbb{P}^{n-1}$. We shall prove by induction on n that

$$H^{2k}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}, \quad 0 \le k \le n, \quad \text{otherwise} \quad H^l(\mathbb{P}^n, \mathbb{Z}) = 0. \tag{15.11}$$

The result is clear for $\mathbb{P}^0$, which is reduced to a single point. For $n \ge 1$, consider the covering (U_1, U_2) of $\mathbb{P}^n$ such that U_1 is the image by f of the open ball B°_{2n} and $U_2 = \mathbb{P}^n \smallsetminus \{f(0)\}$. Then $U_1 \approx B^\circ_{2n}$ is contractible, whereas $U_2 = (B_{2n} \smallsetminus \{0\}) \amalg_{S^{2n-1}} \mathbb{P}^{n-1}$. Moreover $U_1 \cap U_2 \approx B^\circ_{2n} \smallsetminus \{0\}$ can be retracted on the $(2n-1)$-sphere of radius $1/2$. For $q \ge 2$, the Mayer-Vietoris exact sequence IV-3.11 yields

$$\begin{aligned}\cdots\ & H^{q-1}(\mathbb{P}^{n-1},\mathbb{Z}) \longrightarrow H^{q-1}(S^{2n-1},\mathbb{Z}) \\ \longrightarrow H^q(\mathbb{P}^n,\mathbb{Z}) \longrightarrow\ & H^q(\mathbb{P}^{n-1},\mathbb{Z}) \longrightarrow H^q(S^{2n-1},\mathbb{Z}) \quad \cdots .\end{aligned}$$

For $q=1$, the first term has to be replaced by $H^0(\mathbb{P}^{n-1},\mathbb{Z})\oplus\mathbb{Z}$, so that the first arrow is onto. Formula (15.11) follows easily by induction, thanks to our computation of the cohomology groups of spheres in IV-14.6.

We know that $h=c_1(\mathcal{O}(1))\in H^2(\mathbb{P}^n,\mathbb{Z})$. It will follow necessarily that h^k is a generator of $H^{2k}(\mathbb{P}^n,\mathbb{Z})$ if we can prove that h^n is the fundamental class in $H^{2n}(\mathbb{P},\mathbb{Z})$, or equivalently that

$$(15.12)\quad c_1\big(\mathcal{O}(1)\big)^n_{\mathbb{R}} = \int_{\mathbb{P}^n}\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^n = 1.$$

This equality can be verified directly by means of (15.8), but we will avoid this computation. Observe that the element $e_n^\star\in\big(\mathbb{C}^{n+1}\big)^\star$ defines a section $\widetilde{e}_n^\star$ of $H^0(\mathbb{P}^n,\mathcal{O}(1))$ transverse to 0, whose zero set is the hyperplane $\mathbb{P}^{n-1}$. As $\{\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\}=\{[\mathbb{P}^{n-1}]\}$ by Th. 13.4, we get

$$\begin{aligned} c_1(\mathcal{O}(1)) &= \int_{\mathbb{P}^1}[\mathbb{P}^0] = 1 \qquad \text{for}\quad n=1 \quad\text{and}\\ c_1(\mathcal{O}(1))^n &= \int_{\mathbb{P}^n}[\mathbb{P}^{n-1}]\wedge\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^{n-1} = \int_{\mathbb{P}^{n-1}}\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^{n-1}\end{aligned}$$

in general. Since $\mathcal{O}(-1)_{\restriction\mathbb{P}^{n-1}}$ can be identified with the tautological line subbundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ over $\mathbb{P}^{n-1}$, we have $\Theta(\mathcal{O}(1))_{\restriction\mathbb{P}^{n-1}}=\Theta(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and the proof is achieved by induction on n. □

15.C. Tautological Line Bundle Associated to a Vector Bundle

Let E be a holomorphic vector bundle of rank r over a complex manifold X. The projectivized bundle $P(E)$ is the bundle with $\mathbb{P}^{r-1}$ fibers over X defined by $P(E)_x=P(E_x)$ for all $x\in X$. The points of $P(E)$ can thus be identified with the lines in the fibers of E. For any trivialization $\theta_\alpha: E_{\restriction U_\alpha}\to U_\alpha\times\mathbb{C}^r$ of E we have a corresponding trivialization $\widetilde{\theta}_\alpha: P(E)_{\restriction U_\alpha}\to U_\alpha\times\mathbb{P}^{r-1}$, and it is clear that the transition automorphisms are the projectivizations $\widetilde{g}_{\alpha\beta}\in H^0\big(U_\alpha\cap U_\beta, PGL(r,\mathbb{C})\big)$ of the transition automorphisms $g_{\alpha\beta}$ of E.

Similarly, we have a dual projectivized bundle $P(E^\star)$ whose points can be identified with the hyperplanes of E (every hyperplane F in E_x corresponds bijectively to the line of linear forms in $E_x^\star$ which vanish on F); note that $P(E)$ and $P(E^\star)$ coincide only when $r=\operatorname{rk}E=2$. If $\pi: P(E^\star)\to X$ is the natural projection, there is a tautological hyperplane subbundle S of $\pi^\star E$ over $P(E^\star)$ such that $S_{[\xi]}=\xi^{-1}(0)\subset E_x$ for all $\xi\in E_x^\star\smallsetminus\{0\}$.
$\big[$exercise: check that S is actually locally trivial over $P(E^\star)\big]$.

(15.13) Definition. *The quotient line bundle $\pi^\star E/S$ is denoted $\mathcal{O}_E(1)$ and is called the tautological line bundle associated to E. Hence there is an exact sequence*

$$0 \longrightarrow S \longrightarrow \pi^\star E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0$$

of vector bundles over $P(E^\star)$.

Note that (13.3) applied with $V = E_x^\star$ implies that the restriction of $\mathcal{O}_E(1)$ to each fiber $P(E_x^\star) \simeq \mathbb{P}^{r-1}$ coincides with the line bundle $\mathcal{O}(1)$ introduced in Def. 15.2. Theorem 15.5 can then be extended to the present situation and yields:

(15.14) Theorem. *For every $k \in \mathbb{Z}$, the direct image sheaf $\pi_\star \mathcal{O}_E(k)$ on X vanishes for $k < 0$ and is isomorphic to $\mathcal{O}(S^k E)$ for $k \geq 0$.*

Proof. For $k \geq 0$, the k-th symmetric power of the morphism $\pi^\star E \to \mathcal{O}_E(1)$ gives a morphism $\pi^\star S^k E \to \mathcal{O}_E(k)$. This morphism together with the pull-back morphism yield canonical arrows

$$\Phi_U : H^0(U, S^k E) \xrightarrow{\pi^\star} H^0\big(\pi^{-1}(U), \pi^\star S^k E\big) \longrightarrow H^0\big(\pi^{-1}(U), \mathcal{O}_E(k)\big)$$

for any open set $U \subset X$. The right hand side is by definition the space of sections of $\pi_\star \mathcal{O}_E(k)$ over U, hence we get a canonical sheaf morphism

$$\Phi : \mathcal{O}(S^k E) \longrightarrow \pi_\star \mathcal{O}_E(k).$$

It is easy to check that this Φ coincides with the map Φ introduced in the proof of Cor. 15.6 when X is reduced to a point. In order to check that Φ is an isomorphism, we may suppose that U is chosen so small that $E_{\restriction U}$ is trivial, say $E_{\restriction U} = U \times V$ with $\dim V = r$. Then $P(E^\star) = U \times P(V^\star)$ and $\mathcal{O}_E(1) = p^\star \mathcal{O}(1)$ where $\mathcal{O}(1)$ is the tautological line bundle over $P(V^\star)$ and $p : P(E^\star) \to P(V^\star)$ is the second projection. Hence we get

$$\begin{aligned} H^0\big(\pi^{-1}(U), \mathcal{O}_E(k)\big) &= H^0\big(U \times P(V^\star), p^\star \mathcal{O}(1)\big) \\ &= \mathcal{O}_X(U) \otimes H^0\big(P(V^\star), \mathcal{O}(1)\big) \\ &= \mathcal{O}_X(U) \otimes S^k V = H^0(U, S^k E), \end{aligned}$$

as desired; the reason for the second equality is that $p^\star \mathcal{O}(1)$ coincides with $\mathcal{O}(1)$ on each fiber $\{x\} \times P(V^\star)$ of p, thus any section of $p^\star \mathcal{O}(1)$ over $U \times P(V^\star)$ yields a family of sections $H^0\big(\{x\} \times P(V^\star), \mathcal{O}(k)\big)$ depending holomorphically in x. When $k < 0$ there are no non zero such sections, thus $\pi_\star \mathcal{O}_E(k) = 0$. $\square$

Finally, suppose that E is equipped with a hermitian metric. Then the morphism $\pi^\star E \to \mathcal{O}_E(1)$ endows $\mathcal{O}_E(1)$ with a quotient metric. We are going to compute the associated curvature form $\Theta\big(\mathcal{O}_E(1)\big)$.

Fix a point $x_0 \in X$ and $a \in P(E^\star_{x_0})$. Then Prop. 12.10 implies the existence of a normal coordinate frame $(e_\lambda)_{1\le\lambda\le r})$ of E at x_0 such that a is the hyperplane $\langle e_2,\ldots,e_r\rangle = (e_1^\star)^{-1}(0)$ at x_0. Let $(z_1,\ldots,z_n)$ be local coordinates on X near x_0 and let $(\xi_1,\ldots,\xi_r)$ be coordinates on $E^\star$ with respect to the dual frame $(e_1^\star,\ldots,e_r^\star)$. If we assign $\xi_1 = 1$, then $(z_1,\ldots,z_n,\xi_2,\ldots,\xi_r)$ define local coordinates on $P(E^\star)$ near a, and we have a local section of $\mathcal{O}_E(-1) := \mathcal{O}_E(1)^\star \subset \pi^\star E^\star$ defined by

$$\varepsilon(z,\xi) = e_1^\star(z) + \sum_{2\le\lambda\le r} \xi_\lambda\, e_\lambda^\star(z).$$

The hermitian matrix $(\langle e_\lambda^\star, e_\mu^\star\rangle)$ is just the congugate inverse of $(\langle e_\lambda, e_\mu\rangle) = \mathrm{Id} - \big(\sum c_{jk\lambda\mu}\, z_j\overline{z}_k\big) + O(|z|^3)$, hence we get

$$\langle e_\lambda^\star(z), e_\mu^\star(z)\rangle = \delta_{\lambda\mu} + \sum_{1\le j,k\le n} c_{jk\mu\lambda}\, z_j\overline{z}_k + O(|z|^3),$$

where $(c_{jk\lambda\mu})$ are the curvature coefficients of $\Theta(E)$; accordingly we have $\Theta(E^\star) = -\Theta(E)^\dagger$. We infer from this

$$|\varepsilon(z,\xi)|^2 = 1 + \sum_{1\le j,k\le n} c_{jk11}\, z_j\overline{z}_k + \sum_{2\le\lambda\le r} |\xi_\lambda|^2 + O(|z|^3).$$

Since $\Theta\big(\mathcal{O}_E(1)\big) = d'd''\log|\varepsilon(z,\xi)|^2$, we get

$$\Theta\big(\mathcal{O}_E(1)\big)_a = \sum_{1\le j,k\le n} c_{jk11}\, dz_j\wedge d\overline{z}_k + \sum_{2\le\lambda\le r} d\xi_\lambda\wedge d\overline{\xi}_\lambda.$$

Note that the first summation is simply $-\langle\Theta(E^\star)a,a\rangle/|a|^2 = -$ curvature of $E^\star$ in the direction a. A unitary change of variables then gives the slightly more general formula:

(15.15) Formula. *Let (e_λ) be a normal coordinate frame of E at $x_0 \in X$ and let $\Theta(E)_{x_0} = \sum c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu$. At any point $a \in P(E^\star)$ represented by a vector $\sum a_\lambda e_\lambda^\star \in E^\star_{x_0}$ of norm 1, the curvature of $\mathcal{O}_E(1)$ is*

$$\Theta\big(\mathcal{O}_E(1)\big)_a = \sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r} c_{jk\mu\lambda}\, a_\lambda\overline{a}_\mu\, dz_j\wedge d\overline{z}_k + \sum_{1\le\lambda\le r-1} d\zeta_\lambda\wedge d\overline{\zeta}_\lambda,$$

where (ζ_λ) are coordinates near a on $P(E^\star)$, induced by unitary coordinates on the hyperplane $a^\perp \subset E^\star_{x_0}$. □

16. Grassmannians and Universal Vector Bundles

16.A. Universal Subbundles and Quotient Vector Bundles

If V is a complex vector space of dimension d, we denote by $G_r(V)$ the set of all r-codimensional vector subspaces of V. Let $a \in G_r(V)$ and $W \subset V$ be fixed such that

$$V = a \oplus W, \qquad \dim_{\mathbb{C}} W = r.$$

Then any subspace $x \in G_r(V)$ in the open subset

$$\Omega_W = \{x \in G_r(V)\ ;\ x \oplus W = V\}$$

can be represented in a unique way as the graph of a linear map u in $\mathrm{Hom}(a, W)$. This gives rise to a covering of $G_r(V)$ by affine coordinate charts $\Omega_W \simeq \mathrm{Hom}(a, W) \simeq \mathbb{C}^{r(d-r)}$. Indeed, let $(e_1, \ldots, e_r)$ and $(e_{r+1}, \ldots, e_n)$ be respective bases of W and a. Every point $x \in \Omega_W$ is the graph of a linear map

$$(16.1)\quad u:\ a \longrightarrow W, \qquad u(e_k) = \sum_{1 \le j \le r} z_{jk} e_j, \quad r+1 \le k \le d,$$

i.e. $x = \mathrm{Vect}\big(e_k + \sum_{1 \le j \le r} z_{jk} e_j\big)_{r+1 \le k \le d}$. We choose (z_{jk}) as complex coordinates on Ω_W. These coordinates are centered at $a = \mathrm{Vect}(e_{r+1}, \ldots, e_d)$.

(16.2) Proposition. *$G_r(V)$ is a compact complex analytic manifold of dimension $n = r(d-r)$.*

Proof. It is immediate to verify that the coordinate change between two affine charts of $G_r(V)$ is holomorphic. Fix an arbitrary hermitian metric on V. Then the unitary group $U(V)$ is compact and acts transitively on $G_r(V)$. The isotropy subgroup of a point $a \in G_r(V)$ is $U(a) \times U(a^\perp)$, hence $G_r(V)$ is diffeomorphic to the compact quotient space $U(V)/U(a) \times U(a^\perp)$. $\square$

Next, we consider the tautological subbundle $S \subset \underline{V} := G_r(V) \times V$ defined by $S_x = x$ for all $x \in G_r(V)$, and the quotient bundle $Q = \underline{V}/S$ of rank r:

$$(16.3)\quad 0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0.$$

An interesting special case is $r = d-1$, $G_{d-1}(V) = P(V)$, $S = \mathcal{O}(-1)$, $Q = H$. The case $r = 1$ is dual, we have the identification $G_1(V) = P(V^\star)$ because every hyperplane $x \subset V$ corresponds bijectively to the line in $V^\star$ of linear forms $\xi \in V^\star$ that vanish on x. Then the bundles $\mathcal{O}(-1) \subset \underline{V}^\star$ and H on $P(V^\star)$ are given by

$$\begin{aligned} \mathcal{O}(-1)_{[\xi]} &= \mathbb{C}.\xi \simeq (V/x)^\star = Q_x^\star, \\ H_{[\xi]} &= V^\star/\mathbb{C}.\xi \simeq x^\star = S_x^\star, \end{aligned}$$

therefore $S = H^\star$, $Q = \mathcal{O}(1)$. This special case will allow us to compute $H^0(G_r(V), Q)$ in general.

(16.4) Proposition. *There is an isomorphism*

$$V = H^0\big(G_r(V), \underline{V}\big) \xrightarrow{\sim} H^0\big(G_r(V), Q\big).$$

Proof. Let $V = W \oplus W'$ be an arbitrary direct sum decomposition of V with $\operatorname{codim} W = r - 1$. Consider the projective space

$$P(W^\star) = G_1(W) \subset G_r(V),$$

its tautological hyperplane subbundle $H^\star \subset \underline{W} = P(W^\star) \times W$ and the exact sequence $0 \to H^\star \to \underline{W} \to \mathcal{O}(1) \to 0$. Then $S_{\restriction P(W^\star)}$ coincides with $H^\star$ and

$$Q_{\restriction P(W^\star)} = (\underline{W} \oplus \underline{W}')/H^\star = (\underline{W}/H^\star) \oplus \underline{W}' = \mathcal{O}(1) \oplus \underline{W}'.$$

Theorem 15.5 implies $H^0(P(W^\star), \mathcal{O}(1)) = W$, therefore the space

$$H^0(P(W^\star), Q_{\restriction P(W^\star)}) = W \oplus W'$$

is generated by the images of the constant sections of $\underline{V}$. Since W is arbitrary, Prop. 16.4 follows immediately. □

Let us compute the tangent space $TG_r(V)$. The linear group $\mathrm{Gl}(V)$ acts transitively on $G_r(V)$, and the tangent space to the isotropy subgroup of a point $x \in G_r(V)$ is the set of elements $u \in \mathrm{Hom}(V, V)$ in the Lie algebra such that $u(x) \subset x$. We get therefore

$$\begin{aligned} T_x G_r(V) &\simeq \mathrm{Hom}(V, V)/\{u\ ;\ u(x) \subset x\} \\ &\simeq \mathrm{Hom}(V, V/x)/\big\{\widetilde{u}\ ;\ \widetilde{u}(x) = \{0\}\big\} \\ &\simeq \mathrm{Hom}(x, V/x) = \mathrm{Hom}(S_x, Q_x). \end{aligned}$$

(16.5) Corollary. $TG_r(V) = \mathrm{Hom}(S, Q) = S^\star \otimes Q.$ □

16.B. Plücker Embedding

There is a natural map, called the *Plücker embedding*,

(16.6) $\quad j_r : G_r(V) \hookrightarrow P(\Lambda^r V^\star)$

constructed as follows. If $x \in G_r(V)$ is defined by r independent linear forms $\xi_1, \ldots, \xi_r \in V^\star$, we set

$$j_r(x) = [\xi_1 \wedge \cdots \wedge \xi_r].$$

Then x is the subspace of vectors $v \in V$ such that $v \lrcorner (\xi_1 \wedge \cdots \wedge \xi_r) = 0$, so j_r is injective. Since the linear group $\mathrm{Gl}(V)$ acts transitively on $G_r(V)$, the rank of the differential dj_r is a constant. As j_r is injective, the constant rank theorem implies:

(16.7) Proposition. *The map j_r is a holomorphic embedding.* □

Now, we define a commutative diagram

$$\begin{array}{ccc} \Lambda^r Q & \xrightarrow{J_r} & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ G_r(V) & \overset{j_r}{\hookrightarrow} & P(\Lambda^r V^\star) \end{array} \tag{16.8}$$

as follows: for $x = \xi_1^{-1}(0) \cap \cdots \cap \xi_r^{-1}(0) \in G_r(V)$ and $\widetilde{v} = \widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_r \in \Lambda^r Q_x$ where $\widetilde{v}_k \in Q_x = V/x$ is the image of $v_k \in V$ in the quotient, we let $J_r(\widetilde{v}) \in \mathcal{O}(1)_{j_r(x)}$ be the linear form on $\mathcal{O}(-1)_{j_r(x)} = \mathbb{C}.\xi_1 \wedge \ldots \wedge \xi_r$ such that

$$\langle J_r(\widetilde{v}), \lambda \xi_1 \wedge \ldots \wedge \xi_r \rangle = \lambda \det \big(\xi_j(v_k)\big), \qquad \lambda \in \mathbb{C}.$$

Then J_r is an isomorphism on the fibers, so $\Lambda^r Q$ can be identified with the pull-back of $\mathcal{O}(1)$ by j_r.

16.C. Curvature of the Universal Vector Bundles

Assume now that V is a hermitian vector space. We shall generalize our curvature computations of §15.C to the present situation. Let $a \in G_r(V)$ be a given point. We take W to be the orthogonal complement of a in V and select an orthonormal basis $(e_1, \ldots, e_d)$ of V such that $W = \mathrm{Vect}(e_1, \ldots, e_r)$, $a = \mathrm{Vect}(e_{r+1}, \ldots, e_d)$. For any point $x \in G_r(V)$ in Ω_W with coordinates (z_{jk}), we set

$$\varepsilon_k(x) = e_k + \sum_{1 \le j \le r} z_{jk} e_j, \qquad r+1 \le k \le d,$$

$$\widetilde{e}_j(x) = \text{image of } e_j \text{ in } Q_x = V/x, \qquad 1 \le j \le r.$$

Then $(\widetilde{e}_1, \ldots, \widetilde{e}_r)$ and $(\varepsilon_{r+1}, \ldots, \varepsilon_d)$ are holomorphic frames of Q and S respectively. If $g^\star : Q \longrightarrow \underline{V}$ is the orthogonal splitting of $g : \underline{V} \longrightarrow Q$, then

$$g^\star \cdot \widetilde{e}_j = e_j + \sum_{r+1 \le k \le d} \zeta_{jk} \varepsilon_k$$

for some $\zeta_{jk} \in \mathbb{C}$. After an easy computation we find

$$0 = \langle \widetilde{e}_j, g\varepsilon_k \rangle = \langle g^\star \widetilde{e}_j, \varepsilon_k \rangle = \zeta_{jk} + \overline{z}_{jk} + \sum_{l,m} \zeta_{jm} z_{lm} \overline{z}_{lk},$$

so that $\zeta_{jk} = -\overline{z}_{jk} + O(|z|^2)$. Formula (13.3) yields

$$d'' g_a^\star \cdot \widetilde{e}_j = - \sum_{r+1 \le k \le d} d\overline{z}_{jk} \otimes \varepsilon_k,$$

$$\beta_a^\star = \sum_{j,k} d\overline{z}_{jk} \otimes \widetilde{e}_j^\star \otimes \varepsilon_k, \qquad \beta_a = \sum_{j,k} dz_{jk} \otimes \varepsilon_k^\star \otimes \widetilde{e}_j,$$

$$\Theta(Q)_a = (\beta \wedge \beta^\star)_a = \sum_{j,k,l} dz_{jk} \wedge d\overline{z}_{lk} \otimes \widetilde{e}_l^\star \otimes \widetilde{e}_j, \tag{16.9}$$

$$\Theta(S)_a = (\beta^\star \wedge \beta)_a = - \sum_{j,k,l} dz_{jk} \wedge d\overline{z}_{jl} \otimes \varepsilon_k^\star \otimes \varepsilon_l. \tag{16.10}$$