Chapter VI
Hodge Theory

The goal of this chapter is to prove a number of basic facts in the Hodge theory of real or complex manifolds. The theory rests essentially on the fact that the De Rham (or Dolbeault) cohomology groups of a compact manifold can be represented by means of spaces of harmonic forms, once a Riemannian metric has been chosen. At this point, some knowledge of basic results about elliptic differential operators is required. The special properties of compact Kähler manifolds are then investigated in detail: Hodge decomposition theorem, hard Lefschetz theorem, Jacobian and Albanese variety, ...; the example of curves is treated in detail. Finally, the Hodge-Frölicher spectral sequence is applied to get some results on general compact complex manifolds, and it is shown that Hodge decomposition still holds for manifolds in the Fujiki class $(\mathcal{C})$.

§1. Differential Operators on Vector Bundles

We first describe some basic concepts concerning differential operators (symbol, composition, adjunction, ellipticity), in the general setting of vector bundles. Let M be a C^∞ differentiable manifold, $\dim_{\mathbb{R}} M = m$, and let E, F be $\mathbb{K}$-vector bundles over M, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} E = r$, $\operatorname{rank} F = r'$.

(1.1) Definition. *A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : C^\infty(M, E) \to C^\infty(M, F)$, $u \mapsto Pu$ of the form*

$$Pu(x) = \sum_{|\alpha| \le \delta} a_\alpha(x) D^\alpha u(x),$$

where $E_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^r$, $F_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^{r'}$ are trivialized locally on some open chart $\Omega \subset M$ equipped with local coordinates $(x_1, \ldots, x_m)$, and where $a_\alpha(x) = \big(a_{\alpha\lambda\mu}(x)\big)_{1 \le \lambda \le r',\, 1 \le \mu \le r}$ are $r' \times r$-matrices with C^∞ coefficients on Ω. Here $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_m)^{\alpha_m}$ as usual, and $u = (u_\mu)_{1 \le \mu \le r}$, $D^\alpha u = (D^\alpha u_\mu)_{1 \le \mu \le r}$ are viewed as column matrices.

If $t \in \mathbb{K}$ is a parameter, a simple calculation shows that $e^{-tu(x)} P(e^{tu(x)})$ is a polynomial of degree δ in t, of the form

$$e^{-tu(x)} P(e^{tu(x)}) = t^\delta \sigma_P(x, du(x)) + \text{lower order terms } c_j(x) t^j,\ j < \delta,$$

where σ_P is the polynomial map from $T^\star_M \to \mathrm{Hom}(E,F)$ defined by

$$(1.2)\quad T^\star_{M,x} \ni \xi \mapsto \sigma_P(x,\xi) \in \mathrm{Hom}(E_x,F_x), \qquad \sigma_P(x,\xi) = \sum_{|\alpha|=\delta} a_\alpha(x)\xi^\alpha.$$

The formula involving $e^{-tu}P(e^{tu})$ shows that $\sigma_P(x,\xi)$ actually does not depend on the choice of coordinates nor on the trivializations used for E, F. It is clear that $\sigma_P(x,\xi)$ is smooth on $T^\star_M$ as a function of (x,ξ), and is a homogeneous polynomial of degree δ in ξ. We say that σ_P is *the principal symbol* of P. Now, if E, F, G are vector bundles and

$$P : C^\infty(M,E) \to C^\infty(M,F), \qquad Q : C^\infty(M,F) \to C^\infty(M,G)$$

are differential operators of respective degrees δ_P, δ_Q, it is easy to check that $Q\circ P : C^\infty(M,E) \to C^\infty(M,G)$ is a differential operator of degree $\delta_P+\delta_Q$ and that

$$(1.3)\quad \sigma_{Q\circ P}(x,\xi) = \sigma_Q(x,\xi)\sigma_P(x,\xi).$$

Here the product of symbols is computed as a product of matrices.

Now, assume that M is oriented and is equipped with a smooth volume form $dV(x) = \gamma(x)dx_1\wedge\ldots dx_m$, where $\gamma(x)>0$ is a smooth density. If E is a euclidean or hermitian vector bundle, we have a Hilbert space $L^2(M,E)$ of global sections u of E with measurable coefficients, satisfying the L^2 estimate

$$(1.4)\quad \|u\|^2 = \int_M |u(x)|^2\, dV(x) < +\infty.$$

We denote by

$$(1.4')\quad \langle\!\langle u,v\rangle\!\rangle = \int_M \langle u(x),v(x)\rangle\, dV(x), \qquad u,v\in L^2(M,E)$$

the corresponding L^2 inner product.

(1.5) Definition. *If $P : C^\infty(M,E) \to C^\infty(M,F)$ is a differential operator and both E, F are euclidean or hermitian, there exists a unique differential operator*

$$P^\star : C^\infty(M,F) \to C^\infty(M,E),$$

called the formal adjoint of P, such that for all sections $u\in C^\infty(M,E)$ and $v\in C^\infty(M,F)$ there is an identity

$$\langle\!\langle Pu,v\rangle\!\rangle = \langle\!\langle u,P^\star v\rangle\!\rangle, \qquad \text{whenever } \operatorname{Supp} u\cap \operatorname{Supp} v \subset\subset M.$$

Proof. The uniqueness is easy, using the density of the set of elements $u\in C^\infty(M,E)$ with compact support in $L^2(M,E)$. Since uniqueness is clear, it is enough, by a partition of unity argument, to show the existence of $P^\star$ locally. Now, let $Pu(x)=\sum_{|\alpha|\le\delta}a_\alpha(x)D^\alpha u(x)$ be the expansion of P with respect to trivializations of E, F given by orthonormal frames over some coordinate open set $\Omega\subset M$. When $\mathrm{Supp}\,u\cap\mathrm{Supp}\,v\subset\subset\Omega$ an integration by parts yields

$$\begin{aligned}\langle\langle Pu,v\rangle\rangle&=\int_\Omega\sum_{|\alpha|\le\delta,\lambda,\mu}a_{\alpha\lambda\mu}D^\alpha u_\mu(x)\overline{v}_\lambda(x)\,\gamma(x)\,dx_1,\ldots,dx_m\\&=\int_\Omega\sum_{|\alpha|\le\delta,\lambda,\mu}(-1)^{|\alpha|}u_\mu(x)\overline{D^\alpha(\gamma(x)\,\overline{a}_{\alpha\lambda\mu}v_\lambda(x)}\,dx_1,\ldots,dx_m\\&=\int_\Omega\langle u,\sum_{|\alpha|\le\delta}(-1)^{|\alpha|}\gamma(x)^{-1}D^\alpha\big(\gamma(x)\,{}^t\overline{a}_\alpha v(x)\big)\rangle\,dV(x).\end{aligned}$$

Hence we see that $P^\star$ exists and is uniquely defined by

$$P^\star v(x)=\sum_{|\alpha|\le\delta}(-1)^{|\alpha|}\gamma(x)^{-1}D^\alpha\big(\gamma(x)\,{}^t\overline{a}_\alpha v(x)\big).\tag{1.6}$$
□

It follows immediately from (1.6) that the principal symbol of $P^\star$ is

$$\sigma_{P^\star}(x,\xi)=(-1)^\delta\sum_{|\alpha|=\delta}{}^t\overline{a}_\alpha\xi^\alpha=(-1)^\delta\sigma_P(x,\xi)^\star.\tag{1.7}$$

(1.8) Definition. *A differential operator P is said to be elliptic if*

$$\sigma_P(x,\xi)\in\mathrm{Hom}(E_x,F_x)$$

is injective for every $x\in M$ and $\xi\in T^\star_{M,x}\smallsetminus\{0\}$.

§2. Formalism of PseudoDifferential Operators

We assume throughout this section that (M,g) is a compact Riemannian manifold. For any positive integer k and any hermitian bundle $F\to M$, we denote by $W^k(M,F)$ the Sobolev space of sections $s:M\to F$ whose derivatives up to order k are in L^2. Let $\|\ \ \|_k$ be the norm of the Hilbert space $W^k(M,F)$. Let P be an elliptic differential operator of order d acting on $C^\infty(M,F)$. We need the following basic facts of elliptic PDE theory, see e.g. (Hörmander 1963).

(2.1) Sobolev lemma. *For $k>l+\frac{m}{2}$, $W^k(M,F)\subset C^l(M,F)$.*

(2.2) Rellich lemma. *For every integer k, the inclusion*

$$W^{k+1}(M,F)\hookrightarrow W^k(M,F)$$

is a compact linear operator.

(2.3) Gårding's inequality. *Let $\widetilde{P}$ be the extension of P to sections with distribution coefficients. For any $u \in W^0(M,F)$ such that $\widetilde{P}u \in W^k(M,F)$, then $u \in W^{k+d}(M,F)$ and*

$$\|u\|_{k+d} \le C_k(\|\widetilde{P}u\|_k + \|u\|_0),$$

where C_k is a positive constant depending only on k.

(2.4) Corollary. *The operator $P : C^\infty(M,F) \to C^\infty(M,F)$ has the following properties:*

i) $\ker P$ *is finite dimensional.*

ii) $P\big(C^\infty(M,F)\big)$ *is closed and of finite codimension; furthermore, if $P^\star$ is the formal adjoint of P, there is a decomposition*

$$C^\infty(M,F) = P\big(C^\infty(M,F)\big) \oplus \ker P^\star$$

as an orthogonal direct sum in $W^0(M,F) = L^2(M,F)$.

Proof. (i) Gårding's inequality shows that $\|u\|_{k+d} \le C_k\|u\|_0$ for any u in $\ker P$. Thanks to the Sobolev lemma, this implies that $\ker P$ is closed in $W^0(M,F)$. Moreover, the unit closed $\|\ \|_0$-ball of $\ker P$ is contained in the $\|\ \|_d$-ball of radius C_0, thus compact by the Rellich lemma. Riesz' theorem implies that $\dim\ker P < +\infty$.

(ii) We first show that the extension

$$\widetilde{P} : W^{k+d}(M,F) \to W^k(M,F)$$

has a closed range for any k. For every $\varepsilon > 0$, there exists a finite number of elements $v_1, \ldots, v_N \in W^{k+d}(M,F)$, $N = N(\varepsilon)$, such that

$$\|u\|_0 \le \varepsilon\|u\|_{k+d} + \sum_{j=1}^{N} |\langle\!\langle u, v_j\rangle\!\rangle_0| \ ; \tag{2.5}$$

indeed the set

$$K_{(v_j)} = \Big\{u \in W^{k+d}(M,F) \ ; \ \varepsilon\|u\|_{k+d} + \sum_{j=1}^{N} |\langle\!\langle u, v_j\rangle\!\rangle_0| \le 1\Big\}$$

is relatively compact in $W^0(M,F)$ and $\bigcap_{(v_j)} \overline{K}_{(v_j)} = \{0\}$. It follows that there exist elements (v_j) such that $\overline{K}_{(v_j)}$ is contained in the unit ball of $W^0(M,F)$, *QED*. Substitute $||u||_0$ by the upper bound (2.5) in Gårding's inequality; we get

$$(1 - C_k\varepsilon)\|u\|_{k+d} \le C_k\Big(\|\widetilde{P}u\|_k + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0|\Big).$$

Define $G = \big\{u \in W^{k+d}(M, F)\ ;\ u \perp v_j,\ 1 \le j \le n\big\}$ and choose $\varepsilon = 1/2C_k$. We obtain

$$\|u\|_{k+d} \le 2C_k\|\widetilde{P}u\|_k, \quad \forall u \in G.$$

This implies that $\widetilde{P}(G)$ is closed. Therefore

$$\widetilde{P}\big(W^{k+d}(M, F)\big) = \widetilde{P}(G) + \mathrm{Vect}\big(\widetilde{P}(v_1), \ldots, \widetilde{P}(v_N)\big)$$

is closed in $W^k(M, F)$. Take in particular $k = 0$. Since $C^\infty(M, F)$ is dense in $W^d(M, F)$, we see that in $W^0(M, F)$

$$\Big(\widetilde{P}\big(W^d(M, F)\big)\Big)^\perp = \Big(P\big(C^\infty(M, F)\big)\Big)^\perp = \ker \widetilde{P^\star}.$$

We have proved that

$$W^0(M, F) = \widetilde{P}\big(W^d(M, F)\big) \oplus \ker \widetilde{P^\star}. \tag{2.6}$$

Since $P^\star$ is also elliptic, it follows that $\ker \widetilde{P^\star}$ is finite dimensional and that $\ker \widetilde{P^\star} = \ker P^\star$ is contained in $C^\infty(M, F)$. Thanks to Gårding's inequality, the decomposition formula (2.6) yields

$$W^k(M, F) = \widetilde{P}\big(W^{k+d}(M, F)\big) \oplus \ker P^\star, \tag{2.7}$$

$$C^\infty(M, F) = P\big(C^\infty(M, F)\big) \oplus \ker P^\star. \tag{2.8}$$

§3. Hodge Theory of Compact Riemannian Manifolds

§3.1. Euclidean Structure of the Exterior Algebra

Let (M, g) be an oriented Riemannian C^∞-manifold, $\dim_{\mathbb{R}} M = m$, and $E \to M$ a hermitian vector bundle of rank r over M. We denote respectively by $(\xi_1, \ldots, \xi_m)$ and $(e_1, \ldots, e_r)$ orthonormal frames of T_M and E over an open subset $\Omega \subset M$, and by $(\xi_1^\star, \ldots, \xi_m^\star)$, $(e_1^\star, \ldots, e_r^\star)$ the corresponding dual frames of $T_M^\star$, $E^\star$. Let dV stand for the Riemannian volume form on M. The exterior algebra $\Lambda T_M^\star$ has a natural inner product $\langle\bullet, \bullet\rangle$ such that

$$\langle u_1 \wedge \ldots \wedge u_p, v_1 \wedge \ldots \wedge v_p\rangle = \det(\langle u_j, v_k\rangle)_{1 \le j,k \le p}, \qquad u_j, v_k \in T_M^\star \tag{3.1}$$

for all p, with $\Lambda T_M^\star = \bigoplus \Lambda^p T_M^\star$ as an orthogonal sum. Then the covectors $\xi_I^\star = \xi_{i_1}^\star \wedge \cdots \wedge \xi_{i_p}^\star$, $i_1 < i_2 < \cdots < i_p$, provide an orthonormal basis of $\Lambda T_M^\star$. We also denote by $\langle\bullet, \bullet\rangle$ the corresponding inner product on $\Lambda T_M^\star \otimes E$.

(3.2) Hodge Star Operator. *The Hodge-Poincaré-De Rham operator $\star$ is the collection of linear maps defined by*

$$\star : \Lambda^p T^\star_M \to \Lambda^{m-p} T^\star_M, \qquad u \wedge \star\, v = \langle u, v\rangle\, dV, \qquad \forall u, v \in \Lambda^p T^\star_M.$$

The existence and uniqueness of this operator is easily seen by using the duality pairing

$$\begin{aligned} &\Lambda^p T^\star_M \times \Lambda^{m-p} T^\star_M \longrightarrow \mathbb{R} \\ &(u, v) \longmapsto u \wedge v / dV = \sum \varepsilon(I, \complement I)\, u_I v_{\complement I}, \end{aligned} \tag{3.3}$$

where $u = \sum_{|I|=p} u_I\, \xi^\star_I$, $v = \sum_{|J|=m-p} v_J\, \xi^\star_J$, where $\complement I$ stands for the (ordered) complementary multi-index of I and $\varepsilon(I, \complement I)$ for the signature of the permutation $(1, 2, \ldots, m) \longmapsto (I, \complement I)$. From this, we find

$$\star\, v = \sum_{|I|=p} \varepsilon(I, \complement I) v_I\, \xi^\star_{\complement I}. \tag{3.4}$$

More generally, the sesquilinear pairing $\{\bullet, \bullet\}$ defined in (V-7.1) yields an operator $\star$ on vector valued forms, such that

$$\star\ : \Lambda^p T^\star_M \otimes E \to \Lambda^{m-p} T^\star_M \otimes E, \qquad \{s, \star\, t\} = \langle s, t\rangle\, dV, \qquad s, t \in \Lambda^p T^\star_M \otimes E, \tag{3.3$'$}$$

$$\star\, t = \sum_{|I|=p, \lambda} \varepsilon(I, \complement I)\, t_{I,\lambda}\, \xi^\star_{\complement I} \otimes e_\lambda \tag{3.4$'$}$$

for $t = \sum t_{I,\lambda}\, \xi^\star_I \otimes e_\lambda$. Since $\varepsilon(I, \complement I)\varepsilon(\complement I, I) = (-1)^{p(m-p)} = (-1)^{p(m-1)}$, we get immediately

$$\star \star t = (-1)^{p(m-1)} t \quad \text{ on } \ \Lambda^p T^\star_M \otimes E. \tag{3.5}$$

It is clear that $\star$ is an isometry of $\Lambda^\bullet T^\star_M \otimes E$.

We shall also need a variant of the $\star$ operator, namely the conjugate-linear operator

$$\# \ :\ \Lambda^p T^\star_M \otimes E \longrightarrow \Lambda^{m-p} T^\star_M \otimes E^\star$$

defined by $s \wedge \#\, t = \langle s, t\rangle\, dV$, where the wedge product $\wedge$ is combined with the canonical pairing $E \times E^\star \to \mathbb{C}$. We have

$$\#\, t = \sum_{|I|=p, \lambda} \varepsilon(I, \complement I)\, \bar t_{I,\lambda}\, \xi^\star_{\complement I} \otimes e^\star_\lambda. \tag{3.6}$$

(3.7) Contraction by a Vector Field.. *Given a tangent vector $\theta \in T_M$ and a form $u \in \Lambda^p T^\star_M$, the contraction $\theta \lrcorner\ u \in \Lambda^{p-1} T^\star_M$ is defined by*

$$\theta \lrcorner\ u\, (\eta_1, \ldots, \eta_{p-1}) = u(\theta, \eta_1, \ldots, \eta_{p-1}), \qquad \eta_j \in T_M.$$

In terms of the basis (ξ_j), $\bullet \lrcorner \bullet$ is the bilinear operation characterized by

$$\xi_l \lrcorner (\xi_{i_1}^\star \wedge \ldots \wedge \xi_{i_p}^\star) = \begin{cases} 0 & \text{if } l \notin \{i_1, \ldots, i_p\}, \\ (-1)^{k-1} \xi_{i_1}^\star \wedge \ldots \widehat{\xi_{i_k}^\star} \ldots \wedge \xi_{i_p}^\star & \text{if } l = i_k. \end{cases}$$

This formula is in fact valid even when (ξ_j) is non orthonormal. A rather easy computation shows that $\theta \lrcorner \bullet$ is a *derivation* of the exterior algebra, i.e. that

$$\theta \lrcorner (u \wedge v) = (\theta \lrcorner u) \wedge v + (-1)^{\deg u} u \wedge (\theta \lrcorner v).$$

Moreover, if $\widetilde{\theta} = \langle \bullet, \theta \rangle \in T_M^\star$, the operator $\theta \lrcorner \bullet$ is the adjoint map of $\widetilde{\theta} \wedge \bullet$, that is,

$$(3.8) \quad \langle \theta \lrcorner u, v \rangle = \langle u, \widetilde{\theta} \wedge v \rangle, \qquad u, v \in \Lambda T_M^\star.$$

Indeed, this property is immediately checked when $\theta = \xi_l$, $u = \xi_I^\star$, $v = \xi_J^\star$.

§3.2. Laplace-Beltrami Operators

Let us consider the Hilbert space $L^2(M, \Lambda^p T_M^\star)$ of p-forms u on M with measurable coefficients such that

$$\|u\|^2 = \int_M |u|^2 \, dV < +\infty.$$

We denote by $\langle\!\langle \ , \ \rangle\!\rangle$ the global inner product on L^2-forms. The Hilbert space $L^2(M, \Lambda^p T_M^\star \otimes E)$ is defined similarly.

(3.9) Theorem. *The operator $d^\star = (-1)^{mp+1} \star d \star$ is the formal adjoint of the exterior derivative d acting on $C^\infty(M, \Lambda^p T_M^\star \otimes E)$.*

Proof. If $u \in C^\infty(M, \Lambda^p T_M^\star)$, $v \in C^\infty(M, \Lambda^{p+1} T_M^\star \otimes)$ are compactly supported we get

$$\begin{aligned} \langle\!\langle du, v \rangle\!\rangle &= \int_M \langle du, v \rangle \, dV = \int_M du \wedge \star v \\ &= \int_M d(u \wedge \star v) - (-1)^p u \wedge d \star v = -(-1)^p \int_M u \wedge d \star v \end{aligned}$$

by Stokes' formula. Therefore (3.4) implies

$$\langle\!\langle du, v \rangle\!\rangle = -(-1)^p (-1)^{p(m-1)} \int_M u \wedge \star\star d \star v = (-1)^{mp+1} \langle\!\langle u, \star d \star v \rangle\!\rangle. \ \square$$

(3.10) Remark. If m is even, the formula reduces to $d^\star = - \star d \star$.

(3.11) Definition. *The operator $\Delta = dd^\star + d^\star d$ is called the Laplace-Beltrami operator of M.*

Since $d^\star$ is the adjoint of d, the Laplace operator Δ is formally self-adjoint, i.e. $\langle\!\langle \Delta u, v\rangle\!\rangle = \langle\!\langle u, \Delta v\rangle\!\rangle$ when the forms u, v are of class C^∞ and compactly supported.

(3.12) Example. Let M be an open subset of $\mathbb{R}^m$ and $g = \sum_{i=1}^m dx_i^2$. In that case we get

$$u = \sum_{|I|=p} u_I dx_I, \qquad du = \sum_{|I|=p,j} \frac{\partial u_I}{\partial x_j} dx_j \wedge dx_I,$$
$$\langle\!\langle u, v\rangle\!\rangle = \int_M \langle u, v\rangle\, dV = \int_M \sum_I u_I v_I\, dV$$

One can write $dv = \sum dx_j \wedge (\partial v/\partial x_j)$ where $\partial v/\partial x_j$ denotes the form v in which all coefficients v_I are differentiated as $\partial v_I/\partial x_j$. An integration by parts combined with contraction gives

$$\begin{aligned}\langle\!\langle d^\star u, v\rangle\!\rangle = \langle\!\langle u, dv\rangle\!\rangle &= \int_M \langle u, \sum_j dx_j \wedge \frac{\partial v}{\partial x_j}\rangle\, dV \\ &= \int_M \sum_j \langle \frac{\partial}{\partial x_j} \lrcorner\ u,\ \frac{\partial v}{\partial x_j}\rangle\, dV = -\int_M \langle \sum_j \frac{\partial}{\partial x_j} \lrcorner\ \frac{\partial u}{\partial x_j}, v\rangle\, dV,\end{aligned}$$
$$d^\star u = -\sum_j \frac{\partial}{\partial x_j} \lrcorner\ \frac{\partial u}{\partial x_j} = -\sum_{I,j} \frac{\partial u_I}{\partial x_j} \frac{\partial}{\partial x_j} \lrcorner\ dx_I.$$

We get therefore

$$dd^\star u = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} dx_k \wedge \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_I\Big),$$
$$d^\star du = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} \frac{\partial}{\partial x_j} \lrcorner\ (dx_k \wedge dx_I).$$

Since

$$\frac{\partial}{\partial x_j} \lrcorner\ (dx_k \wedge dx_I) = \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_k\Big) dx_I - dx_k \wedge \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_I\Big),$$

we obtain

$$\Delta u = -\sum_I \Big(\sum_j \frac{\partial^2 u_I}{\partial x_j^2}\Big) dx_I.$$

In the case of an arbitrary riemannian manifold (M, g) we have

$$\begin{aligned}
u &= \sum u_I\,\xi_I^\star,\\
du &= \sum_{I,j}(\xi_j\cdot u_I)\,\xi_j^\star\wedge\xi_I^\star+\sum_I u_I\,d\xi_I^\star,\\
d^\star u &= -\sum_{I,j}(\xi_j\cdot u_I)\,\xi_j\,\lrcorner\,\xi_I^\star+\sum_{I,K}\alpha_{I,K}u_I\,\xi_K^\star\,,
\end{aligned}$$

for some C^∞ coefficients $\alpha_{I,K}$, $|I|=p$, $|K|=p-1$. It follows that the principal part of Δ is the same as that of the second order operator

$$u\longmapsto -\sum_I\big(\sum_j\xi_j^2\cdot u_I\big)\xi_I^\star.$$

As a consequence, Δ is *elliptic.*

Assume now that D_E is a hermitian connection on E. The formal adjoint operator of D_E acting on $C^\infty(M,\Lambda^pT_M^\star\otimes E)$ is

(3.13) $\quad D_E^\star=(-1)^{mp+1}\star D_E\star\,.$

Indeed, if $s\in C^\infty(M,\Lambda^pT_M^\star\otimes E)$, $t\in C^\infty(M,\Lambda^{p+1}T_M^\star\otimes E)$ have compact support, we get

$$\begin{aligned}
\langle\!\langle D_Es,t\rangle\!\rangle &= \int_M\langle D_Es,t\rangle\,dV=\int_M\{D_Es,\,\star t\}\\
&= \int_M d\{s,\,\star t\}-(-1)^p\{s,D_E\star t\}=(-1)^{mp+1}\langle\!\langle s,\,\star D_E\star t\rangle\!\rangle.
\end{aligned}$$

(3.14) Definition. *The Laplace-Beltrami operator associated to D_E is the second order operator $\Delta_E=D_ED_E^\star+D_E^\star D_E$.*

Δ_E is a self-adjoint elliptic operator with principal part

$$s\longmapsto -\sum_{I,\lambda}\Big(\sum_j\xi_j^2\cdot s_{I,\lambda}\Big)\xi_I^\star\otimes e_\lambda.$$

§3.3. Harmonic Forms and Hodge Isomorphism

Let E be a hermitian vector bundle over a *compact* Riemannian manifold (M,g). We assume that E possesses a *flat* hermitian connection D_E (this means that $\Theta(D_E)=D_E^2=0$, or equivalently, that E is given by a representation $\pi_1(M)\to U(r)$, cf. § V-6). A fundamental example is of course the trivial bundle $E=M\times\mathbb{C}$ with the connection $D_E=d$. Thanks to our flatness assumption, D_E defines a generalized De Rham complex

$$D_E:C^\infty(M,\Lambda^pT_M^\star\otimes E)\longrightarrow C^\infty(M,\Lambda^{p+1}T_M^\star\otimes E).$$

The cohomology groups of this complex will be denoted by $H^p_{DR}(M,E)$.

The space of *harmonic forms of degree* p with respect to the Laplace-Beltrami operator $\Delta_E = D_E D_E^\star + D_E^\star D_E$ is defined by

$$\text{(3.15)}\quad \mathcal{H}^p(M,E) = \big\{ s \in C^\infty(M, \Lambda^p T^\star_M \otimes E)\ ;\ \Delta_E s = 0 \big\}.$$

Since $\langle\langle \Delta_E s, s\rangle\rangle = ||D_E s||^2 + ||D_E^\star s||^2$, we see that $s \in \mathcal{H}^p(M,E)$ if and only if $D_E s = D_E^\star s = 0$.

(3.16) Theorem. *For any p, there exists an orthogonal decomposition*

$$\begin{aligned} &C^\infty(M, \Lambda^p T^\star_M \otimes E) = \mathcal{H}^p(M,E) \oplus \operatorname{Im} D_E \oplus \operatorname{Im} D_E^\star,\\ &\operatorname{Im} D_E = D_E\big(C^\infty(M, \Lambda^{p-1} T^\star_M \otimes E)\big),\\ &\operatorname{Im} D_E^\star = D_E^\star\big(C^\infty(M, \Lambda^{p+1} T^\star_M \otimes E)\big).\end{aligned}$$

Proof. It is immediate that $\mathcal{H}^p(M,E)$ is orthogonal to both subspaces $\operatorname{Im} D_E$ and $\operatorname{Im} D_E^\star$. The orthogonality of these two subspaces is also clear, thanks to the assumption $D_E^2 = 0$:

$$\langle\langle D_E s, D_E^\star t\rangle\rangle = \langle\langle D_E^2 s, t\rangle\rangle = 0.$$

We apply now Cor. 2.4 to the elliptic operator $\Delta_E = \Delta_E^\star$ acting on p-forms, i.e. on the bundle $F = \Lambda^p T^\star_M \otimes E$. We get

$$\begin{aligned} &C^\infty(M, \Lambda^p T^\star_M \otimes E) = \mathcal{H}^p(M,E) \oplus \Delta_E\big(C^\infty(M, \Lambda^p T^\star_M \otimes E)\big),\\ &\operatorname{Im} \Delta_E = \operatorname{Im}(D_E D_E^\star + D_E^\star D_E) \subset \operatorname{Im} D_E + \operatorname{Im} D_E^\star. \end{aligned}$$

□

(3.17) Hodge isomorphism theorem. *The De Rham cohomology group $H^p_{DR}(M,E)$ is finite dimensional and $H^p_{DR}(M,E) \simeq \mathcal{H}^p(M,E)$.*

Proof. According to decomposition 3.16, we get

$$\begin{aligned} &B^p_{DR}(M,E) = D_E\big(C^\infty(M, \Lambda^{p-1} T^\star_M \otimes E)\big),\\ &Z^p_{DR}(M,E) = \ker D_E = (\operatorname{Im} D_E^\star)^\perp = \mathcal{H}^p(M,E) \oplus \operatorname{Im} D_E. \end{aligned}$$

This shows that every De Rham cohomology class contains a unique harmonic representative. □

(3.18) Poincaré duality. *The bilinear pairing*

$$H^p_{DR}(M,E) \times H^{m-p}_{DR}(M,E^\star) \longrightarrow \mathbb{C}, \qquad (s,t) \longmapsto \int_M s \wedge t$$

is a non degenerate duality.

Proof. First note that there exists a naturally defined flat connection $D_{E^\star}$ such that for any $s_1 \in C^\infty_\bullet(M,E)$, $s_2 \in C^\infty_\bullet(M,E^\star)$ we have

$$d(s_1 \wedge s_2) = (D_E s_1) \wedge s_2 + (-1)^{\deg s_1} s_1 \wedge D_{E^\star} s_2. \tag{3.19}$$

It is then a consequence of Stokes' formula that the map $(s,t) \mapsto \int_M s \wedge t$ can be factorized through cohomology groups. Let $s \in C^\infty(M, \Lambda^p T^\star_M \otimes E)$. We leave to the reader the proof of the following formulas (use (3.19) in analogy with the proof of Th. 3.9):

$$\begin{aligned} D_{E^\star}(\# s) &= (-1)^p \# D^\star_E s, \\ \delta_{E^\star}(\# s) &= (-1)^{p+1} \# D_E s, \\ \Delta_{E^\star}(\# s) &= \# \Delta_E s, \end{aligned} \tag{3.20}$$

Consequently $\# s \in \mathcal{H}^{m-p}(M,E^\star)$ if and only if $s \in \mathcal{H}^p(M,E)$. Since

$$\int_M s \wedge \# s = \int_M |s|^2 \, dV = \|s\|^2,$$

we see that the Poincaré pairing has zero kernel in the left hand factor $\mathcal{H}^p(M,E) \simeq H^p_{DR}(M,E)$. By symmetry, it has also zero kernel on the right. The proof is achieved. □

§4. Hermitian and Kähler Manifolds

Let X be a complex n-dimensional manifold. A *hermitian metric* on X is a positive definite hermitian form of class C^∞ on T_X ; in a coordinate system $(z_1,\ldots,z_n)$, such a form can be written $h(z) = \sum_{1 \le j,k \le n} h_{jk}(z)\, dz_j \otimes d\overline{z}_k$, where (h_{jk}) is a positive hermitian matrix with C^∞ coefficients. According to (III-1.8), the *fundamental* $(1,1)$*-form* associated to h is the positive form of type $(1,1)$

$$\omega = -\mathrm{Im}\, h = \frac{\mathrm{i}}{2} \sum h_{jk} dz_j \wedge d\overline{z}_k, \qquad 1 \le j,k \le n.$$

(4.1) Definition.

a) *A hermitian manifold is a pair (X,ω) where ω is a C^∞ positive definite $(1,1)$-form on X.*

b) *The metric ω is said to be kähler if $d\omega = 0$.*

c) *X is said to be a Kähler manifold if X carries at least one Kähler metric.*

Since ω is real, the conditions $d\omega = 0$, $d'\omega = 0$, $d''\omega = 0$ are all equivalent. In local coordinates we see that $d'\omega = 0$ if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \quad , \quad 1 \le j, k, l \le n.$$

A simple computation gives

$$\frac{\omega^n}{n!} = \det(h_{jk}) \bigwedge_{1 \le j \le n} \Big(\frac{\mathrm{i}}{2} dz_j \wedge d\overline{z}_j\Big) = \det(h_{jk})\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_n = x_n + \mathrm{i} y_n$. Therefore the (n,n)-form

$$(4.2) \quad dV = \frac{1}{n!}\omega^n$$

is positive and coincides with the hermitian volume element of X. If X is compact, then $\int_X \omega^n = n!\ \mathrm{Vol}_\omega(X) > 0$. This simple remark already implies that compact Kähler manifolds must satisfy some restrictive topological conditions:

(4.3) Consequence.

a) *If (X, ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X, \mathbb{R})$, then $\{\omega\}^n \neq 0$.*

b) *If X is compact Kähler, then $H^{2k}(X, \mathbb{R}) \neq 0$ for $0 \le k \le n$. In fact, $\{\omega\}^k$ is a non zero class in $H^{2k}(X, \mathbb{R})$.*

(4.4) Example. The complex projective space $\mathbb{P}^n$ is Kähler. A natural Kähler metric ω on $\mathbb{P}^n$, called the *Fubini-Study metric*, is defined by

$$p^\star \omega = \frac{\mathrm{i}}{2\pi} d'd'' \log\big(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2\big)$$

where $\zeta_0, \zeta_1, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n \subset \mathbb{P}^n$. Then (V-15.8) and (V-15.12) show that

$$\omega = \frac{\mathrm{i}}{2\pi} d'd'' \log(1 + |z|^2) = \frac{\mathrm{i}}{2\pi} c\big(\mathcal{O}(1)\big), \qquad \int_{\mathbb{P}^n} \omega^n = 1.$$

Furthermore $\{\omega\} \in H^2(\mathbb{P}^n, \mathbb{Z})$ is a generator of the cohomology algebra $H^\bullet(\mathbb{P}^n, \mathbb{Z})$ in virtue of Th. V-15.10.

(4.5) Example. A *complex torus* is a quotient $X = \mathbb{C}^n/\Gamma$ by a lattice Γ of rank $2n$. Then X is a compact complex manifold. Any positive definite hermitian form $\omega = \mathrm{i} \sum h_{jk} dz_j \wedge d\overline{z}_k$ with constant coefficients defines a Kähler metric on X.

(4.6) Example. Every (complex) submanifold Y of a Kähler manifold (X, ω) is Kähler with metric $\omega_{\restriction Y}$. Especially, all submanifolds of $\mathbb{P}^n$ are Kähler.

(4.7) Example. Consider the complex surface

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where $\Gamma = \{\lambda^n \,;\, n \in \mathbb{Z}\}$, $\lambda < 1$, acts as a group of homotheties. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}_+^\star \times S^3$, we have $X \simeq S^1 \times S^3$. Therefore $H^2(X, \mathbb{R}) = 0$ by Künneth's formula IV-15.10, and property 4.3 b) shows that X is not Kähler. More generally, one can obtaintake Γ to be an infinite cyclic group generated by a holomorphic contraction of $\mathbb{C}^2$, of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix}, \qquad \text{resp.} \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda z_1 \\ \lambda z_2 + z_1^p \end{pmatrix},$$

where $\lambda, \lambda_1, \lambda_2$ are complex numbers such that $0 < |\lambda_1| \le |\lambda_2| < 1$, $0 < |\lambda| < 1$, and p a positive integer. These non Kähler surfaces are called *Hopf surfaces.*

The following Theorem shows that a hermitian metric ω on X is Kähler if and only if the metric ω is tangent at order 2 to a hermitian metric with constant coefficients at every point of X.

(4.8) Theorem. *Let ω be a C^∞ positive definite $(1,1)$-form on X. In order that ω be Kähler, it is necessary and sufficient that to every point $x_0 \in X$ corresponds a holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0 such that*

$$(4.9) \quad \omega = \mathrm{i} \sum_{1 \le l,m \le n} \omega_{lm}\, dz_l \wedge d\overline{z}_m, \qquad \omega_{lm} = \delta_{lm} + O(|z|^2).$$

If ω is Kähler, the coordinates $(z_j)_{1 \le j \le n}$ can be chosen such that

$$(4.10) \quad \omega_{lm} = \langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} - \sum_{1 \le j,k \le n} c_{jklm}\, z_j \overline{z}_k + O(|z|^3),$$

where (c_{jklm}) are the coefficients of the Chern curvature tensor

$$(4.11) \quad \Theta(T_X)_{x_0} = \sum_{j,k,l,m} c_{jklm}\, dz_j \wedge d\overline{z}_k \otimes \Big(\frac{\partial}{\partial z_l}\Big)^\star \otimes \frac{\partial}{\partial z_m}$$

associated to (T_X, ω) at x_0. Such a system (z_j) will be called a geodesic coordinate system at x_0.

Proof. It is clear that (4.9) implies $d_{x_0}\omega = 0$, so the condition is sufficient. Assume now that ω is Kähler. Then one can choose local coordinates $(x_1, \ldots, x_n)$ such that $(dx_1, \ldots, dx_n)$ is an ω-orthonormal basis of $T^\star_{x_0}X$. Therefore

$$\omega = \mathrm{i}\sum_{1\le l,m\le n} \widetilde{\omega}_{lm}\, dx_l \wedge d\overline{x}_m, \qquad \text{where}$$

$$\widetilde{\omega}_{lm} = \delta_{lm} + O(|x|) = \delta_{lm} + \sum_{1\le j\le n} (a_{jlm}x_j + a'_{jlm}\overline{x}_j) + O(|x|^2). \tag{4.12}$$

Since ω is real, we have $a'_{jlm} = \overline{a}_{jml}$; on the other hand the Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at x_0 implies $a_{jlm} = a_{ljm}$. Set now

$$z_m = x_m + \frac{1}{2}\sum_{j,l} a_{jlm}x_jx_l, \qquad 1 \le m \le n.$$

Then (z_m) is a coordinate system at x_0, and

$$dz_m = dx_m + \sum_{j,l} a_{jlm}x_j dx_l,$$

$$\begin{aligned}
\mathrm{i}\sum_m dz_m \wedge d\overline{z}_m &= \mathrm{i}\sum_m dx_m \wedge d\overline{x}_m + \mathrm{i}\sum_{j,l,m} a_{jlm}x_j\, dx_l \wedge d\overline{x}_m \\
&\qquad\qquad + \mathrm{i}\sum_{j,l,m} \overline{a}_{jlm}\overline{x}_j\, dx_m \wedge d\overline{x}_l + O(|x|^2) \\
&= \mathrm{i}\sum_{l,m} \widetilde{\omega}_{lm}\, dx_l \wedge \overline{d}x_m + O(|x|^2) = \omega + O(|z|^2).
\end{aligned}$$

Condition (4.9) is proved. Suppose the coordinates (x_m) chosen from the beginning so that (4.9) holds with respect to (x_m). Then the Taylor expansion (4.12) can be refined into

$$\begin{aligned}
\widetilde{\omega}_{lm} &= \delta_{lm} + O(|x|^2) \\
&= \delta_{lm} + \sum_{j,k} \big(a_{jklm}x_j\overline{x}_k + a'_{jklm}x_jx_k + a''_{jklm}\overline{x}_j\overline{x}_k\big) + O(|x|^3).
\end{aligned} \tag{4.13}$$

These new coefficients satisfy the relations

$$a'_{jklm} = a'_{kjlm}, \qquad a''_{jklm} = \overline{a}'_{jkml}, \qquad \overline{a}_{jklm} = a_{kjml}.$$

The Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at $x = 0$ gives the equality $a'_{jklm} = a'_{lkjm}$; in particular a'_{jklm} is invariant under all permutations of j, k, l. If we set

$$z_m = x_m + \frac{1}{3}\sum_{j,k,l} a'_{jklm}\, x_jx_kx_l, \qquad 1 \le m \le n,$$

then by (4.13) we find

$$dz_m = dx_m + \sum_{j,k,l} a'_{jklm}\, x_j x_k\, dx_l, \qquad 1 \le m \le n,$$

$$\omega = \mathrm{i} \sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i} \sum_{j,k,l,m} a_{jklm}\, x_j \overline{x}_k\, dx_l \wedge d\overline{x}_m + O(|x|^3),$$

$$\omega = \mathrm{i} \sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i} \sum_{j,k,l,m} a_{jklm}\, z_j \overline{z}_k\, dz_l \wedge d\overline{z}_m + O(|z|^3). \tag{4.14}$$

It is now easy to compute the Chern curvature tensor $\Theta(T_X)_{x_0}$ in terms of the coefficients a_{jklm}. Indeed

$$\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} + \sum_{j,k} a_{jklm}\, z_j \overline{z}_k + O(|z|^3),$$

$$d'\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \Big\{ D'\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \Big\} = \sum_{j,k} a_{jklm}\, \overline{z}_k\, dz_j + O(|z|^2),$$

$$\Theta(T_X)\cdot \frac{\partial}{\partial z_l} = D''D'\Big(\frac{\partial}{\partial z_l}\Big) = -\sum_{j,k,m} a_{jklm}\, dz_j \wedge d\overline{z}_k \otimes \frac{\partial}{\partial z_m} + O(|z|),$$

therefore $c_{jklm} = -a_{jklm}$ and the expansion (4.10) follows from (4.14). $\square$

(4.15) Remark. As a by-product of our computations, we find that on a Kähler manifold the coefficients of $\Theta(T_X)$ satisfy the symmetry relations

$$\overline{c}_{jklm} = c_{kjml}, \qquad c_{jklm} = c_{lkjm} = c_{jmlk} = c_{lmjk}.$$

§5. Basic Results of Kähler Geometry

§5.1. Operators of Hermitian Geometry

Let (X,ω) be a hermitian manifold and let $z_j = x_j + \mathrm{i} y_j$, $1 \le j \le n$, be analytic coordinates at a point $x \in X$ such that $\omega(x) = \mathrm{i}\sum dz_j \wedge d\overline{z}_j$ is diagonalized at this point. The associated hermitian form is the $h(x) = 2\sum dz_j \otimes d\overline{z}_j$ and its real part is the euclidean metric $2\sum (dx_j)^2 + (dy_j)^2$. It follows from this that $|dx_j| = |dy_j| = 1/\sqrt{2}$, $|dz_j| = |d\overline{z}_j| = 1$, and that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $(T^\star_x X, \omega)$. Formula (3.1) with u_j, v_k in the orthogonal sum $(\mathbb{C}\otimes T_X)^\star = T^\star_X \oplus \overline{T^\star_X}$ defines a natural inner product on the exterior algebra $\Lambda^\bullet(\mathbb{C}\otimes T_X)^\star$. The norm of a form

$$u = \sum_{I,J} u_{I,J} dz_I \wedge d\overline{z}_J \in \Lambda(\mathbb{C}\otimes T_X)^\star$$

at the given point x is then equal to

$$(5.1)\quad |u(x)|^2 = \sum_{I,J} |u_{I,J}(x)|^2.$$

The Hodge $\star$ operator (3.2) can be extended to $\mathbb{C}$-valued forms by the formula

$$(5.2)\quad u \wedge \star\,\overline{v} = \langle u, v\rangle\, dV.$$

It follows that $\star$ is a $\mathbb{C}$-linear isometry

$$\star\ :\ \Lambda^{p,q}T_X^\star \longrightarrow \Lambda^{n-q,n-p}T_X^\star.$$

The usual operators of hermitian geometry are the operators d, $\delta = -\star\, d\,\star$, $\Delta = d\delta + \delta d$ already defined, and their complex counterparts

$$(5.3)\quad \begin{cases} d = d' + d'', \\ \delta = d'^\star + d''^\star, \qquad d'^\star = (d')^\star = -\star\, d''\star, \qquad d''^\star = (d'')^\star = -\star\, d'\star, \\ \Delta' = d'd'^\star + d'^\star d', \qquad \Delta'' = d''d''^\star + d''^\star d''. \end{cases}$$

Another important operator is the operator L of type (1,1) defined by

$$(5.4)\quad Lu = \omega \wedge u$$

and its adjoint $\Lambda = \star^{-1} L\,\star$:

$$(5.5)\quad \langle u, \Lambda v\rangle = \langle Lu, v\rangle.$$

§5.2. Commutation Identities

If A, B are endomorphisms of the algebra $C^\infty_{\bullet,\bullet}(X,\mathbb{C})$, their graded commutator (or graded Lie bracket) is defined by

$$(5.6)\quad [A,B] = AB - (-1)^{ab} BA$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following *Jacobi identity* is easy to check:

$$(5.7)\quad (-1)^{ca}\big[A,[B,C]\big] + (-1)^{ab}\big[B,[C,A]\big] + (-1)^{bc}\big[C,[A,B]\big] = 0.$$

For any $\alpha \in \Lambda^{p,q}T_X^\star$, we still denote by α the endomorphism of type (p,q) on $\Lambda^{\bullet,\bullet}T_X^\star$ defined by $u \mapsto \alpha \wedge u$.

Let $\gamma \in \Lambda^{1,1}T_X^\star$ be a real (1,1)-form. There exists an ω-orthogonal basis $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ in T_X which diagonalizes both forms ω and γ:

$$\omega = \mathrm{i} \sum_{1\le j\le n} \zeta_j^\star \wedge \overline{\zeta}_j^\star, \qquad \gamma = \mathrm{i} \sum_{1\le j\le n} \gamma_j\, \zeta_j^\star \wedge \overline{\zeta}_j^\star, \quad \gamma_j \in \mathbb{R}.$$

(5.8) Proposition. *For every form $u = \sum u_{J,K}\, \zeta_J^\star \wedge \overline{\zeta}_K^\star$, one has*

$$[\gamma,\Lambda]u=\sum_{J,K}\Big(\sum_{j\in J}\gamma_j+\sum_{j\in K}\gamma_j-\sum_{1\le j\le n}\gamma_j\Big)u_{J,K}\,\zeta_J^\star\wedge\overline{\zeta}_K^\star.$$

Proof. If u is of type (p,q), a brute-force computation yields

$$\begin{aligned}
\Lambda u&=\mathrm{i}(-1)^p\sum_{J,K,l}u_{J,K}\,(\zeta_l\lrcorner\,\zeta_J^\star)\wedge(\overline{\zeta}_l\lrcorner\,\overline{\zeta}_K^\star),\qquad 1\le l\le n,\\
\gamma\wedge u&=\mathrm{i}(-1)^p\sum_{J,K,m}\gamma_m u_{J,K}\,\zeta_m^\star\wedge\zeta_J^\star\wedge\overline{\zeta}_m^\star\wedge\overline{\zeta}_K^\star,\qquad 1\le m\le n,\\
[\gamma,\Lambda]u&=\sum_{J,K,l,m}\gamma_m\,u_{J,K}\Big(\big(\zeta_l^\star\wedge(\zeta_m\lrcorner\,\zeta_J^\star)\big)\wedge\big(\overline{\zeta}_l^\star\wedge(\overline{\zeta}_m\lrcorner\,\overline{\zeta}_K^\star)\big)\\
&\qquad\qquad-\big(\zeta_m\lrcorner\,(\zeta_l^\star\wedge\zeta_J^\star)\big)\wedge\big(\overline{\zeta}_m\lrcorner\,(\overline{\zeta}_l^\star\wedge\overline{\zeta}_K^\star)\big)\Big)\\
&=\sum_{J,K,m}\gamma_m\,u_{J,K}\Big(\zeta_m^\star\wedge(\zeta_m\lrcorner\,\zeta_J^\star)\wedge\overline{\zeta}_K^\star\\
&\qquad\qquad+\zeta_J^\star\wedge\overline{\zeta}_m^\star\wedge(\overline{\zeta}_m\lrcorner\,\overline{\zeta}_K^\star)-\zeta_J^\star\wedge\overline{\zeta}_K^\star\Big)\\
&=\sum_{J,K}\Big(\sum_{m\in J}\gamma_m+\sum_{m\in K}\gamma_m-\sum_{1\le m\le n}\gamma_m\Big)u_{J,K}\,\zeta_J^\star\wedge\overline{\zeta}_K^\star.
\end{aligned}$$ □

(5.9) Corollary. *For every $u\in\Lambda^{p,q}T_X^\star$, we have*

$$[L,\Lambda]u=(p+q-n)u.$$

Proof. Indeed, if $\gamma=\omega$, we have $\gamma_1=\cdots=\gamma_n=1$. □

This result can be generalized as follows: for every $u\in\Lambda^k(\mathbb{C}\otimes T_X)^\star$, we have

$$(5.10)\qquad [L^r,\Lambda]u=r(k-n+r-1)\,L^{r-1}u.$$

In fact, it is clear that

$$\begin{aligned}
[L^r,\Lambda]u&=\sum_{0\le m\le r-1}L^{r-1-m}[L,\Lambda]L^m u\\
&=\sum_{0\le m\le r-1}(2m+k-n)L^{r-1-m}L^m u=\big(r(r-1)+r(k-n)\big)L^{r-1}u.
\end{aligned}$$

§5.3. Primitive Elements and Hard Lefschetz Theorem

In this subsection, we prove a fundamental decomposition theorem for the representation of the unitary group $U(T_X) \simeq U(n)$ acting on the spaces $\Lambda^{p,q}T_X^\star$ of (p,q)-forms. It turns out that the representation is never irreducible if $0 < p, q < n$.

(5.11) Definition. *A homogeneous element $u \in \Lambda^k(\mathbb{C}\otimes T_X)^\star$ is called primitive if $\Lambda u = 0$. The space of primitive elements of total degree k will be denoted*

$$\mathrm{Prim}^k T_X^\star = \bigoplus_{p+q=k} \mathrm{Prim}^{p,q} T_X^\star.$$

Let $u \in \mathrm{Prim}^k T_X^\star$. Then

$$\Lambda^s L^r u = \Lambda^{s-1}(\Lambda L^r - L^r \Lambda)u = r(n-k-r+1)\Lambda^{s-1}L^{r-1}u.$$

By induction, we get for $r \geq s$

$$\Lambda^s L^r u = r(r-1)\cdots(r-s+1)\cdot(n-k-r+1)\cdots(n-k-r+s)L^{r-s}u. \tag{5.12}$$

Apply (5.12) for $r = n+1$. Then $L^{n+1}u$ is of degree $> 2n$ and therefore we have $L^{n+1}u = 0$. This gives

$$(n+1)\cdots\big(n+1-(s-1)\big)\cdot(-k)(-k+1)\cdots(-k+s-1)L^{n+1-s}u = 0.$$

The integral coefficient is $\neq 0$ if $s \leq k$, hence:

(5.13) Corollary. *If $u \in \mathrm{Prim}^k T_X^\star$, then $L^s u = 0$ for $s \geq (n+1-k)_+$.*

(5.14) Corollary. $\mathrm{Prim}^k T_X^\star = 0$ *for $n+1 \leq k \leq 2n$.*

Proof. Apply Corollary 5.13 with $s = 0$. □

(5.15) Primitive decomposition formula. *For every $u \in \Lambda^k(\mathbb{C}\otimes T_X)^\star$, there is a unique decomposition*

$$u = \sum_{r \geq (k-n)_+} L^r u_r, \qquad u_r \in \mathrm{Prim}^{k-2r} T_X^\star.$$

Furthermore $u_r = \Phi_{k,r}(L,\Lambda)u$ where $\Phi_{k,r}$ is a non commutative polynomial in L, Λ with rational coefficients. As a consequence, there are direct sum decompositions of $U(n)$-representations

$$\Lambda^k(\mathbb{C}\otimes T_X)^\star = \bigoplus_{r\geq(k-n)_+} L^r \operatorname{Prim}^{k-2r} T_X^\star,$$
$$\Lambda^{p,q}T_X^\star = \bigoplus_{r\geq(p+q-n)_+} L^r \operatorname{Prim}^{p-r,q-r} T_X^\star.$$

Proof of the uniqueness of the decomposition Assume that $u = 0$ and that $u_r \neq 0$ for some r. Let s be the largest integer such that $u_s \neq 0$. Then

$$\Lambda^s u = 0 = \sum_{(k-n)_+\leq r\leq s} \Lambda^s L^r u_r = \sum_{(k-n)_+\leq r\leq s} \Lambda^{s-r}\Lambda^r L^r u_r.$$

But formula (5.12) shows that $\Lambda^r L^r u_r = c_{k,r}u_r$ for some non zero integral coefficient $c_{k,r} = r!(n-k+r+1)\cdots(n-k+2r)$. Since u_r is primitive we get $\Lambda^s L^r u_r = 0$ when $r < s$, hence $u_s = 0$, a contradiction.

Proof of the existence of the decomposition We prove by induction on $s \geq (k-n)_+$ that $\Lambda^s u = 0$ implies

$$u = \sum_{(k-n)_+\leq r<s} L^r u_r, \qquad u_r = \Phi_{k,r,s}(L,\Lambda)u \in \operatorname{Prim}^{k-2r} T_X^\star. \tag{5.16}$$

The Theorem will follow from the step $s = n+1$.

Assume that the result is true for s and that $\Lambda^{s+1}u = 0$. Then $\Lambda^s u$ is in $\operatorname{Prim}^{k-2s} T_X^\star$. Since $s \geq (k-n)_+$ we have $c_{k,s} \neq 0$ and we set

$$u_s = \frac{1}{c_{k,s}}\Lambda^s u \in \operatorname{Prim}^{k-2s} T_X^\star,$$
$$u' = u - L^s u_s = \Big(1 - \frac{1}{c_{k,s}}L^s\Lambda^s\Big)u.$$

By formula (5.12), we get

$$\Lambda^s u' = \Lambda^s u - \Lambda^s L^s u_s = \Lambda^s u - c_{k,s}u_s = 0.$$

The induction hypothesis implies

$$u' = \sum_{(k-n)_+\leq r<s} L^r u'_r,\ \ u'_r = \Phi_{k,r,s}(L,\Lambda)u' \in \operatorname{Prim}^{k-2r} T_X^\star,$$

hence $u = \sum_{(k-n)_+\leq r\leq s} L^r u_r$ with

$$\begin{cases} u_r = u'_r = \Phi_{k,r,s}(L,\Lambda)\Big(1 - \frac{1}{c_{k,s}}L^s\Lambda^s\Big)u,\ \ r < s,\\ u_s = \frac{1}{c_{k,s}}\Lambda^s u.\end{cases}$$

It remains to prove the validity of the decomposition 5.16) for the initial step $s = (k-n)_+$, i.e. that $\Lambda^s u = 0$ implies $u = 0$. If $k \leq n$, then $s = 0$ and there is nothing to prove. We are left with the case $k > n$, $\Lambda^{k-n}u = 0$. Then $v = \star u \in \Lambda^{2n-k}(\mathbb{C} \otimes T_X)^\star$ and $2n - k < n$. Since the decomposition exists in degree $\le n$ by what we have just proved, we get

$$v = \star u = \sum_{r\ge 0} L^r v_r, \;\; v_r \in \mathrm{Prim}^{2n-k-2r}\, T_X^\star,$$
$$0 = \star \Lambda^{k-n} u = L^{k-n} \star u = \sum_{r\ge 0} L^{r+k-n} v_r,$$

with degree $(L^{r+k-n}v_r) = 2n - k + 2(k-n) = k$. The uniqueness part shows that $v_r = 0$ for all r, hence $u = 0$. The Theorem is proved. □

(5.17) Corollary. *The linear operators*

$$L^{n-k} : \Lambda^k(\mathbb{C} \otimes T_X)^\star \longrightarrow \Lambda^{2n-k}(\mathbb{C} \otimes T_X)^\star,$$
$$L^{n-p-q} : \Lambda^{p,q}T_X^\star \longrightarrow \Lambda^{n-q,n-p}T_X^\star,$$

are isomorphisms for all integers $k \le n$, $p + q \le n$.

Proof. For every $u \in \Lambda^k_{\mathbb{C}} T_X^\star$, the primitive decomposition $u = \sum_{r\ge 0} L^r u_r$ is mapped bijectively onto that of $L^{n-k}u$:

$$L^{n-k}u = \sum_{r\ge 0} L^{r+n-k} u_r. \qquad \square$$

§6. Commutation Relations

§6.1. Commutation Relations on a Kähler Manifold

Assume first that $X = \Omega \subset \mathbb{C}^n$ is an open subset and that ω is the standard Kähler metric

$$\omega = \mathrm{i} \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j.$$

For any form $u \in C^\infty(\Omega, \Lambda^{p,q}T_X^\star)$ we have

$$(6.1') \qquad d'u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$(6.1'') \qquad d''u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

Since the global L^2 inner product is given by

$$\langle\!\langle u, v\rangle\!\rangle = \int_\Omega \sum_{I,J} u_{I,J}\overline{v}_{I,J}\, dV,$$

easy computations analogous to those of Example 3.12 show that

$$(6.2') \qquad d'^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k}\frac{\partial}{\partial z_k} \lrcorner\ (dz_I \wedge d\overline{z}_J),$$

$$(6.2'') \qquad d''^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k}\frac{\partial}{\partial \overline{z}_k} \lrcorner\ (dz_I \wedge d\overline{z}_J).$$

We first prove a lemma due to (Akizuki and Nakano 1954).

(6.3) Lemma. *In* $\mathbb{C}^n$*, we have* $[d''^\star, L] = \mathrm{i}d'$.

Proof. Formula (6.2″) can be written more briefly

$$d''^\star u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial u}{\partial z_k}\Big).$$

Then we get

$$[d''^\star, L]u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial}{\partial z_k}(\omega \wedge u)\Big) + \omega \wedge \sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial u}{\partial z_k}\Big).$$

Since ω has constant coefficients, we have $\dfrac{\partial}{\partial z_k}(\omega\wedge u) = \omega\wedge\dfrac{\partial u}{\partial z_k}$ and therefore

$$\begin{aligned}[d''^\star, L]\, u &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\omega \wedge \frac{\partial u}{\partial z_k}\Big) - \omega \wedge \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \frac{\partial u}{\partial z_k}\Big)\Big)\\ &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \omega\Big) \wedge \frac{\partial u}{\partial z_k}.\end{aligned}$$

Clearly $\dfrac{\partial}{\partial \overline{z}_k} \lrcorner\ \omega = -\mathrm{i}dz_k$, so

$$[d''^\star, L]\, u = \mathrm{i}\sum_k dz_k \wedge \frac{\partial u}{\partial z_k} = \mathrm{i}d'u. \qquad \square$$

We are now ready to derive the basic commutation relations in the case of an arbitrary Kähler manifold (X, ω).

(6.4) Theorem. *If* (X, ω) *is Kähler, then*

$$\begin{aligned}&[d''^\star, L] = \mathrm{i}d', && [d'^\star, L] = -\mathrm{i}d'',\\ &[\Lambda, d''] = -\mathrm{i}d'^\star, && [\Lambda, d'] = \mathrm{i}d''^\star.\end{aligned}$$

Proof. It is sufficient to verify the first relation, because the second one is the conjugate of the first, and the relations of the second line are the adjoint of those of the first line. If (z_j) is a geodesic coordinate system at a point $x_0 \in X$, then for any (p,q)-forms u, v with compact support in a neighborhood of x_0, (4.9) implies

$$\langle\langle u, v\rangle\rangle = \int_M \Big(\sum_{I,J} u_{IJ}\overline{v}_{IJ} + \sum_{I,J,K,L} a_{IJKL}\, u_{IJ}\overline{v}_{KL} \Big)\, dV,$$

with $a_{IJKL}(z) = O(|z|^2)$ at x_0. An integration by parts as in (3.12) and (6.2$''$) yields

$$d''^{\star} u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner\, (dz_I \wedge d\overline{z}_J) + \sum_{I,J,K,L} b_{IJKL}\, u_{IJ}\, dz_K \wedge d\overline{z}_L,$$

where the coefficients b_{IJKL} are obtained by derivation of the a_{IJKL}'s. Therefore $b_{IJKL} = O(|z|)$. Since $\partial\omega/\partial z_k = O(|z|)$, the proof of Lemma 6.3 implies here $[d''^{\star}, L]u = \mathrm{i}d'u + O(|z|)$, in particular both terms coincide at every given point $x_0 \in X$. □

(6.5) Corollary. *If (X, ω) is Kähler, the complex Laplace-Beltrami operators satisfy*

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

Proof. It will be first shown that $\Delta'' = \Delta'$. We have

$$\Delta'' = [d'', d''^{\star}] = -\mathrm{i}\big[d'', [\Lambda, d']\big].$$

Since $[d', d''] = 0$, Jacobi's identity (5.7) implies

$$-\big[d'', [\Lambda, d']\big] + \big[d', [d'', \Lambda]\big] = 0,$$

hence $\Delta'' = \big[d', -\mathrm{i}[d'', \Lambda]\big] = [d', d'^{\star}] = \Delta'$. On the other hand

$$\Delta = [d' + d'', d'^{\star} + d''^{\star}] = \Delta' + \Delta'' + [d', d''^{\star}] + [d'', d'^{\star}].$$

Thus, it is enough to prove:

(6.6) Lemma. $[d', d''^{\star}] = 0,\ [d'', d'^{\star}] = 0.$

Proof. We have $[d', d''^{\star}] = -\mathrm{i}\big[d', [\Lambda, d']\big]$ and (5.7) implies

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^{\star}] = 0$. The second relation $[d'', d'^{\star}] = 0$ is the adjoint of the first. □

(6.7) Theorem. *Δ commutes with all operators $\star, d', d'', d'^\star, d''^\star, L, \Lambda$.*

Proof. The identities $[d', \Delta'] = [d'^\star, \Delta'] = 0$, $[d'', \Delta''] = [d''^\star, \Delta''] = 0$ and $[\Delta, \star] = 0$ are immediate. Furthermore, the equality $[d', L] = d'\omega = 0$ together with the Jacobi identity implies

$$[L, \Delta'] = \big[L, [d', d'^\star]\big] = -\big[d', [d'^\star, L]\big] = \mathrm{i}[d', d''] = 0.$$

By adjunction, we also get $[\Delta', \Lambda] = 0$. □

§6.2 Commutation Relations on Hermitian Manifolds

We are going to extend the commutation relations of § 6.1 to an arbitrary hermitian manifold (X, ω). In that case ω is no longer tangent to a constant metric, and the commutation relations involve extra terms arising from the *torsion* of ω. Theorem 6.8 below is taken from (Demailly 1984), but the idea was already contained in (Griffiths 1966).

(6.8) Theorem. *Let τ be the operator of type $(1,0)$ and order 0 defined by $\tau = [\Lambda, d'\omega]$. Then*

a) $[d''^\star, L] = \mathrm{i}(d' + \tau)$,
b) $[d'^\star, L] = -\mathrm{i}(d'' + \overline{\tau})$,
c) $[\Lambda, d''] = -\mathrm{i}(d'^\star + \tau^\star)$,
d) $[\Lambda, d'] = \mathrm{i}(d''^\star + \overline{\tau}^\star)$;

$d'\omega$ will be called the torsion form of ω, and τ the torsion operator.

Proof. b) follows from a) by conjugation, whereas c), d) follow from a), b) by adjunction. It is therefore enough to prove relation a).

Let $(z_j)_{1\le j\le n}$ be complex coordinates centered at a point $x_0 \in X$, such that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $T_{x_0}X$ for the metric $\omega(x_0)$. Consider the metric with constant coefficients

$$\omega_0 = \mathrm{i} \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j.$$

The metric ω can then be written

$$\omega = \omega_0 + \gamma \quad \text{with} \quad \gamma = O(|z|).$$

Denote by $\langle\ ,\ \rangle_0$, L_0, Λ_0, $d_0'^\star$, $d_0''^\star$ the inner product and the operators associated to the constant metric ω_0, and let $dV_0 = \omega_0^n/2^n n!$. The proof of relation a) is based on a Taylor expansion of L, Λ, $d'^\star$, $d''^\star$ in terms of the operators with constant coefficients L_0, Λ_0, $d_0'^\star$, $d_0''^\star$.

(6.9) Lemma. *Let $u, v \in C^\infty(X, \Lambda^{p,q}T_X^\star)$. Then in a neighborhood of x_0*

$$\langle u,v\rangle\, dV = \langle u-[\gamma,\Lambda_0]u,v\rangle_0\, dV_0 + O(|z|^2).$$

Proof. In a neighborhood of x_0, let

$$\gamma = \mathrm{i}\sum_{1\le j\le n}\gamma_j\,\zeta_j^\star\wedge\overline{\zeta}_j^\star,\qquad \gamma_1\le\gamma_2\le\cdots\le\gamma_n,$$

be a diagonalization of the (1,1)-form $\gamma(z)$ with respect to an orthonormal basis $(\zeta_j)_{1\le j\le n}$ of T_zX for $\omega_0(z)$. We thus have

$$\omega=\omega_0+\gamma=\mathrm{i}\sum\lambda_j\,\zeta_j^\star\wedge\overline{\zeta}_j^\star$$

with $\lambda_j=1+\gamma_j$ and $\gamma_j=O(|z|)$. Set now

$$J=\{j_1,\ldots,j_p\},\qquad \zeta_J^\star=\zeta_{j_1}^\star\wedge\cdots\wedge\zeta_{j_p}^\star,\qquad \lambda_J=\lambda_{j_1}\cdots\lambda_{j_p},$$

$$u=\sum u_{J,K}\,\zeta_J^\star\wedge\overline{\zeta}_K^\star,\qquad v=\sum v_{J,K}\,\zeta_J^\star\wedge\overline{\zeta}_K^\star$$

where summations are extended to increasing multi-indices J, K such that $|J|=p$, $|K|=q$. With respect to ω we have $\langle\zeta_j^\star,\zeta_j^\star\rangle=\lambda_j^{-1}$, hence

$$\begin{aligned}\langle u,v\rangle\, dV &= \sum_{J,K}\lambda_J^{-1}\lambda_K^{-1}\,u_{J,K}\overline{v}_{J,K}\,\lambda_1\cdots\lambda_n\,dV_0\\ &=\sum_{J,K}\Big(1-\sum_{j\in J}\gamma_j-\sum_{j\in K}\gamma_j+\sum_{1\le j\le n}\gamma_j\Big)u_{J,K}\overline{v}_{J,K}\,dV_0+O(|z|^2).\end{aligned}$$

Lemma 6.9 follows if we take Prop. 5.8 into account. □

(6.10) Lemma. $d''^\star=d_0''^\star+\big[\Lambda_0,[d_0''^\star,\gamma]\big]$ *at point* x_0*, i.e. at this point both operators have the same formal expansion.*

Proof. Since $d''^\star$ is an operator of order 1, Lemma 6.9 shows that $d''^\star$ coincides at point x_0 with the formal adjoint of d'' for the metric

$$\langle\!\langle u,v\rangle\!\rangle_1=\int_X\langle u-[\gamma,\Lambda_0]u,v\rangle_0\,dV_0.$$

For any compactly supported $u\in C^\infty(X,\Lambda^{p,q}T_X^\star)$, $v\in C^\infty(X,\Lambda^{p,q-1}T_X^\star)$ we get by definition

$$\langle\!\langle u,d''v\rangle\!\rangle_1=\int_X\langle u-[\gamma,\Lambda_0]u,d''v\rangle_0\,dV_0=\int_X\langle d_0''^\star u-d_0''^\star[\gamma,\Lambda_0]u,v\rangle_0\,dV_0.$$

Since ω and ω_0 coincide at point x_0 and since $\gamma(x_0)=0$ we obtain at this point

$$\begin{aligned}&d''^\star u=d_0''^\star u-d_0''^\star[\gamma,\Lambda_0]u=d_0''^\star u-\big[d_0''^\star,[\gamma,\Lambda_0]\big]u\ ;\\ &d''^\star=d_0''^\star-\big[d_0''^\star,[\gamma,\Lambda_0]\big].\end{aligned}$$

We have $[\Lambda_0, d_0''^\star] = [d'', L_0]^\star = 0$ since $d''\omega_0 = 0$. The Jacobi identity (5.7) implies

$$\big[d_0''^\star, [\gamma, \Lambda_0]\big] + \big[\Lambda_0, [d_0''^\star, \gamma]\big] = 0,$$

and Lemma 6.10 follows. □

Proof Proof of formula 6.8 a) The equality $L = L_0 + \gamma$ and Lemma 6.10 yield

$$(6.11)\quad [L, d''^\star] = [L_0, d_0''^\star] + \Big[L_0, \big[\Lambda_0, [d_0''^\star, \gamma]\big]\Big] + [\gamma, d_0''^\star]$$

at point x_0, because the triple bracket involving γ twice vanishes at x_0. From the Jacobi identity applied to $C = [d_0''^\star, \gamma]$, we get

$$(6.12)\quad \begin{cases} \big[L_0, [\Lambda_0, C]\big] = -[\Lambda_0, [C, L_0]\big] - \big[C, [L_0, \Lambda_0]\big], \\ [C, L_0] = \big[L_0, [d_0''^\star, \gamma]\big] = \big[\gamma, [L_0, d_0''^\star]\big] \quad (\text{since } \ [\gamma, L_0] = 0). \end{cases}$$

Lemma 6.3 yields $[L_0, d_0''^\star] = -\mathrm{i}d'$, hence

$$(6.13)\quad [C, L_0] = -[\gamma, \mathrm{i}d'] = \mathrm{i}d'\gamma = \mathrm{i}d'\omega.$$

On the other hand, C is of type $(1,0)$ and Cor. 5.9 gives

$$(6.14)\quad \big[C, [L_0, \Lambda_0]\big] = -C = -[d_0''^\star, \gamma].$$

From (6.12), (6.13), (6.14) we get

$$\Big[L_0, \big[\Lambda_0, [d_0''^\star, \gamma]\big]\Big] = -[\Lambda_0, \mathrm{i}d'\omega] + [d_0''^\star, \gamma].$$

This last equality combined with (6.11) implies

$$[L, d''^\star] = [L_0, d_0''^\star] - [\Lambda_0, \mathrm{i}d'\omega] = -\mathrm{i}(d' + \tau)$$

at point x_0. Formula 6.8 a) is proved. □

(6.15) Corollary. *The complex Laplace-Beltrami operators satisfy*

$$\begin{gathered} \Delta'' = \Delta' + [d', \tau^\star] - [d'', \overline{\tau}^\star], \\ [d', d''^\star] = -[d', \overline{\tau}^\star], \qquad [d'', d'^\star] = -[d'', \tau^\star], \\ \Delta = \Delta' + \Delta'' - [d', \overline{\tau}^\star] - [d'', \tau^\star]. \end{gathered}$$

Therefore Δ', Δ'' and $\frac{1}{2}\Delta$ no longer coincide, but they differ by linear differential operators of order 1 only.

Proof. As in the Kähler case (Cor. 6.5 and Lemma 6.6), we find

$$\begin{aligned} \Delta'' &= [d'', d''^\star] = \big[d'', -i[\Lambda, d'] - \overline{\tau}^\star\big] \\ &= \big[d', -\mathrm{i}[d'', \Lambda]\big] - \big[d'', \overline{\tau}^\star\big] = \Delta' + [d', \tau^\star] - [d'', \overline{\tau}^\star], \end{aligned}$$
$$[d', d''^\star + \overline{\tau}^\star] = -\mathrm{i}\big[d', [\Lambda, d']\big] = 0,$$

and the first two lines are proved. The third one is an immediate consequence of the second. □

§7. Groups $\mathcal{H}^{p,q}(X, E)$ and Serre Duality

Let (X, ω) be a *compact hermitian* manifold and E a holomorphic hermitian vector bundle of rank r over X. We denote by D_E the Chern connection of E, by $D_E^\star = - \star D_E \star$ the formal adjoint of D_E, and by $D_E'^\star$, $D_E''^\star$ the components of $D_E^\star$ of type $(-1, 0)$ and $(0, -1)$.

Corollary 6.8 implies that the principal part of the operator $\Delta_E'' = D'' D_E''^\star + D_E''^\star D''$ is one half that of Δ_E. Consequently, the operator Δ_E'' acting on each space $C^\infty(X, \Lambda^{p,q} T_X^\star \otimes E)$ is a self-adjoint elliptic operator. Since $D''^2 = 0$, the following results can be obtained in a way similar to those of § 3.3.

(7.1) Theorem. *For every bidegree (p, q), there exists an orthogonal decomposition*

$$C^\infty(X, \Lambda^{p,q} T_X^\star \otimes E) = \mathcal{H}^{p,q}(X, E) \oplus \mathrm{Im}\, D_E'' \oplus \mathrm{Im}\, D_E''^\star$$

where $\mathcal{H}^{p,q}(X, E)$ is the space of Δ_E''-harmonic forms in $C^\infty(X, \Lambda^{p,q} T_X^\star \otimes E)$.

The above decomposition shows that the subspace of d''-cocycles in $C^\infty(X, \Lambda^{p,q} T_X^\star \otimes E)$ is $\mathcal{H}^{p,q}(X, E) \oplus \mathrm{Im}\, D_E''$. From this, we infer

(7.2) Hodge isomorphism theorem. *The Dolbeault cohomology group $H^{p,q}(X, E)$ is finite dimensional, and there is an isomorphism*

$$H^{p,q}(X, E) \simeq \mathcal{H}^{p,q}(X, E).$$

(7.3) Serre duality theorem. *The bilinear pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^\star) \longrightarrow \mathbb{C}, \qquad (s, t) \longmapsto \int_M s \wedge t$$

is a non degenerate duality.

Proof. Let $s_1 \in C^\infty(X, \Lambda^{p,q} T_X^\star \otimes E)$, $s_2 \in C^\infty(X, \Lambda^{n-p,n-q-1} T_X^\star \otimes E)$. Since $s_1 \wedge s_2$ is of bidegree $(n, n-1)$, we have

$$(7.4) \quad d(s_1 \wedge s_2) = d''(s_1 \wedge s_2) = d'' s_1 \wedge s_2 + (-1)^{p+q} s_1 \wedge d'' s_2.$$

Stokes' formula implies that the above bilinear pairing can be factorized through Dolbeault cohomology groups. The # operator defined in § 3.1 is such that

$$\# : C^\infty(X, \Lambda^{p,q}T_X^\star \otimes E) \longrightarrow C^\infty(X, \Lambda^{n-p,n-q}T_X^\star \otimes E^\star).$$

Furthermore, (3.20) implies

$$d''(\# s) = (-1)^{\deg s} \# D_E''^\star s, \qquad D_{E^\star}''^\star(\# s) = (-1)^{\deg s+1} \# D_E''^\star s,$$

$$\Delta''_{E^\star}(\# s) = \# \Delta''_E s,$$

where $D_{E^\star}$ is the Chern connection of $E^\star$. Consequently, $s \in \mathcal{H}^{p,q}(X,E)$ if and only if $\# s \in \mathcal{H}^{n-p,n-q}(X,E^\star)$. Theorem 7.3 is then a consequence of the fact that the integral $\|s\|^2 = \int_X s \wedge \# s$ does not vanish unless $s = 0$. □

§8. Cohomology of Compact Kähler Manifolds

§8.1. Bott-Chern Cohomology Groups

Let X be for the moment an arbitrary complex manifold. The following "cohomology" groups are helpful to describe Hodge theory on compact complex manifolds which are not necessarily Kähler.

(8.1) Definition. *We define the Bott-Chern cohomology groups of X to be*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) = \big(C^\infty(X, \Lambda^{p,q}T_X^\star) \cap \ker d\big)/d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^\star).$$

Then $H^{\bullet,\bullet}_{\mathrm{BC}}(X,\mathbb{C})$ has the structure of a bigraded algebra, which we call the Bott-Chern cohomology algebra of X.

As the group $d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^\star)$ is contained in the coboundary groups $d''C^\infty(X, \Lambda^{p,q-1}T_X^\star)$ or $dC^\infty(X, \Lambda^{p+q-1}(\mathbb{C}\otimes T_X)^\star)$, there are canonical morphisms

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}), \tag{8.2}$$

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p+q}_{\mathrm{DR}}(X,\mathbb{C}), \tag{8.3}$$

of the Bott-Chern cohomology to the Dolbeault or De Rham cohomology. These morphisms are homomorphisms of $\mathbb{C}$-algebras. It is also clear from the definition that we have the symmetry property $H^{q,p}_{\mathrm{BC}}(X,\mathbb{C}) = \overline{H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})}$. It can be shown from the Hodge-Frölicher spectral sequence (see § 11 and Exercise 13.??) that $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ is always finite dimensional if X is compact.

§8.2. Hodge Decomposition Theorem

We suppose from now on that (X,ω) is a *compact Kähler* manifold. The equality $\Delta = 2\Delta''$ shows that Δ is homogeneous with respect to bidegree and that there is an orthogonal decomposition

$$(8.4) \quad \mathcal{H}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,\mathbb{C}).$$

As $\overline{\Delta''} = \Delta' = \Delta''$, we also have $\mathcal{H}^{q,p}(X,\mathbb{C}) = \overline{\mathcal{H}^{p,q}(X,\mathbb{C})}$. Using the Hodge isomorphism theorems for the De Rham and Dolbeault cohomology, we get:

(8.5) Hodge decomposition theorem. *On a compact Kähler manifold, there are canonical isomorphisms*

$$\begin{aligned} &H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}) &&(\textit{Hodge decomposition}),\\ &H^{q,p}(X,\mathbb{C}) \simeq \overline{H^{p,q}(X,\mathbb{C})} &&(\textit{Hodge symmetry}).\end{aligned}$$

The only point which is not a priori completely clear is that this decomposition is independent of the Kähler metric. In order to show that this is the case, one can use the following Lemma, which allows us to compare all three types of cohomology groups considered in § 8.1.

(8.6) Lemma. *Let u be a d-closed (p,q)-form. The following properties are equivalent:*

a) *u is d-exact;*

b$'$) *u is d'-exact;*

b$''$) *u is d''-exact;*

c) *u is $d'd''$-exact, i.e. u can be written $u = d'd''v$.*

d) *u is orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$.*

Proof. It is obvious that c) implies a), b$'$), b$''$) and that a) or b$'$) or b$''$) implies d). It is thus sufficient to prove that d) implies c). As $du = 0$, we have $d'u = d''u = 0$, and as u is supposed to be orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$, Th. 7.1 implies $u = d''s$, $s \in C^\infty(X, \Lambda^{p,q-1}T_X^\star)$. By the analogue of Th. 7.1 for d', we have $s = h + d'v + d'^\star w$, with $h \in \mathcal{H}^{p,q-1}(X,\mathbb{C})$, $v \in C^\infty(X, \Lambda^{p-1,q-1}T_X^\star)$ and $w \in C^\infty(X, \Lambda^{p+1,q-1}T_X^\star)$. Therefore

$$u = d''d'v + d''d'^\star w = -d'd''v - d'^\star d''w$$

in view of Lemma 6.6. As $d'u = 0$, the component $d'^\star d''w$ orthogonal to $\ker d'$ must be zero. □

From Lemma 8.6 we infer the following Corollary, which in turn implies that the Hodge decomposition does not depend on the Kähler metric.

(8.7) Corollary. *Let X be a compact Kähler manifold. Then the natural morphisms*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}), \qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{DR}}(X,\mathbb{C})$$

are isomorphisms.

Proof. The surjectivity of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C})$ comes from the fact that every class in $H^{p,q}(X,\mathbb{C})$ can be represented by a harmonic (p,q)-form, thus by a d-closed (p,q)-form; the injectivity means nothing more than the equivalence (8.5 b$''$) $\Leftrightarrow$ (8.5 c). Hence $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C}) \simeq \mathcal{H}^{p,q}(X,\mathbb{C})$, and the isomorphism $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{DR}}(X,\mathbb{C})$ follows from (8.4). $\square$

Let us quote now two simple applications of Hodge theory. The first of these is a computation of the Dolbeault cohomology groups of $\mathbb{P}^n$. As $H^{2p}_{\mathrm{DR}}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C}$ and $H^{p,p}(\mathbb{P}^n,\mathbb{C}) \ni \{\omega^p\} \neq 0$, the Hodge decomposition formula implies:

(8.8) Application. *The Dolbeault cohomology groups of $\mathbb{P}^n$ are*

$$H^{p,p}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C} \quad \textit{for } 0 \le p \le n, \qquad H^{p,q}(\mathbb{P}^n,\mathbb{C}) = 0 \quad \textit{for } p \neq q. \qquad \square$$

(8.9) Proposition. *Every holomorphic p-form on a compact Kähler manifold X is d-closed.*

Proof. If u is a holomorphic form of type $(p,0)$ then $d''u = 0$. Furthermore $d''^{\star}u$ is of type $(p,-1)$, hence $d''^{\star}u = 0$. Therefore $\Delta u = 2\Delta''u = 0$, which implies $du = 0$. $\square$

(8.10) Example. Consider the *Heisenberg group* $G \subset \mathrm{Gl}_3(\mathbb{C})$, defined as the subgroup of matrices

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad (x,y,z) \in \mathbb{C}^3.$$

Let Γ be the discrete subgroup of matrices with entries $x,y,z \in \mathbb{Z}[\mathrm{i}]$ (or more generally in the ring of integers of an imaginary quadratic field). Then $X = G/\Gamma$ is a compact complex 3-fold, known as the *Iwasawa manifold.* The equality

$$M^{-1}dM = \begin{pmatrix} 0 & dx & dz - xdy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

shows that dx, dy, $dz - xdy$ are left invariant holomorphic 1-forms on G. These forms induce holomorphic 1-forms on the quotient $X = G/\Gamma$. Since $dz - xdy$ is not d-closed, we see that X cannot be Kähler.

§8.3. Primitive Decomposition and Hard Lefschetz Theorem

We first introduce some standard notation. The *Betti numbers* and *Hodge numbers* of X are by definition

$$(8.11)\qquad b_k = \dim_{\mathbb{C}} H^k(X, \mathbb{C}), \qquad h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X, \mathbb{C}).$$

Thanks to Hodge decomposition, these numbers satisfy the relations

$$(8.12)\qquad b_k = \sum_{p+q=k} h^{p,q}, \qquad h^{q,p} = h^{p,q}.$$

As a consequence, the Betti numbers b_{2k+1} of a compact Kähler manifold are even. Note that the Serre duality theorem gives the additional relation $h^{p,q} = h^{n-p,n-q}$, which holds as soon as X is compact. The existence of primitive decomposition implies other interesting specific features of the cohomology algebra of compact Kähler manifolds.

(8.13) Lemma. *If $u = \sum_{r \geq (k-n)_+} L^r u_r$ is the primitive decomposition of a harmonic k-form u, then all components u_r are harmonic.*

Proof. Since $[\Delta, L] = 0$, we get $0 = \Delta u = \sum_r L^r \Delta u_r$, hence $\Delta u_r = 0$ by uniqueness. □

Let us denote by $\operatorname{Prim} \mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \operatorname{Prim} \mathcal{H}^{p,q}(X, \mathbb{C})$ the spaces of primitive harmonic k-forms and let $b_{k,\mathrm{prim}}$, $h^{p,q}_{\mathrm{prim}}$ be their respective dimensions. Lemma 8.13 yields

$$(8.14)\qquad \mathcal{H}^{p,q}(X, \mathbb{C}) = \bigoplus_{r \geq (p+q-n)_+} L^r \operatorname{Prim} \mathcal{H}^{p-r,q-r}(X, \mathbb{C}),$$

$$(8.15)\qquad h^{p,q} = \sum_{r \geq (p+q-n)_+} h^{p-r,q-r}_{\mathrm{prim}}.$$

Formula (8.15) can be rewritten

$$(8.15')\qquad \begin{cases} \text{if } \ p+q \leq n, & h^{p,q} = h^{p,q}_{\mathrm{prim}} + h^{p-1,q-1}_{\mathrm{prim}} + \cdots \\ \text{if } \ p+q \geq n, & h^{p,q} = h^{n-q,n-p}_{\mathrm{prim}} + h^{n-q-1,n-p-1}_{\mathrm{prim}} + \cdots. \end{cases}$$

(8.16) Corollary. *The Hodge and Betti numbers satisfy the inequalities*

a) *if* $k = p + q \le n$, *then* $h^{p,q} \ge h^{p-1,q-1}$, $b_k \ge b_{k-2}$,

b) *if* $k = p + q \ge n$, *then* $h^{p,q} \ge h^{p+1,q+1}$, $b_k \ge b_{k+2}$. □

Another important result of Hodge theory (which is in fact a direct consequence of Cor. 5.17) is the

(8.17) Hard Lefschetz theorem. *The mappings*

$$\begin{array}{lcll} L^{n-k} & : & H^k(X,\mathbb{C}) \longrightarrow H^{2n-k}(X,\mathbb{C}), & k \le n, \\ L^{n-p-q} & : & H^{p,q}(X,\mathbb{C}) \longrightarrow H^{n-q,n-p}(X,\mathbb{C}), & p+q \le n, \end{array}$$

are isomorphisms. □

§9. Jacobian and Albanese Varieties

§9.1. Description of the Picard Group

An important application of Hodge theory is a description of the Picard group $H^1(X, \mathcal{O}^\star)$ of a compact Kähler manifold. We assume here that X is connected. The exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 1$ gives

$$\begin{aligned} (9.1) \qquad 0 \longrightarrow & H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^\star) \\ \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}) \end{aligned}$$

because the map $\exp(2\pi i \bullet) : H^0(X, \mathcal{O}) = \mathbb{C} \longrightarrow H^0(X, \mathcal{O}^\star) = \mathbb{C}^\star$ is onto. We have $H^1(X, \mathcal{O}) \simeq H^{0,1}(X, \mathbb{C})$ by (V-11.6). The dimension of this group is called the *irregularity of* X and is usually denoted

$$(9.2) \qquad q = q(X) = h^{0,1} = h^{1,0}.$$

Therefore we have $b_1 = 2q$ and

$$(9.3) \qquad H^1(X,\mathcal{O}) \simeq \mathbb{C}^q, \qquad H^0(X,\Omega^1_X) = H^{1,0}(X,\mathbb{C}) \simeq \mathbb{C}^q.$$

(9.4) Lemma. *The image of* $H^1(X, \mathbb{Z})$ *in* $H^1(X, \mathcal{O})$ *is a lattice.*

Proof. Consider the morphisms

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathbb{R}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O})$$

induced by the inclusions $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C} \subset \mathcal{O}$. Since the Čech cohomology groups with values in $\mathbb{Z}$, $\mathbb{R}$ can be computed by finite acyclic coverings, we see that $H^1(X, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module and that the image of $H^1(X, \mathbb{Z})$ in $H^1(X, \mathbb{R})$ is a lattice. It is enough to check that the map $H^1(X, \mathbb{R}) \longrightarrow H^1(X, \mathcal{O})$ is an isomorphism. However, the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathbb{C} \longrightarrow & \mathcal{E}^0 & \xrightarrow{d} & \mathcal{E}^1 & \xrightarrow{d} & \mathcal{E}^2 & \longrightarrow \cdots \\
\downarrow \quad & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow \mathcal{O} \longrightarrow & \mathcal{E}^{0,0} & \xrightarrow{d''} & \mathcal{E}^{0,1} & \xrightarrow{d''} & \mathcal{E}^{0,2} & \longrightarrow \cdots
\end{array}$$

shows that the map $H^1(X, \mathbb{R}) \longrightarrow H^1(X, \mathcal{O})$ corresponds in De Rham and Dolbeault cohomologies to the composite mapping

$$H^1_{DR}(X, \mathbb{R}) \subset H^1_{DR}(X, \mathbb{C}) \longrightarrow H^{0,1}(X, \mathbb{C}).$$

Since $H^{1,0}(X, \mathbb{C})$ and $H^{0,1}(X, \mathbb{C})$ are complex conjugate subspaces in $H^1_{DR}(X, \mathbb{C})$, we conclude that $H^1_{DR}(X, \mathbb{R}) \longrightarrow H^{0,1}(X, \mathbb{C})$ is an isomorphism. □

As a consequence of this lemma, $H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2q}$. The q-dimensional complex torus

$$\text{(9.5)} \quad \mathrm{Jac}(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$$

is called the *Jacobian variety of* X and is isomorphic to the subgroup of $H^1(X, \mathcal{O}^\star)$ corresponding to line bundles of zero first Chern class. On the other hand, the kernel of

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}) = H^{0,2}(X, \mathbb{C})$$

which consists of integral cohomology classes of type $(1, 1)$, is equal to the image of c_1 in $H^2(X, \mathbb{Z})$. This subgroup is called the *Neron-Severi group* of X, and is denoted $NS(X)$. The exact sequence (9.1) yields

$$\text{(9.6)} \quad 0 \longrightarrow \mathrm{Jac}(X) \longrightarrow H^1(X, \mathcal{O}^\star) \xrightarrow{c_1} NS(X) \longrightarrow 0.$$

The Picard group $H^1(X, \mathcal{O}^\star)$ is thus an extension of the complex torus $\mathrm{Jac}(X)$ by the finitely generated $\mathbb{Z}$-module $NS(X)$.

(9.7) Corollary. *The Picard group of* $\mathbb{P}^n$ *is* $H^1(\mathbb{P}^n, \mathcal{O}^\star) \simeq \mathbb{Z}$, *and every line bundle over* $\mathbb{P}^n$ *is isomorphic to one of the line bundles* $\mathcal{O}(k)$, $k \in \mathbb{Z}$.

Proof. We have $H^k(\mathbb{P}^n, \mathcal{O}) = H^{0,k}(\mathbb{P}^n, \mathbb{C}) = 0$ for $k \geq 1$ by Appl. 8.8, thus $\mathrm{Jac}(\mathbb{P}^n) = 0$ and $NS(\mathbb{P}^n) = H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$. Moreover, $c_1\big(\mathcal{O}(1)\big)$ is a generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ in virtue of Th. V-15.10. □

§9.2. Albanese Variety

A proof similar to that of Lemma 9.4 shows that the image of $H^{2n-1}(X, \mathbb{Z})$ in $H^{n-1,n}(X, \mathbb{C})$ via the composite map

$$\text{(9.8)} \quad H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{R}) \to H^{2n-1}(X, \mathbb{C}) \to H^{n-1,n}(X, \mathbb{C})$$

is a lattice. The q-dimensional complex torus

(9.9) $\quad \mathrm{Alb}(X) = H^{n-1,n}(X,\mathbb{C})/\operatorname{Im} H^{2n-1}(X,\mathbb{Z})$

is called the *Albanese variety of* X. We first give a slightly different description of $\mathrm{Alb}(X)$, based on the Serre duality isomorphism

$$H^{n-1,n}(X,\mathbb{C}) \simeq \big(H^{1,0}(X,\mathbb{C})\big)^\star \simeq \big(H^0(X,\Omega^1_X)\big)^\star.$$

(9.10) Lemma. *For any compact oriented differentiable manifold M with $\dim_{\mathbb{R}} M = m$, there is a natural isomorphism*

$$H_1(M,\mathbb{Z}) \to H^{m-1}(M,\mathbb{Z})$$

where $H_1(M,\mathbb{Z})$ is the first homology group of M, that is, the abelianization of $\pi_1(M)$.

Proof. This is a well known consequence of Poincaré duality, see e.g. (Spanier 1966). We will content ourselves with a description of the morphism. Fix a base point $a \in M$. Every homotopy class $[\gamma] \in \pi_1(M,a)$ can be represented by as a composition of closed loops diffeomorphic to S^1. Let γ be such a loop. As every oriented vector bundle over S^1 is trivial, the normal bundle to γ is trivial. Hence $\gamma(S^1)$ has a neighborhood U diffeomorphic to $S^1 \times \mathbb{R}^{m-1}$, and there is a diffeomorphism $\varphi : S^1 \times \mathbb{R}^{m-1} \to U$ with $\varphi_{\restriction S^1\times\{0\}} = \gamma$. Let $\{\delta_0\} \in H^{m-1}_c(\mathbb{R}^{m-1},\mathbb{Z})$ be the fundamental class represented by the Dirac measure $\delta_0 \in \mathcal{D}'_0(\mathbb{R}^{m-1})$ in De Rham cohomology. Then the cartesian product $1 \times \{\delta_0\} \in H^{m-1}_c(S^1\times\mathbb{R}^{m-1},\mathbb{Z})$ is represented by the current $[S^1]\otimes\{\delta_0\} \in \mathcal{D}'_1(S^1\times\mathbb{R}^{m-1})$ and the current of integration over γ is precisely the direct image current

$$I_\gamma := \varphi_\star([S^1]\otimes\delta_0) = (\varphi^{-1})^\star([S^1]\otimes\delta_0).$$

Its cohomology class $\{I_\gamma\} \in H^{m-1}_c(U,\mathbb{R})$ is thus the image of the class $(\varphi^{-1})^\star\big(1\times\{\delta_0\}\big) \in H^{m-1}_c(U,\mathbb{Z})$. Therefore, we have obtained a well defined morphism

$$\pi_1(M,a) \longrightarrow H^{m-1}_c(U,\mathbb{Z}) \longrightarrow H^{m-1}(M,\mathbb{Z}), \qquad [\gamma] \longmapsto (\varphi^{-1})^\star\big(1\times\{\delta_0\}\big)$$

and the image of $[\gamma]$ in $H^{m-1}(M,\mathbb{R})$ is the De Rham cohomology class of the integration current I_γ. □

Thanks to Lemma 9.10, we can reformulate the definition of the Albanese variety as

(9.11) $\quad \mathrm{Alb}(X) = \big(H^0(X,\Omega^1_X)\big)^\star/\operatorname{Im} H_1(X,\mathbb{Z})$

where $H_1(X,\mathbb{Z})$ is mapped to $\big(H^0(X,\Omega^1_X)\big)^\star$ by

$$[\gamma] \longmapsto \widetilde{I}_\gamma = \Big(u \mapsto \int_\gamma u\Big).$$

Observe that the integral only depends on the homotopy class $[\gamma]$ because all holomorphic 1-forms u on X are closed by Prop. 8.9.

We are going to show that there exists a canonical holomorphic map $\alpha : X \to \mathrm{Alb}(X)$. Let a be a base point in X. For any $x \in X$, we select a path ξ from a to x and associate to x the linear form in $\big(H^0(X, \Omega^1_X)\big)^\star$ defined by $\widetilde{I}_\xi$. By construction the class of this linear form mod $\mathrm{Im}\, H_1(X, \mathbb{Z})$ does not depend on ξ, since $\widetilde{I}_{\xi'^{-1}\xi}$ is in the image of $H_1(X, \mathbb{Z})$ for any other path ξ'. It is thus legitimate to define the *Albanese map* as

$$(9.12)\qquad \alpha : X \longrightarrow \mathrm{Alb}(X), \qquad x \longmapsto \Big(u \mapsto \int_a^x u\Big) \ \bmod\ \mathrm{Im}\ H_1(X, \mathbb{Z}).$$

Of course, if we fix a basis $(u_1, \ldots, u_q)$ of $H^0(X, \Omega^1_X)$, the Albanese map can be seen in coordinates as the map

$$(9.13)\qquad \alpha : X \longrightarrow \mathbb{C}^q/\Lambda, \qquad x \longmapsto \Big(\int_a^x u_1, \ldots, \int_a^x u_q\Big) \ \bmod\ \Lambda,$$

where $\Lambda \subset \mathbb{C}^q$ is the *group of periods* of $(u_1, \ldots, u_q)$:

$$(9.13')\qquad \Lambda = \Big\{\Big(\int_\gamma u_1, \ldots, \int_\gamma u_q\Big)\ ;\ [\gamma] \in \pi_1(X, a)\Big\}.$$

It is then clear that α is a holomorphic map. With the original definition (9.9) of the Albanese variety, it is not difficult to see that α is the map given by

$$(9.14)\qquad \alpha : X \longrightarrow \mathrm{Alb}(X), \qquad x \longmapsto \{I_\xi^{n-1,n}\} \ \bmod\ H^{2n-1}(X, \mathbb{Z}),$$

where $\{I_\xi^{n-1,n}\} \in H^{n-1,n}(X, \mathbb{C})$ denotes the $(n-1, n)$-component of the De Rham cohomology class $\{I_\xi\}$.

§10. Complex Curves

We show here how Hodge theory can be used to derive quickly the basic properties of compact manifolds of complex dimension 1 (also called *complex curves* or *Riemann surfaces*). Let X be such a curve. We shall always assume in this section that X is compact and connected. Since every positive $(1,1)$-form on a curve defines a Kähler metric, the results of § 8 and § 9 can be applied.

§10.1. Riemann-Roch Formula

Denoting $g = h^1(X, \mathcal{O})$, we find

$$\text{(10.1)} \quad H^1(X, \mathcal{O}) \simeq \mathbb{C}^g, \qquad H^0(X, \Omega^1_X) \simeq \mathbb{C}^g,$$
$$\text{(10.2)} \quad H^0(X, \mathbb{Z}) = \mathbb{Z}, \qquad H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}, \qquad H^2(X, \mathbb{Z}) = \mathbb{Z}.$$

The classification of oriented topological surfaces shows that X is homeomorphic to a sphere with g handles (= torus with g holes), but this property will not be used in the sequel. The number g is called the *genus* of X.

Any divisor on X can be written $\Delta = \sum m_j a_j$ where (a_j) is a finite sequence of points and $m_j \in \mathbb{Z}$. Let E be a line bundle over X. We shall identify E and the associated locally free sheaf $\mathcal{O}(E)$. According to V-13.2, we denote by $E(\Delta)$ the sheaf of germs of meromorphic sections f of E such that $\operatorname{div} f + \Delta \geq 0$, i.e. which have a pole of order $\leq m_j$ at a_j if $m_j > 0$, and which have a zero of order $\geq |m_j|$ at a_j if $m_j < 0$. Clearly

$$\text{(10.3)} \quad E(\Delta) = E \otimes \mathcal{O}(\Delta), \qquad \mathcal{O}(\Delta + \Delta') = \mathcal{O}(\Delta) \otimes \mathcal{O}(\Delta').$$

For any point $a \in X$ and any integer $m > 0$, there is an exact sequence

$$0 \longrightarrow E \longrightarrow E(m[a]) \longrightarrow \mathcal{S} \longrightarrow 0$$

where $\mathcal{S} = E(m[a])/E$ is a sheaf with only one non zero stalk $\mathcal{S}_a$ isomorphic to $\mathbb{C}^m$. Indeed, if z is a holomorphic coordinate near a, the stalk $\mathcal{S}_a$ corresponds to the polar parts $\sum_{-m \leq k < 0} c_k z^k$ in the power series expansions of germs of meromorphic sections at point a. We get an exact sequence

$$H^0\big(X, E(m[a])\big) \longrightarrow \mathbb{C}^m \longrightarrow H^1(X, E).$$

When m is chosen larger than $\dim H^1(X, E)$, we see that $E(m[a])$ has a non zero section and conclude:

(10.4) Theorem. *Let a be a given point on a curve. Then every line bundle E has non zero meromorphic sections f with a pole at a and no other poles.*

If Δ is the divisor of a meromorphic section f of E, we have $E \simeq \mathcal{O}(\Delta)$, so the map

$$\operatorname{Div}(X) \longrightarrow H^1(X, \mathcal{O}^\star), \qquad \Delta \longmapsto \mathcal{O}(\Delta)$$

is onto (cf. (V-13.8)). On the other hand, Div is clearly a soft sheaf, thus $H^1(X, \operatorname{Div}) = 0$. The long cohomology sequence associated to the exact sequence $1 \to \mathcal{O}^\star \to \mathcal{M}^\star \to \operatorname{Div} \to 0$ implies:

(10.5) Corollary. *On any complex curve, one has $H^1(X, \mathcal{M}^\star) = 0$ and there is an exact sequence*

$$0 \longrightarrow \mathbb{C}^\star \longrightarrow \mathcal{M}^\star(X) \longrightarrow \operatorname{Div}(X) \longrightarrow H^1(X, \mathcal{O}^\star) \longrightarrow 0.$$

The first Chern class $c_1(E) \in H^2(X, \mathbb{Z})$ can be interpreted as an integer. This integer is called the *degree* of E. If $E \simeq \mathcal{O}(\Delta)$ with $\Delta = \sum m_j a_j$, formula V-13.6 shows that the image of $c_1(E)$ in $H^2(X,\mathbb{R})$ is the De Rham cohomology class of the associated current $[\Delta]=\sum m_j\delta_{a_j}$, hence

$$(10.6)\quad c_1(E)=\int_X[\Delta]=\sum m_j.$$

If $\sum m_j a_j$ is the divisor of a meromorphic function, we have $\sum m_j = 0$ because the associated bundle $E=\mathcal{O}(\sum m_j a_j)$ is trivial.

(10.7) Theorem. *Let E be a line bundle on a complex curve X. Then*

a) $H^0(X,E)=0$ *if* $c_1(E)<0$ *or if* $c_1(E)=0$ *and E is non trivial;*

b) *For every positive $(1,1)$-form ω on X with $\int_X\omega=1$, E has a hermitian metric such that $\frac{\mathrm{i}}{2\pi}\Theta(E)=c_1(E)\,\omega$. In particular, E has a metric of positive (resp. negative) curvature if and only if $c_1(E)>0$ (resp. if and only if $c_1(E)<0$).*

Proof. a) If E has a non zero holomorphic section f, then its degree is $c_1(E)=\int_X\operatorname{div}f\geq 0$. In fact, we even have $c_1(E)>0$ unless f does not vanish, in which case E is trivial.

b) Select an arbitrary hermitian metric h on E. Then $c_1(E)\,\omega-\frac{\mathrm{i}}{2\pi}\Theta_h(E)$ is a real $(1,1)$-form cohomologous to zero (the integral over X is zero), so Lemma 8.6 c) implies

$$c_1(E)\,\omega-\frac{\mathrm{i}}{2\pi}\Theta_h(E)=\mathrm{i}d'd''\varphi$$

for some real function $\varphi\in C^\infty(X,\mathbb{R})$. If we replace the initial metric of E by $h'=h\,e^{-\varphi}$, we get a metric of constant curvature $c_1(E)\,\omega$. □

(10.8) Riemann-Roch formula. *Let E be a holomorphic line bundle and let $h^q(E)=\dim H^q(X,E)$. Then*

$$h^0(E)-h^1(E)=c_1(E)-g+1.$$

Moreover $h^1(E)=h^0(K\otimes E^\star)$, where $K=\Omega^1_X$ is the canonical line bundle of X.

Proof. We claim that for every line bundle F and every divisor Δ we have the equality

$$(10.9)\quad h^0\big(F(\Delta)\big)-h^1\big(F(\Delta)\big)=h^0(F)-h^1(F)+\int_X[\Delta].$$

If we write $E=\mathcal{O}(\Delta)$ and apply the above equality with $F=\mathcal{O}$, the Riemann-Roch formula results from (10.6), (10.9) and from the equalities

$$h^0(\mathcal{O})=\dim H^0(X,\mathcal{O})=1,\qquad h^1(\mathcal{O})=\dim H^1(X,\mathcal{O})=g.$$

However, (10.9) need only be proved when $\Delta \geq 0$: otherwise we are reduced to this case by writing $\Delta = \Delta_1 - \Delta_2$ with $\Delta_1, \Delta_2 \geq 0$ and by applying the result to the pairs (F, Δ_1) and $\big(F(\Delta), \Delta_2\big)$. If $\Delta = \sum m_j a_j \geq 0$, there is an exact sequence

$$0 \longrightarrow F \longrightarrow F(\Delta) \longrightarrow \mathcal{S} \longrightarrow 0$$

where $\mathcal{S}_{a_j} \simeq \mathbb{C}^{m_j}$ and the other stalks are zero. Let $m = \sum m_j = \int_X [\Delta]$. The sheaf $\mathcal{S}$ is acyclic, because its support $\{a_j\}$ is of dimension 0. Hence there is an exact sequence

$$0 \longrightarrow H^0(F) \longrightarrow H^0\big(F(\Delta)\big) \longrightarrow \mathbb{C}^m \longrightarrow H^1(F) \longrightarrow H^1\big(F(\Delta)\big) \longrightarrow 0$$

and (10.9) follows. The equality $h^1(E) = h^0(K \otimes E^\star)$ is a consequence of the Serre duality theorem

$$\big(H^{0,1}(X, E)\big)^\star \simeq H^{1,0}(X, E^\star), \quad \text{i.e.} \quad \big(H^1(X, E)\big)^\star \simeq H^0(X, K \otimes E^\star). \ \square$$

(10.10) Corollary (Hurwitz' formula). $c_1(K) = 2g - 2$.

Proof. Apply Riemann-Roch to $E = K$ and observe that

$$(10.11) \qquad \begin{aligned} h^0(K) &= \dim H^0(X, \Omega^1_X) = g \\ h^1(K) &= \dim H^1(X, \Omega^1_X) = h^{1,1} = b_2 = 1 \end{aligned}$$

(10.12) Corollary. *For every $a \in X$ and every $m \in \mathbb{Z}$*

$$h^0\big(K(-m[a])\big) = h^1\big(\mathcal{O}(m[a])\big) = h^0\big(\mathcal{O}(m[a])\big) - m + g - 1.$$

§10.2. Jacobian of a Curve

By the Neron-Severi sequence (9.6), there is an exact sequence

$$(10.13) \quad 0 \longrightarrow \mathrm{Jac}(X) \longrightarrow H^1(X, \mathcal{O}^\star) \xrightarrow{c_1} \mathbb{Z} \longrightarrow 0,$$

where the Jacobian $\mathrm{Jac}(X)$ is a g-dimensional torus. Choose a base point $a \in X$. For every point $x \in X$, the line bundle $\mathcal{O}([x] - [a])$ has zero first Chern class, so we have a well-defined map

$$(10.14) \quad \Phi_a : X \longrightarrow \mathrm{Jac}(X), \qquad x \longmapsto \mathcal{O}([x] - [a]).$$

Observe that the Jacobian $\mathrm{Jac}(X)$ of a curve coincides by definition with the Albanese variety $\mathrm{Alb}(X)$.

(10.15) Lemma. *The above map Φ_a coincides with the Albanese map $\alpha : X \longrightarrow \mathrm{Alb}(X)$ defined in* (9.12).

Proof. By holomorphic continuation, it is enough to prove that $\Phi_a(x) = \alpha(x)$ when x is near a. Let z be a complex coordinate and let $D' \subset\subset D$ be open disks centered at a. Relatively to the covering

$$U_1 = D, \qquad U_2 = X \setminus \overline{D'},$$

the line bundle $\mathcal{O}([x]-[a])$ is defined by the Čech cocycle $c \in C^1(\mathcal{U}, \mathcal{O}^\star)$ such that

$$c_{12}(z) = \frac{z-x}{z-a} \quad \text{on} \quad U_{12} = D \setminus \overline{D'}.$$

On the other hand, we compute $\alpha(x)$ by Formula (9.14). The path integral current $I_{[a,x]} \in \mathcal{D}'_1(X)$ is equal to 0 on U_2. Lemma I-2.10 implies $d''(dz/2\pi \mathrm{i} z) = \delta_0\, d\overline{z} \wedge dz/2i = \delta_0$ according to the usual identification of distributions and currents of degree 0, thus

$$I^{0,1}_{[a,x]} = d''\Big(\frac{dz}{2\pi iz} \star I^{0,1}_{[a,x]}\Big) \quad \text{on} \quad U_1.$$

Therefore $\{I^{0,1}_{[a,x]}\} \in H^{0,1}(X, \mathbb{C})$ is equal to the Čech cohomology class $\{c'\}$ in $H^1(X, \mathcal{O})$ represented by the cocycle

$$c'_{12}(z) = \frac{dw}{2\pi iw} \star I^{0,1}_{[a,x]} = \frac{1}{2\pi i}\int_a^x \frac{dw}{w-z} = \frac{1}{2\pi \mathrm{i}} \log \frac{z-x}{z-a} \quad \text{on} \quad U_{12}$$

and we have $c = \exp(2\pi \mathrm{i} c')$ in $H^1(X, \mathcal{O}^\star)$. □

The nature of Φ_a depends on the value of the genus g. A careful examination of Φ_a will enable us to determine all curves of genus 0 and 1.

(10.16) Theorem. *The following properties are equivalent:*

a) $g = 0$;

b) *X has a meromorphic function f having only one simple pole p;*

c) *X is biholomorphic to $\mathbb{P}^1$.*

Proof. c) $\Longrightarrow$ a) is clear.

a) $\Longrightarrow$ b). Since $g = 0$, we have $\mathrm{Jac}(X) = 0$. If $p, p' \in X$ are distinct points, the bundle $\mathcal{O}([p'] - [p])$ has zero first Chern class, therefore it is trivial and there exists a meromorphic function f with $\mathrm{div}\, f = [p'] - [p]$. In particular p is the only pole of f, and this pole is simple.

b) $\Longrightarrow$ c). We may consider f as a map $X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. For every value $w \in \mathbb{C}$, the function $f - w$ must have exactly one simple zero $x \in X$ because $\int_X \mathrm{div}(f - w) = 0$ and p is a simple pole. Therefore $f : X \to \mathbb{P}^1$ is bijective and X is biholomorphic to $\mathbb{P}^1$. □

(10.17) Theorem. *The map Φ_a is always injective for $g \geq 1$.*

a) *If $g = 1$, Φ_a is a biholomorphism. In particular every curve of genus 1 is biholomorphic to a complex torus $\mathbb{C}/\Gamma$.*

b) *If $g \geq 2$, Φ_a is an embedding.*

Proof. If Φ_a is not injective, there exist points $x_1 \neq x_2$ such that $\mathcal{O}([x_1]-[x_2])$ is trivial; then there is a meromorphic function f such that $\operatorname{div} f = [x_1]-[x_2]$ and Th. 10.16 implies that $g = 0$.

When $g = 1$, Φ_a is an injective map $X \longrightarrow \operatorname{Jac}(X) \simeq \mathbb{C}/\Gamma$, thus Φ_a is open. It follows that $\Phi_a(X)$ is a compact open subset of $\mathbb{C}/\Gamma$, so $\Phi_a(X) = \mathbb{C}/\Gamma$ and Φ_a is a biholomorphism of X onto $\mathbb{C}/\Gamma$.

In order to prove that Φ_a is an embedding when $g \geq 2$, it is sufficient to show that the holomorphic 1-forms $u_1, \ldots, u_g$ do not all vanish at a given point $x \in X$. In fact, X has no non constant meromorphic function having only a simple pole at x, thus $h^0\big(\mathcal{O}([x])\big) = 1$ and Cor. 10.12 implies

$$h^0\big(K(-[x])\big) = g - 1 < h^0(K) = g.$$

Hence K has a section u which does not vanish at x. □

§10.3. Weierstrass Points of a Curve

We want to study how many meromorphic functions have a unique pole of multiplicity $\leq m$ at a given point $a \in X$, i.e. we want to compute $h^0\big(\mathcal{O}(m[a])\big)$. As we shall see soon, these numbers may depend on a only if m is small. We have $c_1\big(K(-m[a])\big) = 2g-2-m$, so the degree is < 0 and $h^0\big(K(-m[a])\big) = 0$ for $m \geq 2g-1$ by 10.7 a). Cor. 10.12 implies

$$(10.18) \quad h^0\big(\mathcal{O}(m[a])\big) = m - g + 1 \quad \text{for} \quad m \geq 2g-1.$$

It remains to compute $h^0\big(K(-m[a])\big)$ for $0 \leq m \leq 2g-2$ and $g \geq 1$. Let $u_1, \ldots, u_g$ be a basis of $H^0(X, K)$ and let z be a complex coordinate centered at a. Any germ $u \in \mathcal{O}(K)_a$ can be written $u = U(z)\,dz$ with $U(z) = \sum_{m \in \mathbb{N}} \frac{1}{m!} U^{(m)}(a) z^m \, dz$. The unique non zero stalk of the quotient sheaf $\mathcal{O}\big(K(-m[a])\big)/\mathcal{O}\big(K(-(m+1)[a])\big)$ is canonically isomorphic to K_a^{m+1} via the map $u \mapsto U^{(m)}(a)(dz)^{m+1}$, which is independant of the choice of z. Hence $\bigwedge^g \big(\mathcal{O}(K)/\mathcal{O}(K - g[a])\big) \simeq K_a^{1+2+\cdots+g}$ and the *Wronskian*

$$(10.19) \quad W(u_1, \ldots, u_g) = \begin{vmatrix} U_1(z) & \ldots & U_g(z) \\ U_1'(z) & \ldots & U_g'(z) \\ \vdots & & \vdots \\ U_1^{(g-1)}(z) & \ldots & U_g^{(g-1)}(z) \end{vmatrix} dz^{1+2+\cdots+g}$$

defines a global section $W(u_1, \ldots, u_g) \in H^0(X, K^{g(g+1)/2})$. At the given point a, we can find linear combinations $\widetilde{u}_1, \ldots, \widetilde{u}_g$ of $u_1, \ldots, u_g$ such that

$$\widetilde{u}_j(z) = \big(z^{s_j-1} + \mathrm{O}(z^{s_j})\big)dz, \qquad s_1 < \ldots < s_g.$$

We know that not all sections of K vanish at a and that $c_1(K) = 2g-2$, thus $s_1 = 1$ and $s_g \le 2g-1$. We have $W(\widetilde{u}_1,\ldots,\widetilde{u}_g) \sim W(z^{s_1-1}dz,\ldots,z^{s_g-1}dz)$ at point a, and an easy induction on $\sum s_j$ combined with differentiation in z yields

$$W(z^{s_1-1}dz,\ldots,z^{s_g-1}dz) = C\, z^{s_1+\ldots+s_g-g(g+1)/2}\, dz^{g(g+1)/2}$$

for some positive integer constant C. In particular, $W(u_1,\ldots,u_g)$ is not identically zero and vanishes at a with multiplicity

$$(10.20)\qquad \mu_a = s_1 + \ldots + s_g - g(g+1)/2 > 0$$

unless $s_1 = 1$, $s_2 = 2$, …, $s_g = g$. Now, we have

$$h^0\big(K(-m[a])\big) = \mathrm{card}\{j\,;\, s_j > m\} = g - \mathrm{card}\{j\,;\, s_j \le m\}$$

and Cor. 10.12 gives

$$(10.21)\qquad h^0\big(\mathcal{O}(m[a])\big) = m + 1 - \mathrm{card}\{j\,;\, s_j \le m\}.$$

If a is not a zero of $W(u_1,\ldots,u_g)$, we find

$$(10.22)\qquad \begin{cases} h^0\big(\mathcal{O}(m[a])\big) = 1 & \text{for}\ \ m \le g,\\ h^0\big(\mathcal{O}(m[a])\big) = m+1-g & \text{for}\ \ m > g.\end{cases}$$

The zeroes of $W(u_1,\ldots,u_g)$ are called the *Weierstrass points* of X, and the associated *Weierstrass sequence* is the sequence $w_m = h^0\big(\mathcal{O}(m[a])\big)$, $m \in \mathbb{N}$. We have $w_{m-1} \le w_m \le w_{m-1}+1$ and $s_1 < \ldots < s_g$ are precisely the integers $m \ge 1$ such that $w_m = w_{m-1}$. The numbers $s_j \in \{1, 2, \ldots, 2g-1\}$ are called the *gaps* and μ_a the *weight* of the Weierstrass point a. Since $W(u_1,\ldots,u_g)$ is a section of $K^{g(g+1)/2}$, Hurwitz' formula implies

$$(10.23)\qquad \sum_{a\in X} \mu_a = c_1(K^{g(g+1)/2}) = g(g+1)(g-1).$$

In particular, a curve of genus g has at most $g(g+1)(g-1)$ Weierstrass points.

§11. Hodge-Frölicher Spectral Sequence

Let X be a *compact* complex n-dimensional manifold. We consider the double complex $K^{p,q} = C^\infty(X, \Lambda^{p,q}T^\star_X)$, $d = d' + d''$. The Hodge-Frölicher spectral sequence is by definition the spectral sequence associated to this double complex (cf. IV-11.9). It starts with

$$(11.1)\quad E_1^{p,q} = H^{p,q}(X, \mathbb{C})$$

and the limit term $E_\infty^{p,q}$ is the graded module associated to a filtration of the De Rham cohomology group $H^k(X,\mathbb{C})$, $k = p+q$. In particular, if the numbers b_k and $h^{p,q}$ are still defined as in (8.11), we have

$$(11.2)\quad b_k = \sum_{p+q=k} \dim E_\infty^{p,q} \le \sum_{p+q=k} \dim E_1^{p,q} = \sum_{p+q=k} h^{p,q}.$$

The equality is equivalent to the degeneration of the spectral sequence at $E_1^\bullet$. As a consequence, the Hodge-Frölicher spectral sequence of a compact Kähler manifold degenerates in $E_1^\bullet$.

(11.3) Theorem and Definition. *The existence of an isomorphism*

$$H^k_{\mathrm{DR}}(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$$

is equivalent to the degeneration of the Hodge-Frölicher spectral sequence at E_1. In this case, the isomorphism is canonically defined and we say that X admits a Hodge decomposition. □

In general, interesting informations can be deduced from the spectral sequence. Theorem IV-11.8 shows in particular that

$$(11.4)\quad b_1 \ge \dim E_2^{1,0} + (\dim E_2^{0,1} - \dim E_2^{2,0})_+.$$

However, $E_2^{1,0}$ is the central cohomology group in the sequence

$$d_1 = d' : E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0},$$

and as $E_1^{0,0}$ is the space of holomorphic functions on X, the first map d_1 is zero (by the maximum principle, holomorphic functions are constant on each connected component of X). Hence $\dim E_2^{1,0} \ge h^{1,0} - h^{2,0}$. Similarly, $E_2^{0,1}$ is the kernel of a map $E_1^{0,1} \to E_1^{1,1}$, thus $\dim E_2^{0,1} \ge h^{0,1} - h^{1,1}$. By (11.4) we obtain

$$(11.5)\quad b_1 \ge (h^{1,0} - h^{2,0})_+ + (h^{0,1} - h^{1,1} - h^{2,0})_+.$$

Another interesting relation concerns the topological Euler-Poincaré characteristic

$$\chi_{\mathrm{top}}(X) = b_0 - b_1 + \ldots - b_{2n-1} + b_{2n}.$$

We need the following simple lemma.

(11.6) Lemma. *Let $(C^\bullet, d)$ a bounded complex of finite dimensional vector spaces over some field. Then, the Euler characteristic*

$$\chi(C^\bullet) = \sum (-1)^q \dim C^q$$

is equal to the Euler characteristic $\chi\big(H^\bullet(C^\bullet)\big)$ of the cohomology module.

Proof. Set

$$c_q = \dim C^q, \qquad z_q = \dim Z^q(C^\bullet), \qquad b_q = \dim B^q(C^\bullet), \qquad h_q = \dim H^q(C^\bullet).$$

Then

$$c_q = z_q + b_{q+1}, \qquad h_q = z_q - b_q.$$

Therefore we find

$$\sum (-1)^q\, c_q = \sum (-1)^q\, z_q - \sum (-1)^q\, b_q = \sum (-1)^q\, h_q. \qquad \square$$

In particular, if the term $E_r^\bullet$ of the spectral sequence of a filtered complex $K^\bullet$ is a bounded and finite dimensional complex, we have

$$\chi(E_r^\bullet) = \chi(E_{r+1}^\bullet) = \ldots = \chi(E_\infty^\bullet) = \chi\big(H^\bullet(K^\bullet)\big)$$

because $E_{r+1}^\bullet = H^\bullet(E_r^\bullet)$ and $\dim E_\infty^l = \dim H^l(K^\bullet)$. In the Hodge-Frölicher spectral sequence, we have $\dim E_1^l = \sum_{p+q=l} h^{p,q}$, hence:

(11.7) Theorem. *For any compact complex manifold X, one has*

$$\chi_{\mathrm{top}}(X) = \sum_{0 \le k \le 2n} (-1)^k b_k = \sum_{0 \le p,q \le n} (-1)^{p+q} h^{p,q}.$$

§12. Effect of a Modification on Hodge Decomposition

In this section, we show that the existence of a Hodge decomposition on a compact complex manifold X is guaranteed as soon as there exists such a decomposition on a modification $\widetilde{X}$ of X (see II-??.?? for the Definition). This leads us to extend Hodge theory to a class of manifolds which are non necessarily Kähler, the so called Fujiki class $(\mathcal{C})$ of manifolds bimeromorphic to Kähler manifolds.

§12.1. Sheaf Cohomology Reinterpretation of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$

We first give a description of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ in terms of the hypercohomology of a suitable complex of sheaves. This interpretation, combined with the analogue of the Hodge-Frölicher spectral sequence, will imply in particular that $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ is always finite dimensional when X is compact. Let us denote by $\mathcal{E}^{p,q}$ the sheaf of germs of C^∞ forms of bidegree (p,q), and by Ω^p the sheaf of germs of holomorphic p-forms on X. For a fixed bidegree (p_0,q_0), we let $k_0=p_0+q_0$ and we introduce a complex of sheaves $(\mathcal{L}^\bullet_{p_0,q_0},\delta)$, also denoted $\mathcal{L}^\bullet$ for simplicity, such that

$$\mathcal{L}^k=\bigoplus_{p+q=k,p<p_0,q<q_0}\mathcal{E}^{p,q}\quad\text{for}\quad k\le k_0-2,$$
$$\mathcal{L}^{k-1}=\bigoplus_{p+q=k,p\ge p_0,q\ge q_0}\mathcal{E}^{p,q}\quad\text{for}\quad k\ge k_0.$$

The differential δ^k on $\mathcal{L}^k$ is chosen equal to the exterior derivative d for $k\ne k_0-2$ (in the case $k\le k_0-3$, we neglect the components which fall outside $\mathcal{L}^{k+1}$), and we set

$$\delta^{k_0-2}=d'd'':\mathcal{L}^{k_0-2}=\mathcal{E}^{p_0-1,q_0-1}\longrightarrow\mathcal{L}^{k_0-1}=\mathcal{E}^{p_0,q_0}.$$

We find in particular $H^{p_0,q_0}_{\mathrm{BC}}(X,\mathbb{C})=H^{k_0-1}\big(\mathcal{L}^\bullet(X)\big)$. We observe that $\mathcal{L}^\bullet$ has subcomplexes $(\mathcal{S}'^\bullet,d')$ and $(\mathcal{S}''^\bullet,d'')$ defined by

$$\mathcal{S}'^k=\Omega^k_X\quad\text{for}\quad 0\le k\le p_0-1,\qquad \mathcal{S}'^k=0\quad\text{otherwise},$$
$$\mathcal{S}''^k=\overline{\Omega^k_X}\quad\text{for}\quad 0\le k\le q_0-1,\qquad \mathcal{S}''^k=0\quad\text{otherwise}.$$

If $p_0=0$ or $q_0=0$ we set instead $\mathcal{S}'^0=\mathbb{C}$ or $\mathcal{S}''^0=\mathbb{C}$, and take the other components to be zero. Finally, we let $\mathcal{S}^\bullet=\mathcal{S}'^\bullet+\mathcal{S}''^\bullet\subset\mathcal{L}^\bullet$ (the sum is direct except for $\mathcal{S}^0$); we denote by $\mathcal{M}^\bullet$ the sheaf complex defined in the same way as $\mathcal{L}^\bullet$, except that the sheaves $\mathcal{E}^{p,q}$ are replaced by the sheaves of currents $\mathcal{D}'_{n-p,n-q}$.

(12.1) Lemma. *The inclusions $\mathcal{S}^\bullet\subset\mathcal{L}^\bullet\subset\mathcal{M}^\bullet$ induce isomorphisms*

$$\mathcal{H}^k(\mathcal{S}^\bullet)\simeq\mathcal{H}^k(\mathcal{L}^\bullet)\simeq\mathcal{H}^k(\mathcal{M}^\bullet),$$

and these cohomology sheaves vanish for $k\ne 0,p_0-1,q_0-1$.

Proof. We will prove the result only for the inclusion $\mathcal{S}^\bullet\subset\mathcal{L}^\bullet$, the other case $\mathcal{S}^\bullet\subset\mathcal{M}^\bullet$ is identical. Let us denote by $\mathcal{Z}^{p,q}$ the sheaf of d''-closed differential forms of bidegree (p,q). We consider the filtration

$$F_p(\mathcal{L}^k)=\mathcal{L}^k\cap\bigoplus_{r\ge p}\mathcal{E}^{r,\bullet}$$

and the induced filtration on $\mathcal{S}^\bullet$. In the case of $\mathcal{L}^\bullet$, the first spectral sequence has the following terms $E_0^\bullet$ and $E_1^\bullet$:

$$\begin{array}{lll}
\text{if } p < p_0 & E_0^{p,\bullet} : & 0 \longrightarrow \mathcal{E}^{p,0} \xrightarrow{d''} \mathcal{E}^{p,1} \longrightarrow \cdots \xrightarrow{d''} \mathcal{E}^{p,q_0-1} \longrightarrow 0, \\
\text{if } p \geq p_0 & E_0^{p,\bullet} : & 0 \longrightarrow \mathcal{E}^{p,q_0} \xrightarrow{d''} \mathcal{E}^{p,q_0+1} \longrightarrow \cdots \longrightarrow \mathcal{E}^{p,q} \xrightarrow{d''} \cdots, \\
\text{if } p < p_0 & \multicolumn{2}{l}{E_1^{p,0} = \Omega_X^p, \quad E_1^{p,q_0-1} \simeq \mathcal{Z}^{p,q_0}, \quad E_1^{p,q} = 0 \quad \text{for} \quad q \neq 0, q_0 - 1,} \\
\text{if } p \geq p_0 & \multicolumn{2}{l}{E_1^{p,q_0-1} = \mathcal{Z}^{p,q_0}, \quad E_1^{p,q} = 0 \quad \text{for} \quad q \neq q_0 - 1.}
\end{array}$$

The isomorphism in the third line is given by

$$\mathcal{E}^{p,q_0-1}/d''\mathcal{E}^{p,q_0-2} \simeq d''\mathcal{E}^{p,q_0-1} \simeq \mathcal{Z}^{p,q_0}.$$

The map $d_1 : E_1^{p_0-1,q_0-1} \longrightarrow E_1^{p_0,q_0-1}$ is induced by $d'd''$ acting on $\mathcal{E}^{p_0-1,q_0-1}$, but thanks to the previous identification, this map becomes d' acting on $\mathcal{Z}^{p_0-1,q_0}$. Hence $E_1^\bullet$ consists of two sequences

$$\begin{aligned}
&E_1^{\bullet,0} : \quad 0 \longrightarrow \Omega_X^0 \xrightarrow{d'} \Omega_X^1 \longrightarrow \cdots \xrightarrow{d'} \Omega_X^{p_0-1} \longrightarrow 0, \\
&E_1^{\bullet,q_0-1} : \quad 0 \longrightarrow \mathcal{Z}^{0,q_0} \xrightarrow{d'} \mathcal{Z}^{1,q_0} \longrightarrow \cdots \longrightarrow \mathcal{Z}^{p,q_0} \xrightarrow{d'} \cdots ;
\end{aligned}$$

if these sequences overlap ($q_0 = 1$), only the second one has to be considered. The term $E_1^\bullet$ in the spectral sequence of $\mathcal{S}^\bullet$ has the same first line, but the second is reduced to $E_1^{0,q_0-1} = \overline{d\Omega_X^{q_0-2}}$ (resp. $= \mathbb{C}$ for $q_0 = 1$). Thanks to Lemma 12.2 below, we see that the two spectral sequences coincide in $E_2^\bullet$, with at most three non zero terms:

$$E_2^{0,0} = \mathbb{C}, \qquad E_2^{p_0-1,0} = d\Omega_X^{p_0-2} \quad \text{for} \quad p_0 \geq 2, \qquad E_2^{0,q_0-1} = \overline{d\Omega_X^{q_0-2}} \quad \text{for} \quad q_0 \geq 2.$$

Hence $\mathcal{H}^k(\mathcal{S}^\bullet) \simeq \mathcal{H}^k(\mathcal{L}^\bullet)$ and these sheaves vanish for $k \neq 0, p_0 - 1, q_0 - 1$. $\square$

(12.2) Lemma. *The complex of sheaves*

$$0 \longrightarrow \mathcal{Z}^{0,q_0} \xrightarrow{d'} \mathcal{Z}^{1,q_0} \longrightarrow \cdots \longrightarrow \mathcal{Z}^{p,q_0} \xrightarrow{d'} \cdots$$

is a resolution of $\overline{d\Omega_X^{q_0-1}}$ *for* $q_0 \geq 1$, *resp. of* $\mathbb{C}$ *for* $q_0 = 0$.

Proof. Embed $\mathcal{Z}^{\bullet,q_0}$ in the double complex

$$K^{p,q} = \mathcal{E}^{p,q} \quad \text{for} \quad q < q_0, \qquad K^{p,q} = 0 \quad \text{for} \quad q \geq q_0.$$

For the first fitration of $K^\bullet$, we find

$$E_1^{p,q_0-1} = \mathcal{Z}^{p,q_0}, \qquad E_1^{p,q} = 0 \quad \text{for} \quad q \neq q_0 - 1$$

The second fitration gives $\widetilde{E}_1^{p,q} = 0$ for $q \geq 1$ and

$$\widetilde{E}_1^{p,0} = H^0(K^{\bullet,p}) = \begin{cases} \overline{H^0(\mathcal{E}^{p,\bullet})} = \overline{\Omega_X^p} & \text{for } p \leq q_0 - 1 \\ 0 & \text{for } p \geq q_0, \end{cases}$$

thus the cohomology of $\mathcal{Z}^{\bullet,q_0}$ coincides with that of $(\overline{\Omega_X^p}, d)_{0\le p<q_0}$. □

Lemma IV-11.10 and formula (IV-12.9) imply

$$\begin{aligned}(12.3)\quad \mathbb{H}^k(X,\mathcal{S}^\bullet) &\simeq \mathbb{H}^k(X,\mathcal{L}^\bullet) \simeq \mathbb{H}^k(X,\mathcal{M}^\bullet)\\ &\simeq H^k\big(\mathcal{L}^\bullet(X)\big) \simeq H^k\big(\mathcal{M}^\bullet(X)\big)\end{aligned}$$

because the sheaves $\mathcal{L}^k$ and $\mathcal{M}^k$ are soft. In particular, the group $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ can be computed either by means of C^∞ differential forms or by means of currents. This property also holds for the De Rham or Dolbeault groups $H^k(X,\mathbb{C})$, $H^{p,q}(X,\mathbb{C})$, as was already remarked in §IV-6. Another important consequence of (12.3) is:

(12.4) Theorem. *If X is compact, then* $\dim H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) < +\infty$.

Proof. We show more generally that the hypercohomology groups $\mathbb{H}^k(X,\mathcal{S}^\bullet)$ are finite dimensional. As there is an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{S}'^\bullet \oplus \mathcal{S}''^\bullet \longrightarrow \mathcal{S}^\bullet \longrightarrow 0$$

and a corresponding long exact sequence for hypercohomology groups, it is enough to show that the groups $\mathbb{H}^k(X,\mathcal{S}'^\bullet)$ are finite dimensional. This property is proved for $\mathcal{S}'^\bullet = \mathcal{S}'^\bullet_{p_0}$ by induction on p_0. For $p_0 = 0$ or 1, the complex $\mathcal{S}'^\bullet$ is reduced to its term $\mathcal{S}'^0$, thus

$$\mathbb{H}^k(X,\mathcal{S}^\bullet) = H^k(X,\mathcal{S}'^0) = \begin{cases} H^k(X,\mathbb{C}) & \text{for } p_0 = 0\\ H^k(X,\mathcal{O}) & \text{for } p_0 = 1\end{cases}$$

and this groups are finite dimensional. In general, we have an exact sequence

$$0 \longrightarrow \Omega_X^{p_0} \longrightarrow \mathcal{S}^\bullet_{p_0+1} \longrightarrow \mathcal{S}^\bullet_{p_0} \longrightarrow 0$$

where $\Omega_X^{p_0}$ denotes the subcomplex of $\mathcal{S}^\bullet_{p_0+1}$ reduced to one term in degree p_0. As

$$\mathbb{H}^k(X,\Omega_X^{p_0}) = H^{k-p_0}(X,\Omega_X^{p_0}) = H^{p_0,k-p_0}(X,\mathbb{C})$$

is finite dimensional, the Theorem follows. □

(12.5) Definition. *We say that a compact manifold admits a strong Hodge decomposition if the natural maps*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C})$$

are isomorphisms.

This implies of course that there are natural isomorphisms

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}), \qquad H^{q,p}(X,\mathbb{C}) \simeq \overline{H^{p,q}(X,\mathbb{C})}$$

and that the Hodge-Frölicher spectral sequence degenerates in $E_1^\bullet$. It follows from § 8 that all Kähler manifolds admit a strong Hodge decomposition.

§12.2. Direct and Inverse Image Morphisms

Let $F : X \longrightarrow Y$ be a holomorphic map between complex analytic manifolds of respective dimensions n, m, and $r = n - m$. We have pull-back morphisms

$$(12.6)\qquad \begin{aligned} F^\star &: H^k(Y,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C}),\\ F^\star &: H^{p,q}(Y,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\\ F^\star &: H^{p,q}_{\mathrm{BC}}(Y,\mathbb{C}) \longrightarrow H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}),\end{aligned}$$

commuting with the natural morphisms (8.2), (8.3).

Assume now that F is *proper*. Theorem I-1.14 shows that one can define direct image morphisms

$$F_\star \ :\ \mathcal{D}'_k(X) \longrightarrow \mathcal{D}'_k(Y), \qquad F_\star \ :\ \mathcal{D}'_{p,q}(X) \longrightarrow \mathcal{D}'_{p,q}(Y),$$

commuting with d', d''. To $F_\star$ therefore correspond cohomology morphisms

$$(12.7)\qquad \begin{aligned} F_\star &: H^k(X,\mathbb{C}) \longrightarrow H^{k-2r}(Y,\mathbb{C}),\\ F_\star &: H^{p,q}(X,\mathbb{C}) \longrightarrow H^{p-r,q-r}(Y,\mathbb{C}),\\ F_\star &: H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p-r,q-r}_{\mathrm{BC}}(Y,\mathbb{C}),\end{aligned}$$

which commute also with (8.2), (8.3). In addition, I-1.14 c) implies the *adjunction formula*

$$(12.8)\qquad F_\star(\alpha \smile F^\star\beta) = (F_\star\alpha) \smile \beta$$

whenever α is a cohomology class (of any of the three above types) on X, and β a cohomology class (of the same type) on Y.

§12.3. Modifications and the Fujiki Class ($\mathcal{C}$)

Recall that a modification of a compact manifold X is a holomorphic map $\mu : \widetilde{X} \longrightarrow X$ such that

i) $\widetilde{X}$ is a compact complex manifold of the same dimension as X ;

ii) there exists an analytic subset $S \subset X$ of codimension ≥ 1 such that $\mu : \widetilde{X} \setminus \mu^{-1}(S) \longrightarrow X \setminus S$ is a biholomorphism.

(12.9) Theorem. *If $\widetilde{X}$ admits a strong Hodge decomposition, and if $\mu : \widetilde{X} \longrightarrow X$ is a modification, then X also admits a strong Hodge decomposition.*

Proof. We first observe that $\mu_\star\mu^\star f = f$ for every smooth form f on Y. In fact, this property is equivalent to the equality

$$\int_Y (\mu_\star\mu^\star f)\wedge g = \int_X \mu^\star(f\wedge g) = \int_Y f\wedge g$$

for every smooth form g on Y, and this equality is clear because μ is a bi-holomorphism outside sets of Lebesgue measure 0. Consequently, the induced cohomology morphism $\mu_\star$ is surjective and $\mu^\star$ is injective (but these maps need not be isomorphisms). Now, we have commutative diagrams

$$\begin{array}{ccc} H^{p,q}_{\mathrm{BC}}(\widetilde X,\mathbb{C}) & \longrightarrow & H^{p,q}(\widetilde X,\mathbb{C}), \\ \mu_\star\big\downarrow\big\uparrow\mu^\star & & \mu_\star\big\downarrow\big\uparrow\mu^\star \\ H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^{p,q}(X,\mathbb{C}), \end{array} \qquad \begin{array}{ccc} \bigoplus\limits_{p+q=k} H^{p,q}_{\mathrm{BC}}(\widetilde X,\mathbb{C}) & \longrightarrow & H^k(\widetilde X,\mathbb{C}) \\ \mu_\star\big\downarrow\big\uparrow\mu^\star & & \mu_\star\big\downarrow\big\uparrow\mu^\star \\ \bigoplus\limits_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^k(X,\mathbb{C}) \end{array}$$

with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows. □

(12.10) Definition. *A manifold X is said to be in the Fujiki class $(\mathcal{C})$ if X admits a Kähler modification $\widetilde X$.*

By Th. 12.9, Hodge decomposition still holds for a manifold in the class $(\mathcal{C})$. We will see later that there exist non-Kähler manifolds in $(\mathcal{C})$, for example all non projective Moišezon manifolds (cf. §?.?). The class $(\mathcal{C})$ has been first introduced in (Fujiki 1978).