Chapter VII
Positive Vector Bundles and Vanishing Theorems

In this chapter, we prove a few vanishing theorems for hermitian vector bundles over *compact* complex manifolds. All these theorems are based on an a priori inequality for (p,q)-forms with values in a vector bundle, known as the Bochner-Kodaira-Nakano inequality. This inequality naturally leads to several positivity notions for the curvature of a vector bundle (Kodaira 1953, 1954), (Griffiths 1969) and (Nakano 1955, 1973). The corresponding algebraic notion of ampleness introduced by (Grothendieck 196?) and (Hartshorne 1966) is also discussed. The differential geometric techniques yield optimal vanishing results in the case of line bundles (Kodaira-Akizuki-Nakano and Girbau vanishing theorems) and also some partial results in the case of vector bundles (Nakano vanishing theorem). As an illustration, we compute the cohomology groups $H^{p,q}(\mathbb{P}^n, \mathcal{O}(k))$; much finer results will be obtained in chapters 8–11. Finally, the Kodaira vanishing theorem is combined with a blowing-up technique in order to establish the projective embedding theorem for manifolds admitting a Hodge metric.

1. Bochner-Kodaira-Nakano Identity

Let (X,ω) be a hermitian manifold, $\dim_{\mathbb{C}} X = n$, and let E be a hermitian holomorphic vector bundle of rank r over X. We denote by $D = D' + D''$ its Chern connection (or D_E if we want to specify the bundle), and by $\delta = \delta' + \delta''$ the formal adjoint operator of D. The operators L, Λ of chapter 6 are extended to vector valued forms in $\Lambda^{p,q}T^\star X \otimes E$ by taking their tensor product with Id_E. The following result extends the commutation relations of chapter 6 to the case of bundle valued operators.

(1.1) Theorem. *If τ is the operator of type $(1,0)$ defined by $\tau = [\Lambda, d'\omega]$ on $C^\infty_{\bullet,\bullet}(X,E)$, then*

a) $[\delta''_E, L] = \mathrm{i}(D'_E + \tau)$,
b) $[\delta'_E, L] = -\mathrm{i}(D''_E + \overline{\tau})$,
c) $[\Lambda, D''_E] = -\mathrm{i}(\delta'_E + \tau^\star)$,
d) $[\Lambda, D'_E] = \mathrm{i}(\delta''_E + \overline{\tau}^\star)$.

Proof. Fix a point x_0 in X and a coordinate system $z = (z_1, \ldots, z_n)$ centered at x_0. Then Prop. V-12.?? shows the existence of a normal coordinate frame

(e_λ) at x_0. Given any section $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda \in C^\infty_{p,q}(X,E)$, it is easy to check that the operators D_E, δ''_E, ... have Taylor expansions of the type

$$D_E s = \sum_\lambda d\sigma_\lambda \otimes e_\lambda + O(|z|), \qquad \delta''_E s = \sum_\lambda \delta''\sigma_\lambda \otimes e_\lambda + O(|z|), \ \ldots$$

in terms of the scalar valued operators d, δ, ... Here the terms $O(|z|)$ depend on the curvature coefficients of E. The proof of Th. 1.1 is then reduced to the case of scalar valued operators, which is granted by Th. VI-10.1. □

The Bochner-Kodaira-Nakano identity expresses the antiholomorphic Laplace operator $\Delta'' = D''\delta'' + \delta''D''$ acting on $C^\infty_{\bullet,\bullet}(X,E)$ in terms of its conjugate operator $\Delta' = D'\delta' + \delta'D'$, plus some extra terms involving the curvature of E and the torsion of the metric ω (in case ω is not Kähler). Such identities appear frequently in riemannian geometry (Weitzenböck formula).

(1.2) Theorem. $\Delta'' = \Delta' + [\mathrm{i}\Theta(E),\Lambda] + [D',\tau^\star] - [D'',\overline{\tau}^\star]$.

Proof. Equality 1.1 d) yields $\delta'' = -\mathrm{i}[\Lambda,D'] - \overline{\tau}^\star$, hence

$$\Delta'' = [D'',\delta''] = -\mathrm{i}[D'',[\Lambda,D']] - [D'',\overline{\tau}^\star].$$

The Jacobi identity VI-10.2 and relation 1.1 c) imply

$$\big[D'',[\Lambda,D']\big] = \big[\Lambda,[D',D'']\big] + \big[D',[D'',\Lambda]\big] = [\Lambda,\Theta(E)] + \mathrm{i}[D',\delta'+\tau^\star],$$

taking into account that $[D',D''] = D^2 = \Theta(E)$. Theorem 1.2 follows. □

(1.3) Corollary (Akizuki-Nakano 1955). *If ω is Kähler, then*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(E),\Lambda].$$

In the latter case, $\Delta'' - \Delta'$ is therefore an operator of order 0 closely related to the curvature of E. When ω is not Kähler, Formula 1.2 is not really satisfactory, because it involves the first order operators $[D',\tau^\star]$ and $[D'',\overline{\tau}^\star]$. In fact, these operators can be combined with Δ' in order to yield a new positive self-adjoint operator Δ'_τ.

(1.4) Theorem (Demailly 1985). *The operator $\Delta'_\tau = [D'+\tau,\delta'+\tau^\star]$ is a positive and formally self-adjoint operator with the same principal part as the Laplace operator Δ'. Moreover*

$$\Delta'' = \Delta'_\tau + [\mathrm{i}\Theta(E),\Lambda] + T_\omega,$$

where T_ω is an operator of order 0 depending only on the torsion of the hermitian metric ω :

$$T_\omega = \Big[\Lambda, \big[\Lambda, \frac{\mathrm{i}}{2} d'd''\omega\big]\Big] - \big[d'\omega, (d'\omega)^\star\big].$$

Proof. The first assertion is clear, because the equality $(D' + \tau)^\star = \delta' + \tau^\star$ implies the self-adjointness of Δ'_τ and

$$\langle\!\langle \Delta'_\tau u, u\rangle\!\rangle = \|D'u + \tau u\|^2 + \|\delta' u + \tau^\star u\|^2 \geq 0$$

for any compactly supported form $u \in C^\infty_{p,q}(X, E)$. In order to prove the formula, we need two lemmas.

(1.5) Lemma. a) $\quad [L, \tau] = 3d'\omega, \qquad$ b) $\quad [\Lambda, \tau] = -2\mathrm{i}\overline{\tau}^\star.$

Proof. a) Since $[L, d'\omega] = 0$, the Jacobi identity implies

$$[L, \tau] = \big[L, [\Lambda, d'\omega]\big] = -\big[d'\omega, [L, \Lambda]\big] = 3d'\omega,$$

taking into account Cor. VI-10.4 and the fact that $d'\omega$ is of degree 3.

b) By 1.1 a) we have $\tau = -\mathrm{i}[\delta'', L] - D'$, hence

$$[\Lambda, \tau] = -i\big[\Lambda, [\delta'', L]\big] - [\Lambda, D'] = -i\big(\big[\Lambda, [\delta'', L]\big] + \delta'' + \overline{\tau}^\star\big).$$

Using again VI-10.4 and the Jacobi identity, we get

$$\begin{aligned}\big[\Lambda, [\delta'', L]\big] &= -\big[L, [\Lambda, \delta'']\big] - \big[\delta'', [L, \Lambda]\big]\\ &= -\big[[d'', L], \Lambda\big]^\star - \delta'' = -[d''\omega, \Lambda]^\star - \delta'' = \overline{\tau}^\star - \delta''.\end{aligned}$$

A substitution in the previous equality gives $[\Lambda, \tau] = -2\mathrm{i}\overline{\tau}^\star$. □

(1.6) Lemma. *The following identities hold:*

a) $[D', \overline{\tau}^\star] = -[D', \delta''] = [\tau, \delta'']$,

b) $-[D'', \overline{\tau}^\star] = [\tau, \delta' + \tau^\star] + T_\omega$.

Proof. a) The Jacobi identity implies

$$-\big[D', [\Lambda, D']\big] + \big[D', [D', \Lambda]\big] + \big[\Lambda, [D', D']\big] = 0,$$

hence $-2\big[D', [\Lambda, D']\big] = 0$ and likewise $\big[\delta'', [\delta'', L]\big] = 0$. Assertion a) is now a consequence of 1.1 a) and d).

b) In order to verify b), we start from the equality $\overline{\tau}^\star = \frac{\mathrm{i}}{2}[\Lambda, \tau]$ provided by Lemma 1.5 b). It follows that

$$(1.7) \quad [D'', \overline{\tau}^\star] = \frac{\mathrm{i}}{2}\big[D'', [\Lambda, \tau]\big].$$

The Jacobi identity will now be used several times. One obtains

$$(1.8)\qquad \big[D'',[\Lambda,\tau]\big]=\big[\Lambda,[\tau,D'']\big]+\big[\tau,[D'',\Lambda]\big]\ ;$$

$$(1.9)\qquad \begin{aligned}[\tau,D'']=[D'',\tau]=\big[D'',[\Lambda,d'\omega]\big]&=\big[\Lambda,[d'\omega,D'']\big]+\big[d'\omega,[D'',\Lambda]\big]\\ &=[\Lambda,d''d'\omega]+[d'\omega,A]\end{aligned}$$

with $A=[D'',\Lambda]=\mathrm{i}(\delta'+\tau^\star)$. From (1.9) we deduce

$$(1.10)\qquad \big[\Lambda,[\tau,D'']\big]=\big[\Lambda,[\Lambda,d''d'\omega]\big]+\big[\Lambda,[d'\omega,A]\big].$$

Let us compute now the second Lie bracket in the right hand side of (1.10:

$$(1.11)\qquad \big[\Lambda,[d'\omega,A]\big]=\big[A,[\Lambda,d'\omega]\big]-\big[d'\omega,[A,\Lambda]\big]=[\tau,A]+\big[d'\omega,[\Lambda,A]\big]\ ;$$

$$(1.12)\qquad [\Lambda,A]=\mathrm{i}[\Lambda,\delta'+\tau^\star]=\mathrm{i}[D'+\tau,L]^\star.$$

Lemma 1.5 b) provides $[\tau,L]=-3d'\omega$, and it is clear that $[D',L]=d'\omega$. Equalities (1.12) and (1.11) yield therefore

$$[\Lambda,A]=-2\mathrm{i}(d'\omega)^\star,$$

$$(1.13)\qquad \big[\Lambda,[d'\omega,A]\big]=\big[\tau,[D'',\Lambda]\big]-2\mathrm{i}[d'\omega,(d'\omega)^\star].$$

Substituting (1.10) and (1.13) in (1.8) we get

$$(1.14)\qquad \begin{aligned}\big[D'',[\Lambda,\tau]\big]&=\big[\Lambda,[\Lambda,d''d'\omega]\big]+2\big[\tau,[D'',\Lambda]\big]-2\mathrm{i}\big[d'\omega,(d'\omega)^\star\big]\\ &=2\mathrm{i}\big(T_\omega+[\tau,\delta'+\tau^\star]\big).\end{aligned}$$

Formula b) is a consequence of (1.7) and (1.14). □

Theorem 1.4 follows now from Th. 1.2 if Formula 1.6 b) is rewritten

$$\Delta'+[D',\tau^\star]-[D'',\overline{\tau}^\star]=[D'+\tau,\delta'+\tau^\star]+T_\omega.$$

When ω is Kähler, then $\tau=T_\omega=0$ and Lemma 1.6 a) shows that $[D',\delta'']=0$. Together with the adjoint relation $[D'',\delta']=0$, this equality implies

$$(1.15)\qquad \Delta=\Delta'+\Delta''.$$

When ω is not Kähler, Lemma 1.6 a) can be written $[D'+\tau,\delta'']=0$ and we obtain more generally

$$[D+\tau,\delta+\tau^\star]=\big[(D'+\tau)+D'',(\delta'+\tau^\star)+\delta''\big]=\Delta'_\tau+\Delta''.$$

(1.16) Proposition. *Set $\Delta_\tau=[D+\tau,\delta+\tau^\star]$. Then $\Delta_\tau=\Delta'_\tau+\Delta''$.*

2. Basic a Priori Inequality

Let (X, ω) be a *compact* hermitian manifold, $\dim_{\mathbb{C}} X = n$, and E a hermitian holomorphic vector bundle over X. For any section $u \in C^\infty_{p,q}(X, E)$ we have $\langle\!\langle \Delta'' u, u \rangle\!\rangle = \|D'' u\|^2 + \|\delta'' u\|^2$ and the similar formula for Δ'_τ gives $\langle\!\langle \Delta'_\tau u, u \rangle\!\rangle \geq 0$. Theorem 1.4 implies therefore

$$\text{(2.1)} \quad \|D'' u\|^2 + \|\delta'' u\|^2 \geq \int_X \big(\langle [\mathrm{i}\Theta(E), \Lambda] u, u \rangle + \langle T_\omega u, u \rangle\big) dV.$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality. When u is Δ''-harmonic, we get in particular

$$\text{(2.2)} \quad \int_X \big(\langle [\mathrm{i}\Theta(E), \Lambda] u, u \rangle + \langle T_\omega u, u \rangle\big) dV \leq 0.$$

These basic a priori estimates are the starting point of all vanishing theorems. Observe that $[\mathrm{i}\Theta(E), \Lambda] + T_\omega$ is a hermitian operator acting pointwise on $\Lambda^{p,q} T^\star X \otimes E$ (the hermitian property can be seen from the fact that this operator coincides with $\Delta'' - \Delta'_\tau$ on smooth sections). Using Hodge theory (Cor. VI-11.2), we get:

(2.3) Corollary. *If the hermitian operator* $[\mathrm{i}\Theta(E), \Lambda] + T_\omega$ *is positive definite on* $\Lambda^{p,q} T^\star X \otimes E$*, then* $H^{p,q}(X, E) = 0$. □

In some circumstances, one can improve Cor. 2.3 thanks to the following "analytic continuation lemma" due to (Aronszajn 1957):

(2.4) Lemma. *Let M be a connected C^∞-manifold, F a vector bundle over M, and P a second order elliptic differential operator acting on $C^\infty(M, F)$. Then any section $\alpha \in \ker P$ vanishing on a non-empty open subset of M vanishes identically on M.*

(2.5) Corollary. *Assume that X is compact and connected. If*

$$[\mathrm{i}\Theta(E), \Lambda] + T_\omega \in \mathrm{Herm}\big(\Lambda^{p,q} T^\star X \otimes E\big)$$

is semi-positive on X and positive definite in at least one point $x_0 \in X$, then $H^{p,q}(X, E) = 0$.

Proof. By (2.2) every Δ''-harmonic (p, q)-form u must vanish in the neighborhood of x_0 where $[\mathrm{i}\Theta(E), \Lambda] + T_\omega > 0$, thus $u \equiv 0$. Hodge theory implies $H^{p,q}(X, E) = 0$. □

3. Kodaira-Akizuki-Nakano Vanishing Theorem

The main goal of vanishing theorems is to find natural geometric or algebraic conditions on a bundle E that will ensure that some cohomology groups with values in E vanish. In the next three sections, we prove various vanishing theorems for cohomology groups of a hermitian *line bundle* E over a *compact* complex manifold X.

(3.1) Definition. *A hermitian holomorphic line bundle E on X is said to be positive (resp. negative) if the hermitian matrix $\big(c_{jk}(z)\big)$ of its Chern curvature form*

$$\mathrm{i}\Theta(E) = \mathrm{i}\sum_{1\le j,k\le n} c_{jk}(z)\,dz_j\wedge d\overline{z}_k$$

is positive (resp. negative) definite at every point $z\in X$.

Assume that X has a Kähler metric ω. Let

$$\gamma_1(x)\le\ldots\le\gamma_n(x)$$

be the eigenvalues of $\mathrm{i}\Theta(E)_x$ with respect to ω_x at each point $x\in X$, and let

$$\mathrm{i}\Theta(E)_x = \mathrm{i}\sum_{1\le j\le n}\gamma_j(x)\,\zeta_j\wedge\overline{\zeta}_j,\qquad \zeta_j\in T^\star_x X$$

be a diagonalization of $\mathrm{i}\Theta(E)_x$. By Prop. VI-8.3 we have

$$\begin{aligned}\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle &= \sum_{J,K}\Big(\sum_{j\in J}\gamma_j+\sum_{j\in K}\gamma_j-\sum_{1\le j\le n}\gamma_j\Big)|u_{J,K}|^2\\ &\ge(\gamma_1+\ldots+\gamma_q-\gamma_{p+1}-\ldots-\gamma_n)|u|^2\end{aligned}\tag{3.2}$$

for any form $u=\sum_{J,K}u_{J,K}\,\zeta_J\wedge\overline{\zeta}_K\in\Lambda^{p,q}T^\star X$.

(3.3) Akizuki-Nakano vanishing theorem (1954). *Let E be a holomorphic line bundle on X.*

a) *If E is positive, then $H^{p,q}(X,E)=0$ for $p+q\ge n+1$.*

b) *If E is negative, then $H^{p,q}(X,E)=0$ for $p+q\le n-1$.*

Proof. In case a), choose $\omega=\mathrm{i}\Theta(E)$ as a Kähler metric on X. Then we have $\gamma_j(x)=1$ for all j and x, so that

$$\langle\!\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle\!\rangle\ge(p+q-n)||u||^2$$

for any $u\in\Lambda^{p,q}T^\star X\otimes E$. Assertion a) follows now from Corollary 2.3. Property b) is proved similarly, by taking $\omega=-\mathrm{i}\Theta(E)$. One can also derive b) from a) by Serre duality (Theorem VI-11.3). □

When $p = 0$ or $p = n$, Th. 3.3 can be generalized to the case where $\mathrm{i}\Theta(E)$ degenerates at some points. We use here the standard notations

$$\text{(3.4)} \quad \Omega_X^p = \Lambda^p T^\star X, \qquad K_X = \Lambda^n T^\star X, \qquad n = \dim_{\mathbb{C}} X \ ;$$

K_X is called the *canonical line bundle* of X.

(3.5) Theorem (Grauert-Riemenschneider 1970). *Let (X, ω) be a compact and connected Kähler manifold and E a line bundle on X.*

a) *If $\mathrm{i}\Theta(E) \geq 0$ on X and $\mathrm{i}\Theta(E) > 0$ in at least one point $x_0 \in X$, then*
$$H^q(X, K_X \otimes E) = 0 \quad \text{for} \quad q \geq 1.$$
b) *If $\mathrm{i}\Theta(E) \leq 0$ on X and $\mathrm{i}\Theta(E) < 0$ in at least one point $x_0 \in X$, then*
$$H^q(X, E) = 0 \quad \text{for} \quad q \leq n - 1.$$

It will be proved in Volume II, by means of holomorphic Morse inequalities, that the Kähler assumption is in fact unnecessary. This improvement is a deep result first proved by (Siu 1984) with a different ad hoc method.

Proof. For $p = n$, formula (3.2) gives

$$\text{(3.6)} \quad \langle\!\langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle\!\rangle \geq (\gamma_1 + \ldots + \gamma_q)|u|^2$$

and a) follows from Cor. 2.5. Now b) is a consequence of a) by Serre duality. □

4. Girbau's Vanishing Theorem

Let E be a line bundle over a compact connected Kähler manifold (X, ω). Girbau's theorem deals with the (possibly everywhere) degenerate semi-positive case. We first state the corresponding generalization of Th. 4.5.

(4.1) Theorem. *If $\mathrm{i}\Theta(E)$ is semi-positive and has at least $n - s + 1$ positive eigenvalues at a point $x_0 \in X$ for some integer $s \in \{1, \ldots, n\}$, then*

$$H^q(X, K_X \otimes E) = 0 \quad \text{for} \quad q \geq s.$$

Proof. Apply 2.5 and inequality (3.6), and observe that $\gamma_q(x_0) > 0$ for all $q \geq s$. □

(4.2) Theorem (Girbau 1976). *If $\mathrm{i}\Theta(E)$ is semi-positive and has at least $n - s + 1$ positive eigenvalues at every point $x \in X$, then*

$$H^{p,q}(X, E) = 0 \quad \text{for} \quad p + q \geq n + s.$$

Proof. Let us consider on X the new Kähler metric

$$\omega_\varepsilon = \varepsilon\omega + \mathrm{i}\Theta(E), \qquad \varepsilon > 0,$$

and let $\mathrm{i}\Theta(E) = \mathrm{i}\sum \gamma_j\, \zeta_j \wedge \overline{\zeta}_j$ be a diagonalization of $\mathrm{i}\Theta(E)$ with respect to ω and with $\gamma_1 \le \ldots \le \gamma_n$. Then

$$\omega_\varepsilon = \mathrm{i}\sum\, (\varepsilon + \gamma_j)\, \zeta_j \wedge \overline{\zeta}_j.$$

The eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε are given therefore by

$$(4.3) \qquad \gamma_{j,\varepsilon} = \gamma_j/(\varepsilon + \gamma_j) \in [0, 1[, \qquad 1 \le j \le n.$$

On the other hand, the hypothesis is equivalent to $\gamma_s > 0$ on X. For $j \ge s$ we have $\gamma_j \ge \gamma_s$, thus

$$(4.4) \qquad \gamma_{j,\varepsilon} = \frac{1}{1 + \varepsilon/\gamma_j} \ge \frac{1}{1 + \varepsilon/\gamma_s} \ge 1 - \varepsilon/\gamma_s, \qquad s \le j \le n.$$

Let us denote the operators and inner products associated to ω_ε with ε as an index. Then inequality (3.2) combined with (4.4) implies

$$\begin{aligned}\langle[\mathrm{i}\Theta(E), \Lambda_\varepsilon]u, u\rangle_\varepsilon &\ge \Big(\big(q - s + 1\big)\big)(1 - \varepsilon/\gamma_s) - (n - p)\Big)|u|^2 \\ &= \big(p + q - n - s + 1 - (q - s + 1)\varepsilon/\gamma_s\big)|u|^2.\end{aligned}$$

Theorem 4.2 follows now from Cor. 2.3 if we choose

$$\varepsilon < \frac{p + q - n - s + 1}{q - s + 1}\ \min_{x \in X} \gamma_s(x). \qquad \square$$

(4.5) Remark. The following example due to (Ramanujam 1972, 1974) shows that Girbau's result is no longer true for $p < n$ when $\mathrm{i}\Theta(E)$ is only assumed to have $n - s + 1$ positive eigenvalues on a dense open set.

Let V be a hermitian vector space of dimension $n+1$ and X the manifold obtained from $P(V) \simeq \mathbb{P}^n$ by blowing-up one point a. The manifold X may be described as follows: if $P(V/\mathbb{C}a)$ is the projective space of lines ℓ containing a, then

$$X = \big\{(x, \ell) \in P(V) \times P(V/\mathbb{C}a)\ ;\ x \in \ell\big\}.$$

We have two natural projections

$$\begin{aligned}\pi_1 &: X \longrightarrow P(V) \simeq \mathbb{P}^n, \\ \pi_2 &: X \longrightarrow Y = P(V/\mathbb{C}a) \simeq \mathbb{P}^{n-1}.\end{aligned}$$

It is clear that the preimage $\pi_1^{-1}(x)$ is the single point $\big(x, \ell = (ax)\big)$ if $x \neq a$ and that $\pi_1^{-1}(a) = \{a\} \times Y \simeq \mathbb{P}^{n-1}$, therefore

$$\pi_1 \;:\; X \setminus (\{a\} \times Y) \longrightarrow P(V) \setminus \{a\}$$

is an isomorphism. On the other hand, π_2 is a locally trivial fiber bundle over Y with fiber $\pi_2^{-1}(\ell) = \ell \simeq \mathbb{P}^1$, in particular X is smooth and n-dimensional. Consider now the line bundle $E = \pi_1^\star \mathcal{O}(1)$ over X, with the hermitian metric induced by that of $\mathcal{O}(1)$. Then E is semi-positive and $\mathrm{i}\Theta(E)$ has n positive eigenvalues at every point of $X \setminus (\{a\} \times Y)$, hence the assumption of Th. 4.2 is satisfied on $X \setminus (\{a\} \times Y)$. However, we will see that

$$H^{p,p}(X, E) \neq 0, \qquad 0 \leq p \leq n-1,$$

in contradiction with the expected generalization of (4.2) when $2p \geq n+1$. Let $j : Y \simeq \{a\} \times Y \longrightarrow X$ be the inclusion. Then $\pi_1 \circ j \;:\; Y \to \{a\}$ and $\pi_2 \circ j = \mathrm{Id}_Y$; in particular $j^\star E = (\pi_1 \circ j)^\star \mathcal{O}(1)$ is the trivial bundle $Y \times \mathcal{O}(1)_a$. Consider now the composite morphism

$$\begin{aligned} H^{p,p}(Y, \mathbb{C}) \otimes H^0\big(P(V), \mathcal{O}(1)\big) &\longrightarrow H^{p,p}(X, E) \xrightarrow{\;j^\star\;} H^{p,p}(Y, \mathbb{C}) \otimes \mathcal{O}(1)_a \\ u \otimes s \qquad\qquad &\longmapsto \pi_2^\star u \otimes \pi_1^\star s, \end{aligned}$$

given by $u \otimes s \longmapsto (\pi_2 \circ j)^\star u \otimes (\pi_1 \circ j)^\star s = u \otimes s(a)$; it is surjective and $H^{p,p}(Y, \mathbb{C}) \neq 0$ for $0 \leq p \leq n-1$, so we have $H^{p,p}(X, E) \neq 0$. □

5. Vanishing Theorem for Partially Positive Line Bundles

Even in the case when the curvature form $\mathrm{i}\Theta(E)$ is not semi-positive, some cohomology groups of high tensor powers E^k still vanish under suitable assumptions. The prototype of such results is the following assertion, which can be seen as a consequence of the Andreotti-Grauert theorem (Andreotti-Grauert 1962), see IX-?.?; the special case where E is > 0 (that is, $s = 1$) is due to (Kodaira 1953) and (Serre 1956).

(5.1) Theorem. *Let F be a holomorphic vector bundle over a compact complex manifold X, s a positive integer and E a hermitian line bundle such that $\mathrm{i}\Theta(E)$ has at least $n - s + 1$ positive eigenvalues at every point $x \in X$. Then there exists an integer $k_0 \geq 0$ such that*

$$H^q(X, E^k \otimes F) = 0 \qquad \text{for} \quad q \geq s \quad \text{and} \quad k \geq k_0.$$

Proof. The main idea is to construct a hermitian metric ω_ε on X in such a way that all negative eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε will be of small absolute value. Let ω denote a fixed hermitian metric on X and let $\gamma_1 \leq \ldots \leq \gamma_n$ be the corresponding eigenvalues of $\mathrm{i}\Theta(E)$.

(5.2) Lemma. *Let $\psi \in C^\infty(\mathbb{R},\mathbb{R})$. If A is a hermitian $n\times n$ matrix with eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ and corresponding eigenvectors $v_1,\ldots,v_n$, we define $\psi[A]$ as the hermitian matrix with eigenvalues $\psi(\lambda_j)$ and eigenvectors v_j, $1\le j\le n$. Then the map $A \longmapsto \psi[A]$ is C^∞ on $\mathrm{Herm}(\mathbb{C}^n)$.*

Proof. Although the result is very well known, we give here a short proof. Without loss of generality, we may assume that ψ is compactly supported. Then we have

$$\psi[A] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\psi}(t) e^{itA} dt$$

where $\widehat{\psi}$ is the rapidly decreasing Fourier transform of ψ. The equality $\int_0^t (t-u)^p u^q\,du = p!\,q!/(p+q+1)!$ and obvious power series developments yield

$$D_A(e^{itA})\cdot B = i\int_0^t e^{\mathrm{i}(t-u)A}\,B\,e^{iuA}du.$$

Since e^{iuA} is unitary, we get $\|D_A(e^{itA})\| \le |t|$. A differentiation under the integral sign and Leibniz' formula imply by induction on k the bound $\|D_A^k(e^{itA})\| \le |t|^k$. Hence $A\longmapsto \psi[A]$ is smooth. □

Let us consider now the positive numbers

$$t_0 = \inf_X \gamma_s > 0, \qquad M = \sup_X \max_j |\gamma_j| > 0.$$

We select a function $\psi_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\psi_\varepsilon(t) = t \ \text{ for } t\ge t_0,\quad \psi_\varepsilon(t)\ge t \ \text{ for } 0\le t\le t_0,\quad \psi_\varepsilon(t) = M/\varepsilon \ \text{ for } t\le 0.$$

By Lemma 5.2, $\omega_\varepsilon := \psi_\varepsilon[\mathrm{i}\Theta(E)]$ is a smooth hermitian metric on X. Let us write

$$\mathrm{i}\Theta(E) = \mathrm{i}\sum_{1\le j\le n} \gamma_j\,\zeta_j\wedge\overline{\zeta}_j,\qquad \omega_\varepsilon = \mathrm{i}\sum_{1\le j\le n}\psi_\varepsilon(\gamma_j)\,\zeta_j\wedge\overline{\zeta}_j$$

in an orthonormal basis $(\zeta_1,\ldots,\zeta_n)$ of $T^\star X$ for ω. The eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε are given by $\gamma_{j,\varepsilon} = \gamma_j/\psi_\varepsilon(\gamma_j)$ and the construction of ψ_ε shows that $-\varepsilon \le \gamma_{j,\varepsilon}\le 1$, $1\le j\le n$, and $\gamma_{j,\varepsilon}=1$ for $s\le j\le n$. Now, we have

$$H^q(X, E^k\otimes F) \simeq H^{n,q}(X, E^k\otimes G)$$

where $G = F\otimes K_X^\star$. Let e, $(g_\lambda)_{1\le\lambda\le r}$ and $(\zeta_j)_{1\le j\le n}$ denote orthonormal frames of E, G and $(T^\star X,\omega_\varepsilon)$ respectively. For

$$u = \sum_{|J|=q,\lambda} u_{J,\lambda}\,\zeta_1\wedge\ldots\wedge\zeta_n\wedge\overline{\zeta}_J\otimes e^k\otimes g_\lambda \in \Lambda^{n,q}T^\star X\otimes E^k\otimes G,$$

inequality (3.2) yields

$$\langle[\mathrm{i}\Theta(E),\Lambda_\varepsilon]u,u\rangle_\varepsilon=\sum_{J,\lambda}\Big(\sum_{j\in J}\gamma_{j,\varepsilon}\Big)|u_{J,\lambda}|^2\ge\big(q-s+1-(s-1)\varepsilon\big)|u|^2.$$

Choosing $\varepsilon=1/s$ and $q\ge s$, the right hand side becomes $\ge(1/s)|u|^2$. Since $\Theta(E^k\otimes G)=k\Theta(E)\otimes\mathrm{Id}_G+\Theta(G)$, there exists an integer k_0 such that

$$\big[\mathrm{i}\Theta(E^k\otimes G),\Lambda_\varepsilon\big]+T_{\omega_\varepsilon}\quad\text{acting on}\quad\Lambda^{n,q}T^\star X\otimes E^k\otimes G$$

is positive definite for $q\ge s$ and $k\ge k_0$. The proof is complete. □

6. Positivity Concepts for Vector Bundles

Let E be a hermitian holomorphic vector bundle of rank r over X, where $\dim_{\mathbb{C}}X=n$. Denote by $(e_1,\ldots,e_r)$ an orthonormal frame of E over a coordinate patch $\Omega\subset X$ with complex coordinates $(z_1,\ldots,z_n)$, and

$$(6.1)\quad \mathrm{i}\Theta(E)=\mathrm{i}\sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r}c_{jk\lambda\mu}\,dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu,\qquad \overline{c}_{jk\lambda\mu}=c_{kj\mu\lambda}$$

the Chern curvature tensor. To $\mathrm{i}\Theta(E)$ corresponds a natural hermitian form θ_E on $TX\otimes E$ defined by

$$\theta_E=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}(dz_j\otimes e_\lambda^\star)\otimes(\overline{dz_k\otimes e_\mu^\star}),$$

and such that

$$\theta_E(u,u)=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}(x)\,u_{j\lambda}\overline{u}_{k\mu},\qquad u\in T_xX\otimes E_x.(6.2)$$

(6.3) Definition (Nakano 1955). *E is said to be Nakano positive (resp. Nakano semi-negative) if θ_E is positive definite (resp. semi-negative) as a hermitian form on $TX\otimes E$, i.e. if for every $u\in TX\otimes E$, $u\ne 0$, we have*

$$\theta_E(u,u)>0\quad(\textit{resp.}\le 0).$$

We write $>_{\mathrm{Nak}}$ (resp. $\le_{\mathrm{Nak}}$) for Nakano positivity (resp. semi-negativity).

(6.4) Definition (Griffiths 1969). *E is said to be Griffiths positive (resp. Griffiths semi-negative) if for all $\xi\in T_xX$, $\xi\ne 0$ and $s\in E_x$, $s\ne 0$ we have*

$$\theta_E(\xi\otimes s,\xi\otimes s)>0\quad(\textit{resp.}\le 0).$$

We write $>_{\mathrm{Grif}}$ (resp. $\le_{\mathrm{Grif}}$) for Griffiths positivity (resp. semi-negativity).

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if $r = 1$ (in the case of a line bundle, E is merely said to be positive). One can generalize further by introducing additional concepts of positivity which interpolate between Griffiths positivity and Nakano positivity.

(6.5) Definition. *Let T and E be complex vector spaces of dimensions n, r respectively, and let Θ be a hermitian form on $T \otimes E$.*

a) *A tensor $u \in T \otimes E$ is said to be of rank m if m is the smallest ≥ 0 integer such that u can be written*

$$u = \sum_{j=1}^{m} \xi_j \otimes s_j, \qquad \xi_j \in T,\ s_j \in E.$$

b) *Θ is said to be m-positive (resp. m-semi-negative) if $\Theta(u,u) > 0$ (resp. $\Theta(u,u) \leq 0$) for every tensor $u \in T \otimes E$ of rank $\leq m$, $u \neq 0$. In this case, we write*

$$\Theta >_m 0 \qquad (\textit{resp.}\ \ \Theta \leq_m 0).$$

We say that the bundle E is m-positive if $\theta_E >_m 0$. Griffiths positivity corresponds to $m = 1$ and Nakano positivity to $m \geq \min(n, r)$.

(6.6) Proposition. *A bundle E is Griffiths positive if and only if $E^\star$ is Griffiths negative.*

Proof. By (V-4.3′) we get $\mathrm{i}\Theta(E^\star) = -\mathrm{i}\Theta(E)^\dagger$, hence

$$\theta_{E^\star}(\xi_1 \otimes \overline{s}_2, \xi_2 \otimes \overline{s}_1) = -\theta_E(\xi_1 \otimes s_1, \xi_2 \otimes s_2), \qquad \forall \xi_1, \xi_2 \in TX,\ \forall s_1, s_2 \in E,$$

where $\overline{s}_j = \langle \bullet, s_j \rangle \in E^\star$. Proposition 6.6 follows immediately. □

It should be observed that the corresponding duality property for Nakano positive bundles is *not true.* In fact, using (6.1) we get

$$\mathrm{i}\Theta(E^\star) = -\mathrm{i} \sum_{j,k,\lambda,\mu} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes e_\lambda^{\star\star} \otimes e_\mu^\star,$$

$$(6.7) \quad \theta_{E^\star}(v,v) = -\sum_{j,k,\mu,\lambda} c_{jk\mu\lambda} v_{j\lambda} \overline{v}_{k\mu},$$

for any $v = \sum v_{j\lambda}\,(\partial/\partial z_j) \otimes e_\lambda^\star \in TX \otimes E^\star$. The following example shows that Nakano positivity or negativity of θ_E and $\theta_{E^\star}$ are unrelated.

(6.8) Example. Let H be the rank n bundle over $\mathbb{P}^n$ defined in § V-15. For any $u = \sum u_{j\lambda}(\partial/\partial z_j) \otimes \widetilde{e}_\lambda \in TX \otimes H$, $v = \sum v_{j\lambda}(\partial/\partial z_j) \otimes \widetilde{e}_\lambda^\star \in TX \otimes H^\star$, $1 \leq j, \lambda \leq n$, formula (V-15.9) implies

$$(6.9)\qquad \begin{cases} \theta_H(u,u) = \sum u_{j\lambda}\overline{u}_{\lambda j} \\ \theta_{H^\star}(v,v) = \sum v_{jj}\overline{v}_{\lambda\lambda} = \big|\sum v_{jj}\big|^2. \end{cases}$$

It is then clear that $H \geq_{\mathrm{Grif}} 0$ and $H^\star \leq_{\mathrm{Nak}} 0$, but H is neither $\geq_{\mathrm{Nak}} 0$ nor $\leq_{\mathrm{Nak}} 0$.

(6.10) Proposition. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then*

a) $E \geq_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad Q \geq_{\mathrm{Grif}} 0$,

b) $E \leq_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad S \leq_{\mathrm{Grif}} 0$,

c) $E \leq_{\mathrm{Nak}} 0 \quad\Longrightarrow\quad S \leq_{\mathrm{Nak}} 0$,

and analogous implications hold true for strict positivity.

Proof. If β is written $\sum dz_j \otimes \beta_j$, $\beta_j \in \hom(S,Q)$, then formulas (V-14.6) and (V-14.7) yield

$$\mathrm{i}\Theta(S) = \mathrm{i}\Theta(E)_{\restriction S} - \sum dz_j \wedge d\overline{z}_k \otimes \beta_k^\star\beta_j,$$
$$\mathrm{i}\Theta(Q) = \mathrm{i}\Theta(E)_{\restriction Q} + \sum dz_j \wedge d\overline{z}_k \otimes \beta_j\beta_k^\star.$$

Since $\beta\cdot(\xi\otimes s) = \sum \xi_j\beta_j\cdot s$ and $\beta^\star\cdot(\xi\otimes s) = \sum \overline{\xi}_k\beta_k^\star\cdot s$ we get

$$\theta_S(\xi\otimes s,\xi'\otimes s') = \theta_E(\xi\otimes s,\xi'\otimes s') - \sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_j\cdot s,\beta_k\cdot s'\rangle,$$

$$\theta_S(u,u) = \theta_E(u,u) - |\beta\cdot u|^2,$$

$$\theta_Q(\xi\otimes s,\xi'\otimes s') = \theta_E(\xi\otimes s,\xi'\otimes s') + \sum_{j,k}\xi_j\overline{\xi}'_k\langle\beta_k^\star\cdot s,\beta_j^\star\cdot s'\rangle,$$

$$\theta_Q(\xi\otimes s,\xi\otimes s) = \theta_E(\xi\otimes s,\xi\otimes s) + |\beta^\star\cdot(\xi\otimes s)|^2. \qquad\square$$

Since H is a quotient bundle of the trivial bundle $\underline{V}$, Example 6.8 shows that $E \geq_{\mathrm{Nak}} 0$ *does not imply* $Q \geq_{\mathrm{Nak}} 0$.

7. Nakano Vanishing Theorem

Let (X,ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} X = n$, and $E \longrightarrow X$ a hermitian vector bundle of rank r. We are going to compute explicitly the hermitian operator $[\mathrm{i}\Theta(E),\Lambda]$ acting on $\Lambda^{p,q}T^\star X\otimes E$. Let $x_0\in X$ and $(z_1,\ldots,z_n)$ be local coordinates such that $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ is an orthonormal basis of (TX,ω) at x_0. One can write

$$\begin{aligned}\omega_{x_0} &= \mathrm{i}\sum_{1\le j\le n} dz_j\wedge d\overline{z}_j,\\ \mathrm{i}\Theta(E)_{x_0} &= \mathrm{i}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu\end{aligned}$$

where $(e_1,\ldots,e_r)$ is an orthonormal basis of E_{x_0}. Let

$$u=\sum_{|J|=p,\,|K|=q,\,\lambda} u_{J,K,\lambda}\, dz_J\wedge d\overline{z}_K\otimes e_\lambda \in \big(\Lambda^{p,q}T^\star X\otimes E\big)_{x_0}.$$

A simple computation as in the proof of Prop. VI-8.3 gives

$$\begin{aligned}\Lambda u &= \mathrm{i}(-1)^p\sum_{J,K,\lambda,s} u_{J,K,\lambda}\Big(\frac{\partial}{\partial z_s}\lrcorner\, dz_J\Big)\wedge\Big(\frac{\partial}{\partial\overline{z}_s}\lrcorner\, d\overline{z}_K\Big)\otimes e_\lambda,\\ \mathrm{i}\Theta(E)\wedge u &= \mathrm{i}(-1)^p\sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_j\wedge dz_J\wedge d\overline{z}_k\wedge d\overline{z}_K\otimes e_\mu,\\ [\mathrm{i}\Theta(E),\Lambda]u &= \sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_j\wedge\Big(\frac{\partial}{\partial z_k}\lrcorner\, dz_J\Big)\wedge d\overline{z}_K\otimes e_\mu\\ &\quad+\sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\,u_{J,K,\lambda}\,dz_J\wedge d\overline{z}_k\wedge\Big(\frac{\partial}{\partial\overline{z}_j}\lrcorner\, d\overline{z}_K\Big)\otimes e_\mu\\ &\quad-\sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\,u_{J,K,\lambda}\,dz_J\wedge d\overline{z}_K\otimes e_\mu.\end{aligned}$$

We extend the definition of $u_{J,K,\lambda}$ to non increasing multi-indices $J=(j_s)$, $K=(k_s)$ by deciding that $u_{J,K,\lambda}=0$ if J or K contains identical components repeated and that $u_{J,K,\lambda}$ is alternate in the indices (j_s), (k_s). Then the above equality can be written

$$\begin{aligned}\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle &= \sum c_{jk\lambda\mu}\,u_{J,jS,\lambda}\overline{u}_{J,kS,\mu}\\ &\quad+\sum c_{jk\lambda\mu}\,u_{kR,K,\lambda}\overline{u}_{jR,K,\mu}\\ &\quad-\sum c_{jj\lambda\mu}\,u_{J,K,\lambda}\overline{u}_{J,K,\mu},\end{aligned}$$

extended over all indices j,k,λ,μ,J,K,R,S with $|R|=p-1$, $|S|=q-1$. This hermitian form appears rather difficult to handle for general (p,q) because of sign compensation. Two interesting cases are $p=n$ and $q=n$.

• For $u = \sum u_{K,\lambda}\, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_K \otimes e_\lambda$ of type (n,q), we get

$$(7.1)\quad \langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle = \sum_{|S|=q-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, u_{jS,\lambda} \overline{u}_{kS,\mu},$$

because of the equality of the second and third summations in the general formula. Since $u_{jS,\lambda} = 0$ for $j \in S$, the rank of the tensor $(u_{jS,\lambda})_{j,\lambda} \in \mathbb{C}^n \otimes \mathbb{C}^r$ is in fact $\leq \min\{n-q+1, r\}$. We obtain therefore:

(7.2) Lemma. *Assume that $E >_m 0$ in the sense of Def. 6.5. Then the hermitian operator $[\mathrm{i}\Theta(E), \Lambda]$ is positive definite on $\Lambda^{n,q}T^\star X \otimes E$ for $q \geq 1$ and $m \geq \min\{n-q+1, r\}$.*

(7.3) Theorem. *Let X be a compact connected Kähler manifold of dimension n and E a hermitian vector bundle of rank r. If $\theta_E \geq_m 0$ on X and $\theta_E >_m 0$ in at least one point, then*

$$H^{n,q}(X, E) = H^q(X, K_X \otimes E) = 0 \qquad \text{for} \quad q \geq 1 \quad \text{and} \quad m \geq \min\{n-q+1, r\}.$$

• Similarly, for $u = \sum u_{J,\lambda}\, dz_J \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \otimes e_\lambda$ of type (p,n), we get

$$\langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle = \sum_{|R|=p-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, u_{kR,\lambda} \overline{u}_{jR,\mu},$$

because of the equality of the first and third summations in the general formula. The indices j,k are twisted, thus $[\mathrm{i}\Theta(E), \Lambda]$ defines a positive hermitian form under the assumption $\mathrm{i}\Theta(E)^\dagger >_m 0$, i.e. $\mathrm{i}\Theta(E^\star) <_m 0$, with $m \geq \min\{n-p+1, r\}$. Serre duality $\big(H^{p,0}(X,E)\big)^\star = H^{n-p,n}(X, E^\star)$ gives:

(7.4) Theorem. *Let X and E be as above. If $\theta_E \leq_m 0$ on X and $\theta_E <_m 0$ in at least one point, then*

$$H^{p,0}(X, E) = H^0(X, \Omega^p_X \otimes E) = 0 \qquad \text{for} \quad p < n \quad \text{and} \quad m \geq \min\{p+1, r\}.$$

The special case $m = r$ yields:

(7.5) Corollary. *For X and E as above:*

a) *Nakano vanishing theorem (1955):*

$E \geq_{\mathrm{Nak}} 0$, *strictly in one point* $\Longrightarrow$ $H^{n,q}(X,E) = 0$ *for* $q \geq 1$.

b) $E \leq_{\mathrm{Nak}} 0$, *strictly in one point* $\Longrightarrow$ $H^{p,0}(X,E) = 0$ *for* $p < n$.

8. Relations Between Nakano and Griffiths Positivity

It is clear that Nakano positivity implies Griffiths positivity. The main result of § 8 is the following "converse" to this property (Demailly-Skoda 1979).

(8.1) Theorem. *For any hermitian vector bundle E,*

$$E >_{\mathrm{Grif}} 0 \quad \Longrightarrow \quad E \otimes \det E >_{\mathrm{Nak}} 0.$$

To prove this result, we first use (V-4.2′) and (V-4.6). If $\mathrm{End}(E \otimes \det E)$ is identified to $\hom(E, E)$, one can write

$$\Theta(E \otimes \det E) = \Theta(E) + \mathrm{Tr}_E(\Theta(E)) \otimes \mathrm{Id}_E,$$

$$\theta_{E \otimes \det E} = \theta_E + \mathrm{Tr}_E\, \theta_E \otimes h,$$

where h denotes the hermitian metric on E and where $\mathrm{Tr}_E\, \theta_E$ is the hermitian form on TX defined by

$$\mathrm{Tr}_E\, \theta_E(\xi, \xi) = \sum_{1 \le \lambda \le r} \theta_E(\xi \otimes e_\lambda, \xi \otimes e_\lambda),\ \ \xi \in TX,$$

for any orthonormal frame $(e_1, \ldots, e_r)$ of E. Theorem 8.1 is now a consequence of the following simple property of hermitian forms on a tensor product of complex vector spaces.

(8.2) Proposition. *Let T, E be complex vector spaces of respective dimensions n, r, and h a hermitian metric on E. Then for every hermitian form Θ on $T \otimes E$*

$$\Theta >_{\mathrm{Grif}} 0 \quad \Longrightarrow \quad \Theta + \mathrm{Tr}_E\, \Theta \otimes h >_{\mathrm{Nak}} 0.$$

We first need a lemma analogous to Fourier inversion formula for discrete Fourier transforms.

(8.3) Lemma. *Let q be an integer ≥ 3, and $x_\lambda,\ y_\mu,\ 1 \le \lambda, \mu \le r$, be complex numbers. Let σ describe the set U_q^r of r-tuples of q-th roots of unity and put*

$$x'_\sigma = \sum_{1 \le \lambda \le r} x_\lambda \overline{\sigma}_\lambda, \qquad y'_\sigma = \sum_{1 \le \mu \le r} y_\mu \overline{\sigma}_\mu, \qquad \sigma \in U_q^r.$$

Then for every pair (α, β), $1 \le \alpha, \beta \le r$, the following identity holds:

$$q^{-r} \sum_{\sigma \in U_q^r} x'_\sigma \overline{y}'_\sigma \sigma_\alpha \overline{\sigma}_\beta = \begin{cases} x_\alpha \overline{y}_\beta & \text{if } \ \alpha \ne \beta, \\ \displaystyle\sum_{1 \le \mu \le r} x_\mu \overline{y}_\mu & \text{if } \ \alpha = \beta. \end{cases}$$

Proof. The coefficient of $x_\lambda \overline{y}_\mu$ in the summation $q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma \overline{y}'_\sigma \sigma_\alpha \overline{\sigma}_\beta$ is given by

$$q^{-r}\sum_{\sigma\in U_q^r} \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu.$$

This coefficient equals 1 when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are equal (in which case $\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = 1$ for any one of the q^r elements of U_q^r). Hence, it is sufficient to prove that

$$\sum_{\sigma\in U_q^r} \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = 0$$

when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are distinct.

If $\{\alpha,\mu\}\neq\{\beta,\lambda\}$, then one of the elements of one of the pairs does not belong to the other pair. As the four indices α,β,λ,μ play the same role, we may suppose for example that $\alpha\notin\{\beta,\lambda\}$. Let us apply to σ the substitution $\sigma\mapsto\tau$, where τ is defined by

$$\tau_\alpha = e^{2\pi\mathrm{i}/q}\sigma_\alpha,\ \tau_\nu=\sigma_\nu \quad \text{for} \quad \nu\neq\alpha.$$

We get

$$\sum_\sigma \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = \sum_\tau = \begin{cases} e^{2\pi\mathrm{i}/q}\displaystyle\sum_\sigma & \text{if } \alpha\neq\mu,\\ e^{4\pi\mathrm{i}/q}\displaystyle\sum_\sigma & \text{if } \alpha=\mu,\end{cases}$$

Since $q\geq 3$ by hypothesis, it follows that

$$\sum_\sigma \sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu = 0.$$

Proof of Proposition 8.2. Let $(t_j)_{1\leq j\leq n}$ be a basis of T, $(e_\lambda)_{1\leq\lambda\leq r}$ an orthonormal basis of E and $\xi=\sum_j \xi_j t_j\in T$, $u=\sum_{j,\lambda} u_{j\lambda}\, t_j\otimes e_\lambda\in T\otimes E$. The coefficients $c_{jk\lambda\mu}$ of Θ with respect to the basis $t_j\otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$, and we have the formulas

$$\begin{aligned}
\Theta(u,u) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu},\\
\mathrm{Tr}_E\,\Theta(\xi,\xi) &= \sum_{j,k,\lambda} c_{jk\lambda\lambda}\xi_j\overline{\xi}_k,\\
(\Theta+\mathrm{Tr}_E\,\Theta\otimes h)(u,u) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu} + c_{jk\lambda\lambda} u_{j\mu}\overline{u}_{k\mu}.
\end{aligned}$$

For every $\sigma\in U_q^r$ (cf. Lemma 8.3), put

$$u'_{j\sigma} = \sum_{1\le\lambda\le r} u_{j\lambda}\overline{\sigma}_\lambda \in \mathbb{C},$$
$$\widehat{u}_\sigma = \sum_j u'_{j\sigma} t_j \in T \quad , \quad \widehat{e}_\sigma = \sum_\lambda \sigma_\lambda e_\lambda \in E.$$

Lemma 8.3 implies

$$\begin{aligned} q^{-r} \sum_{\sigma\in U_q^r} \Theta(\widehat{u}_\sigma \otimes \widehat{e}_\sigma, \widehat{u}_\sigma \otimes \widehat{e}_\sigma) &= q^{-r} \sum_{\sigma\in U_q^r} c_{jk\lambda\mu} u'_{j\sigma}\overline{u}'_{k\sigma}\sigma_\lambda\overline{\sigma}_\mu \\ &= \sum_{j,k,\lambda\ne\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} u_{j\mu}\overline{u}_{k\mu}. \end{aligned}$$

The Griffiths positivity assumption shows that the left hand side is ≥ 0, hence

$$(\Theta + \mathrm{Tr}_E\,\Theta\otimes h)(u,u) \ge \sum_{j,k,\lambda} c_{jk\lambda\lambda} u_{j\lambda}\overline{u}_{k\lambda} \ge 0$$

with strict positivity if $\Theta >_{\mathrm{Grif}} 0$ and $u \ne 0$. □

(8.4) Example. Take $E = H$ over $\mathbb{P}^n = P(V)$. The exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow H \longrightarrow 0$$

implies $\det \underline{V} = \det H \otimes \mathcal{O}(-1)$. Since $\det \underline{V}$ is a trivial bundle, we get (non canonical) isomorphisms

$$\det H \simeq \mathcal{O}(1),$$
$$T\mathbb{P}^n = H \otimes \mathcal{O}(1) \simeq H \otimes \det H.$$

We already know that $H \ge_{\mathrm{Grif}} 0$, hence $T\mathbb{P}^n \ge_{\mathrm{Nak}} 0$. A direct computation based on (6.9) shows that

$$\begin{aligned} \theta_{T\mathbb{P}^n}(u,u) &= (\theta_H + \mathrm{Tr}_H\,\theta_H \otimes h)(u,u) \\ &= \sum_{1\le j,k\le n} u_{jk}\overline{u}_{kj} + u_{jk}\overline{u}_{jk} = \frac{1}{2}\sum_{1\le j,k\le n} |u_{jk} + u_{kj}|^2. \end{aligned}$$

In addition, we have $T\mathbb{P}^n >_{\mathrm{Grif}} 0$. However, the Serre duality theorem gives

$$\begin{aligned} H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T\mathbb{P}^n)^\star &\simeq H^{n-q}(\mathbb{P}^n, T^\star\mathbb{P}^n) \\ &= H^{1,n-q}(\mathbb{P}^n, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } q = n-1, \\ 0 & \text{if } q \ne n-1. \end{cases} \end{aligned}$$

For $n \ge 2$, Th. 7.3 implies that $T\mathbb{P}^n$ has no hermitian metric such that $\theta_{T\mathbb{P}^n} \ge_2 0$ on $\mathbb{P}^n$ and $\theta_{T\mathbb{P}^n} >_2 0$ in one point. This shows that the notion of 2-positivity is actually stronger than 1-positivity (i.e. Griffiths positivity).

(8.5) Remark. Since $\mathrm{Tr}_H\,\theta_H = \theta_{\mathcal{O}(1)}$ is positive and $\theta_{T\mathbb{P}^n}$ is not $>_{\mathrm{Nak}} 0$ when $n \ge 2$, we see that Prop. 8.2 is best possible in the sense that there cannot exist any constant $c < 1$ such that

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad \Theta + c\,\mathrm{Tr}_E\,\Theta \otimes h \ge_{\mathrm{Nak}} 0.$$

9. Applications to Griffiths Positive Bundles

We first need a preliminary result.

(9.1) Proposition. *Let T be a complex vector space and (E, h) a hermitian vector space of respective dimensions n, r with $r \ge 2$. Then for any hermitian form Θ on $T \otimes E$ and any integer $m \ge 1$*

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad m\,\mathrm{Tr}_E\,\Theta \otimes h - \Theta >_m 0.$$

Proof. Let us distinguish two cases.

a) $m = 1$. Let $u \in T \otimes E$ be a tensor of rank 1. Then u can be written $u = \xi_1 \otimes e_1$ with $\xi_1 \in T$, $\xi_1 \ne 0$, and $e_1 \in E$, $|e_1| = 1$. Complete e_1 into an orthonormal basis $(e_1, \ldots, e_r)$ of E. One gets immediately

$$\begin{aligned}(\mathrm{Tr}_E\,\Theta \otimes h)(u, u) = \mathrm{Tr}_E\,\Theta(\xi_1, \xi_1) &= \sum_{1\le\lambda\le r} \Theta(\xi_1 \otimes e_\lambda, \xi_1 \otimes e_\lambda)\\ &> \Theta(\xi_1 \otimes e_1, \xi_1 \otimes e_1) = \Theta(u, u).\end{aligned}$$

b) $m \ge 2$. Every tensor $u \in T \otimes E$ of rank $\le m$ can be written

$$u = \sum_{1\le\lambda\le q} \xi_\lambda \otimes e_\lambda \quad,\quad \xi_\lambda \in T,$$

with $q = \min(m, r)$ and $(e_\lambda)_{1\le\lambda\le r}$ an orthonormal basis of E. Let F be the vector subspace of E generated by $(e_1, \ldots, e_q)$ and Θ_F the restriction of Θ to $T \otimes F$. The first part shows that

$$\Theta' := \mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_{\mathrm{Grif}} 0.$$

Proposition 9.2 applied to Θ' on $T \otimes F$ yields

$$\Theta' + \mathrm{Tr}_F\,\Theta' \otimes h = q\,\mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_q 0.$$

Since $u \in T \otimes F$ is of rank $\le q \le m$, we get (for $u \ne 0$)

$$\begin{aligned}\Theta(u,u) = \Theta_F(u,u) &< q(\mathrm{Tr}_F\,\Theta_F \otimes h)(u,u)\\ &= q\sum_{1\le j,\lambda\le q}\Theta(\xi_j\otimes e_\lambda,\xi_j\otimes e_\lambda) \le m\,\mathrm{Tr}_E\,\Theta\otimes h(u,u)\square\end{aligned}$$

Proposition 9.1 is of course also true in the semi-positive case. From these facts, we deduce

(9.2) Theorem. *Let E be a Griffiths (semi-)positive bundle of rank $r \ge 2$. Then for any integer $m \ge 1$*

$$E^\star \otimes (\det E)^m >_m 0 \qquad (\textit{resp.}\quad \ge_m 0).$$

Proof. Apply Prop. 8.1 to $\Theta = -\theta_{E^\star} >_{\mathrm{Grif}} 0$ and observe that

$$\theta_{\det E} = -\theta_{\det E^\star} = \mathrm{Tr}_{E^\star}\,\Theta.$$

(9.3) Theorem. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then for any $m \ge 1$*

$$E >_m 0 \quad\Longrightarrow\quad S\otimes(\det Q)^m >_m 0.$$

Proof. Formulas (V-14.6) and (V-14.7) imply

$$\mathrm{i}\Theta(S) >_m \mathrm{i}\beta^\star\wedge\beta \quad,\quad \mathrm{i}\Theta(Q) >_m \mathrm{i}\beta\wedge\beta^\star,$$

$$\mathrm{i}\Theta(\det Q) = \mathrm{Tr}_Q(\mathrm{i}\Theta(Q)) > \mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^\star).$$

If we write $\beta = \sum dz_j\otimes\beta_j$ as in the proof of Prop. 6.10, then

$$\begin{aligned}\mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^\star) &= \sum idz_j\wedge d\overline{z}_k\,\mathrm{Tr}_Q(\beta_j\beta_k^\star)\\ &= \sum idz_j\wedge d\overline{z}_k\,\mathrm{Tr}_S(\beta_k^\star\beta_j) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star\wedge\beta).\end{aligned}$$

Furthermore, it has been already proved that $-\mathrm{i}\beta^\star\wedge\beta \ge_{\mathrm{Nak}} 0$. By Prop. 8.1 applied to the corresponding hermitian form Θ on $TX\otimes S$, we get

$$m\,\mathrm{Tr}_S(-\mathrm{i}\beta^\star\wedge\beta)\otimes\mathrm{Id}_S + \mathrm{i}\beta^\star\wedge\beta \ge_m 0,$$

and Th. 9.3 follows.

(9.4) Corollary. *Let X be a compact n-dimensional complex manifold, E a vector bundle of rank $r \ge 2$ and $m \ge 1$ an integer. Then*

a) $E >_{\mathrm{Grif}} 0 \Longrightarrow H^{n,q}(X, E\otimes\det E) = 0$ *for* $q \ge 1$;

b) $E >_{\mathrm{Grif}} 0 \Longrightarrow H^{n,q}\big(X, E^\star\otimes(\det E)^m\big) = 0$ *for* $q \ge 1$ *and* $m \ge \min\{n-q+1, r\}$;

c) *Let* $0 \to S \to E \to Q \to 0$ *be an exact sequence of vector bundles and* $m = \min\{n-q+1, \operatorname{rk} S\}$, $q \geq 1$. *If* $E >_m 0$ *and if* L *is a line bundle such that* $L \otimes (\det Q)^{-m} \geq 0$, *then*

$$H^{n,q}(X, S \otimes L) = 0.$$

Proof. Immediate consequence of Theorems 7.3, 8.1, 9.2 and 9.3. □

Note that under our hypotheses $\omega = \mathrm{i}\,\mathrm{Tr}_E\,\Theta(E) = \mathrm{i}\Theta(\Lambda^r E)$ is a Kähler metric on X. Corollary 2.5 shows that it is enough in a), b), c) to assume semi-positivity and strict positivity in one point (this is true a priori only if X is supposed in addition to be Kähler, but this hypothesis can be removed by means of Siu's result mentioned after (4.5).

a) is in fact a special case of a result of (Griffiths 1969), which we will prove in full generality in volume II (see the chapter on vanishing theorems for ample vector bundles); property b) will be also considerably strengthened there. Property c) is due to (Skoda 1978) for $q = 0$ and to (Demailly 1982c) in general. Let us take the tensor product of the exact sequence in c) with $(\det Q)^l$. The corresponding long cohomology exact sequence implies that the natural morphism

$$H^{n,q}\big(X, E \otimes (\det Q)^l\big) \longrightarrow H^{n,q}\big(X, Q \otimes (\det Q)^l\big)$$

is surjective for $q \geq 0$ and $l, m \geq \min\{n-q, \operatorname{rk} S\}$, bijective for $q \geq 1$ and $l, m \geq \min\{n-q+1, \operatorname{rk} S\}$.

10. Cohomology Groups of $\mathcal{O}(k)$ over $\mathbb{P}^n$

As an illustration of the above results, we compute now the cohomology groups of all line bundles $\mathcal{O}(k) \to \mathbb{P}^n$. This precise evaluation will be needed in the proof of a general vanishing theorem for vector bundles, due to Le Potier (see volume II). As in §V-15, we consider a complex vector space V of dimension $n+1$ and the exact sequence

$$(10.1) \quad 0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow H \longrightarrow 0$$

of vector bundles over $\mathbb{P}^n = P(V)$. We thus have $\det \underline{V} = \det H \otimes \mathcal{O}(-1)$, and as $TP(V) = H \otimes \mathcal{O}(1)$ by Th. V-15.7, we find

$$(10.2) \quad K_{P(V)} = \det T^\star P(V) = \det H^\star \otimes \mathcal{O}(-n) = \det \underline{V}^\star \otimes \mathcal{O}(-n-1)$$

where $\det \underline{V}$ is a trivial line bundle.

Before going further, we need some notations. For every integer $k \in \mathbb{N}$, we consider the homological complex $C^{\bullet,k}(V^\star)$ with differential γ such that

$$(10.3)\quad \begin{cases} C^{p,k}(V^\star) = \Lambda^p V^\star \otimes S^{k-p}V^\star, & 0 \le p \le k, \\ \qquad\qquad = 0 & \text{otherwise}, \\ \gamma \;:\; \Lambda^p V^\star \otimes S^{k-p}V^\star \longrightarrow \Lambda^{p-1}V^\star \otimes S^{k-p+1}V^\star, \end{cases}$$

where γ is the linear map obtained by contraction with the Euler vector field $\mathrm{Id}_V \in V \otimes V^\star$, through the obvious maps $V \otimes \Lambda^p V^\star \longrightarrow \Lambda^{p-1}V^\star$ and $V^\star \otimes S^{k-p}V^\star \longrightarrow S^{k-p+1}V^\star$. If $(z_0, \ldots, z_n)$ are coordinates on V, the module $C^{p,k}(V^\star)$ can be identified with the space of p-forms

$$\alpha(z) = \sum_{|I|=p} \alpha_I(z)\, dz_I$$

where the α_I 's are homogeneous polynomials of degree $k-p$. The differential γ is given by contraction with the Euler vector field $\xi = \sum_{0\le j\le n} z_j\, \partial/\partial z_j$.

Let us denote by $Z^{p,k}(V^\star)$ the space of p-cycles of $C^{\bullet,k}(V^\star)$, i.e. the space of forms $\alpha \in C^{p,k}(V^\star)$ such that $\xi \lrcorner\, \alpha = 0$. The exterior derivative d also acts on $C^{\bullet,k}(V^\star)$; we have

$$d \;:\; C^{p,k}(V^\star) \longrightarrow C^{p+1,k}(V^\star),$$

and a trivial computation shows that $d\gamma + \gamma d = k \cdot \mathrm{Id}_{C^{\bullet,k}(V^\star)}$.

(10.4) Theorem. *For $k \ne 0$, $C^{\bullet,k}(V^\star)$ is exact and there exist canonical isomorphisms*

$$C^{\bullet,k}(V^\star) = \Lambda^p V^\star \otimes S^{k-p}V^\star \simeq Z^{p,k}(V^\star) \oplus Z^{p-1,k}(V^\star).$$

Proof. The identity $d\gamma + \gamma d = k \cdot \mathrm{Id}$ implies the exactness. The isomorphism is given by $\frac{1}{k}\gamma d \oplus \gamma$ and its inverse by $\mathcal{P}_1 + \frac{1}{k} d \circ \mathcal{P}_2$. □

Let us consider now the canonical mappings

$$\pi \;:\; V \setminus \{0\} \longrightarrow P(V), \qquad \mu' \;:\; V \setminus \{0\} \longrightarrow \mathcal{O}(-1)$$

defined in §V-15. As $T_{[z]}P(V) \simeq V/\mathbb{C}\xi(z)$ for all $z \in V \setminus \{0\}$, every form $\alpha \in Z^{p,k}(V^\star)$ defines a holomorphic section of $\pi^\star\big(\Lambda^p T^\star P(V)\big)$, $\alpha(z)$ being homogeneous of degree k with respect to z. Hence $\alpha(z) \otimes \mu'(z)^{-k}$ is a holomorphic section of $\pi^\star\big(\Lambda^p T^\star P(V) \otimes \mathcal{O}(k)\big)$, and since its homogeneity degree is 0, it induces a holomorphic section of $\Lambda^p T^\star P(V) \otimes \mathcal{O}(k)$. We thus have an injective morphism

$$(10.5)\quad Z^{p,k}(V^\star) \longrightarrow H^{p,0}\big(P(V), \mathcal{O}(k)\big).$$

(10.6) Theorem. *The groups $H^{p,0}\big(P(V), \mathcal{O}(k)\big)$ are given by*

a) $H^{p,0}\big(P(V), \mathcal{O}(k)\big) \simeq Z^{p,k}(V^\star)$ *for* $k \ge p \ge 0$,

b) $H^{p,0}\big(P(V), \mathcal{O}(k)\big) = 0$ *for* $k \le p$ *and* $(k,p) \ne (0,0)$.

Proof. Let s be a holomorphic section of $\Lambda^p T^\star P(V) \otimes \mathcal{O}(k)$. Set

$$\alpha(z) = (d\pi_z)^\star\big(s([z]) \otimes \mu'(z)^k\big), \qquad z \in V \setminus \{0\}.$$

Then α is a holomorphic p-form on $V \setminus \{0\}$ such that $\xi \;\lrcorner\; \alpha = 0$, and the coefficients of α are homogeneous of degree $k - p$ on $V \setminus \{0\}$ (recall that $d\pi_{\lambda z} = \lambda^{-1} d\pi_z$). It follows that $\alpha = 0$ if $k < p$ and that $\alpha \in Z^{p,k}(V^\star)$ if $k \ge p$. The injective morphism (10.5) is therefore also surjective. Finally, $Z^{p,p}(V^\star) = 0$ for $p = k \ne 0$, because of the exactness of $C^{\bullet,k}(V^\star)$ when $k \ne 0$. The proof is complete. □

(10.7) Theorem. *The cohomology groups $H^{p,q}\big(P(V), \mathcal{O}(k)\big)$ vanish in the following cases:*

a) $q \ne 0, n, p\,$;

b) $q = 0$, $\;k \le p\;$ *and* $\;(k,p) \ne (0,0)\,$;

c) $q = n$, $\;k \ge -n + p\;$ *and* $\;(k,p) \ne (0,n)\,$;

d) $q = p \ne 0, n$, $\;k \ne 0$.

The remaining non vanishing groups are:

$\overline{\text{b}}$) $H^{p,0}\big(P(V), \mathcal{O}(k)\big) \simeq Z^{p,k}(V^\star)$ *for* $k > p\,$;

$\overline{\text{c}}$) $H^{p,n}\big(P(V), \mathcal{O}(k)\big) \simeq Z^{n-p,-k}(V)$ *for* $k < -n + p\,$;

$\overline{\text{d}}$) $H^{p,p}\big(P(V), \mathbb{C}\big) = \mathbb{C}$, $\quad 0 \le p \le n$.

Proof. • $\overline{\text{d}}$) is already known, and so is a) when $k = 0$ (Th. VI-13.3).

• b) and $\overline{\text{b}}$) follow from Th. 10.6, and c), $\overline{\text{c}}$) are equivalent to b), $\overline{\text{b}}$) via Serre duality:

$$H^{p,q}\big(P(V), \mathcal{O}(k)\big)^\star = H^{n-p,n-q}\big(P(V), \mathcal{O}(-k)\big),$$

thanks to the canonical isomorphism $\big(Z^{p,k}(V)\big)^\star = Z^{p,k}(V^\star)$.

• Let us prove now property a) when $k \ne 0$ and property d). By Serre duality, we may assume $k > 0$. Then

$$\Lambda^p T^\star P(V) \simeq K_{P(V)} \otimes \Lambda^{n-p} TP(V).$$

It is very easy to verify that $E \ge_{\mathrm{Nak}} 0$ implies $\Lambda^s E \ge_{\mathrm{Nak}} 0$ for every integer s. Since $TP(V) \ge_{\mathrm{Nak}} 0$, we get therefore

$$F = \Lambda^{n-p} TP(V) \otimes \mathcal{O}(k) >_{\mathrm{Nak}} 0 \quad \text{for } \; k > 0,$$

and the Nakano vanishing theorem implies

$$\begin{aligned} H^{p,q}\big(P(V), \mathcal{O}(k)\big) &= H^q\big(P(V), \Lambda^p T^\star P(V) \otimes \mathcal{O}(k)\big) \\ &= H^q\big(P(V), K_{P(V)} \otimes F\big) = 0, \qquad q \ge 1. \end{aligned}$$ □

11. Ample Vector Bundles

11.A. Globally Generated Vector Bundles

All definitions concerning ampleness are purely algebraic and do not involve differential geometry. We shall see however that ampleness is intimately connected with the differential geometric notion of positivity. For a general discussion of properties of ample vector bundles in arbitrary characteristic, we refer to (Hartshorne 1966).

(11.1) Definition. *Let $E \to X$ be a holomorphic vector bundle over an arbitrary complex manifold X.*

a) *E is said to be globally generated if for every $x \in X$ the evaluation map $H^0(X, E) \to E_x$ is onto.*

b) *E is said to be semi-ample if there exists an integer k_0 such that $S^k E$ is globally generated for $k \geq k_0$.*

Any quotient of a trivial vector bundle is globally generated, for example the tautological quotient vector bundle Q over the Grassmannian $G_r(V)$ is globally generated. Conversely, every globally generated vector bundle E of rank r is isomorphic to the quotient of a trivial vector bundle of rank $\leq n+r$, as shown by the following result.

(11.2) Proposition. *If a vector bundle E of rank r is globally generated, then there exists a finite dimensional subspace $V \subset H^0(X, E)$, $\dim V \leq n+r$, such that V generates all fibers E_x, $x \in X$.*

Proof. Put an arbitrary hermitian metric on E and consider the Fréchet space $\mathcal{F} = \big(H^0(X, E)\big)^{n+r}$ of $(n+r)$-tuples of holomorphic sections of E, endowed with the topology of uniform convergence on compact subsets of X. For every compact set $K \subset X$, we set

$$A(K) = \{(s_1, \ldots, s_{n+r}) \in \mathcal{F} \text{ which do not generate } E \text{ on } K\}.$$

It is enough to prove that $A(K)$ is of first category in $\mathcal{F}$: indeed, Baire's theorem will imply that $A(X) = \bigcup A(K_\nu)$ is also of first category, if (K_ν) is an exhaustive sequence of compact subsets of X. It is clear that $A(K)$ is closed, because $A(K)$ is characterized by the closed condition

$$\min_K \sum_{i_1 < \cdots < i_r} |s_{i_1} \wedge \cdots \wedge s_{i_r}| = 0.$$

It is therefore sufficient to prove that $A(K)$ has no interior point. By hypothesis, each fiber E_x, $x \in K$, is generated by r global sections $s'_1, \ldots, s'_r$. We have in fact $s'_1 \wedge \cdots \wedge s'_r \neq 0$ in a neighborhood U_x of x. By compactness of K, there exist finitely many sections $s'_1, \ldots, s'_N$ which generate E in a neighborhood Ω of the set K.

If T is a complex vector space of dimension r, define $R_k(T^p)$ as the set of p-tuples $(x_1, \ldots, x_p) \in T^p$ of rank k. Given $a \in R_k(T^p)$, we can reorder the p-tuple in such a way that $a_1 \wedge \cdots \wedge a_k \neq 0$. Complete these k vectors into a basis $(a_1, \ldots, a_k, b_1, \ldots, b_{r-k})$ of T. For every point $x \in T^p$ in a neighborhood of a, then $(x_1, \ldots, x_k, b_1, \ldots, b_{r-k})$ is again a basis of T. Therefore, we will have $x \in R_k(T^p)$ if and only if the coordinates of x_l, $k+1 \leq l \leq N$, relative to $b_1, \ldots, b_{r-k}$ vanish. It follows that $R_k(T^p)$ is a (non closed) submanifold of T^p of codimension $(r-k)(p-k)$.

Now, we have a surjective affine bundle-morphism

$$\begin{aligned} \Phi :\ \Omega \times \mathbb{C}^{N(n+r)} &\longrightarrow E^{n+r} \\ (x, \lambda) &\longmapsto \big(s_j(x) + \sum_{1 \leq k \leq N} \lambda_{jk} s'_k(x)\big)_{1 \leq j \leq n+r}. \end{aligned}$$

Therefore $\Phi^{-1}(R_k(E^{n+r}))$ is a locally trivial differentiable bundle over Ω, and the codimension of its fibers in $\mathbb{C}^{N(n+r)}$ is $(r-k)(n+r-k) \geq n+1$ if $k < r$; it follows that the dimension of the total space $\Phi^{-1}(R_k(E^{n+r}))$ is $\leq N(n+r) - 1$. By Sard's theorem

$$\bigcup_{k<r} \mathcal{P}_2\Big(\Phi^{-1}\big(R_k(E^{n+r})\big)\Big)$$

is of zero measure in $\mathbb{C}^{N(n+r)}$. This means that for almost every value of the parameter λ the vectors $s_j(x) + \sum_k \lambda_{jk} s'_k(x) \in E_x$, $1 \leq j \leq n+r$, are of maximum rank r at each point $x \in \Omega$. Therefore $A(K)$ has no interior point. □

Assume now that $V \subset H^0(X, E)$ generates E on X. Then there is an exact sequence

$$(11.3) \quad 0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow E \longrightarrow 0$$

of vector bundles over X, where $S_x = \{s \in V \ ;\ s(x) = 0\}$, $\operatorname{codim}_V S_x = r$. One obtains therefore a commutative diagram

$$(11.4) \quad \begin{array}{ccc} E & \xrightarrow{\Psi_V} & Q \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi_V} & G_r(V) \end{array}$$

where ψ_V, Ψ_V are the holomorphic maps defined by

$$\begin{aligned} &\psi_V(x) = S_x, \quad x \in X, \\ &\Psi_V(u) = \{s \in V \ ;\ s(x) = u\} \in V/S_x, \quad u \in E_x. \end{aligned}$$

In particular, we see that every globally generated vector bundle E of rank r is the pull-back of the tautological quotient vector bundle Q of rank r over the Grassmannian by means of some holomorphic map $X \longrightarrow G_r(V)$. In the special case when $\operatorname{rk} E = r = 1$, the above diagram becomes

$$(11.4') \qquad \begin{array}{ccc} E & \xrightarrow{\Psi_V} & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi_V} & P(V^\star) \end{array}$$

(11.5) Corollary. *If E is globally generated, then E possesses a hermitian metric such that $E \geq_{\mathrm{Grif}} 0$ (and also $E^\star \leq_{\mathrm{Nak}} 0$).*

Proof. Apply Prop. 6.11 to the exact sequence (11.3), where $\underline{V}$ is endowed with an arbitrary hermitian metric. □

When E is of rank $r = 1$, then $S^k E = E^{\otimes k}$ and any hermitian metric of $E^{\otimes k}$ yields a metric on E after extracting k-th roots. Thus:

(11.6) Corollary. *If E is a semi-ample line bundle, then $E \geq 0$.* □

In the case of vector bundles ($r \geq 2$) the answer is unknown, mainly because there is no known procedure to get a Griffiths semipositive metric on E from one on $S^k E$.

11.B. Ampleness

We are now turning ourselves to the definition of ampleness. If $E \longrightarrow X$ is a holomorphic vector bundle, we define the bundle $J^k E$ of *k-jets of sections* of E by $(J^k E)_x = \mathcal{O}_x(E)/\big(\mathcal{M}_x^{k+1} \cdot \mathcal{O}_x(E)\big)$ for every $x \in X$, where $\mathcal{M}_x$ is the maximal ideal of $\mathcal{O}_x$. Let $(e_1, \ldots, e_r)$ be a holomorphic frame of E and $(z_1, \ldots, z_n)$ analytic coordinates on an open subset $\Omega \subset X$. The fiber $(J^k E)_x$ can be identified with the set of Taylor developments of order k :

$$\sum_{1 \leq \lambda \leq r, |\alpha| \leq k} c_{\lambda,\alpha}(z - x)^\alpha \, e_\lambda(z),$$

and the coefficients $c_{\lambda,\alpha}$ define coordinates along the fibers of $J^k E$. It is clear that the choice of another holomorphic frame (e_λ) would yield a linear change of coordinates $(c_{\lambda,\alpha})$ with holomorphic coefficients in x. Hence $J^k E$ is a holomorphic vector bundle of rank $r\binom{n+k}{n}$.

(11.7) Definition.

a) *E is said to be very ample if all evaluation maps $H^0(X, E) \to (J^1 E)_x$, $H^0(X, E) \to E_x \oplus E_y$, $x, y \in X$, $x \neq y$, are surjective.*

b) *E is said to be ample if there exists an integer k_0 such that S^kE is very ample for $k \geq k_0$.*

(11.8) Example. $\mathcal{O}(1) \to \mathbb{P}^n$ is a very ample line bundle (immediate verification). Since the pull-back of a (very) ample vector bundle by an embedding is clearly also (very) ample, diagram (V-16.8) shows that $\Lambda^r Q \to G_r(V)$ is very ample. However, Q itself cannot be very ample if $r \geq 2$, because $\dim H^0(G_r(V), Q) = \dim V = d$, whereas

$$\operatorname{rank}(J^1Q) = (\operatorname{rank} Q)\big(1 + \dim G_r(V)\big) = r\big(1 + r(d-r)\big) > d \quad \text{if} \quad r \geq 2.$$

(11.9) Proposition. *If E is very ample of rank r, there exists a subspace V of $H^0(X, E)$, $\dim V \leq \max\big(nr+n+r, 2(n+r)\big)$, such that all the evaluation maps $V \to E_x \oplus E_y$, $x \neq y$, and $V \to (J^1E)_x$, $x \in X$, are surjective.*

Proof. The arguments are exactly the same as in the proof of Prop. 11.4, if we consider instead the bundles $J^1E \longrightarrow X$ and $E \times E \longrightarrow X \times X \setminus \Delta_X$ of respective ranks $r(n+1)$ and $2r$, and sections $s'_1, \ldots, s'_N \in H^0(X, E)$ generating these bundles. □

(11.10) Proposition. *Let $E \to X$ be a holomorphic vector bundle.*

a) *If $V \subset H^0(X, E)$ generates $J^1E \longrightarrow X$ and $E \times E \longrightarrow X \times X \setminus \Delta_X$, then ψ_V is an embedding.*

b) *Conversely, if $\operatorname{rank} E = 1$ and if there exists $V \subset H^0(X, E)$ generating E such that ψ_V is an embedding, then E is very ample.*

Proof. b) is immediate, because $E = \psi_V^\star(\mathcal{O}(1))$ and $\mathcal{O}(1)$ is very ample. Note that the result is false for $r \geq 2$ as shown by the example $E = Q$ over $X = G_r(V)$.

a) Under the assumption of a), it is clear since $S_x = \{s \in V \,;\, s(x) = 0\}$ that $S_x = S_y$ implies $x = y$, hence ψ_V is injective. Therefore, it is enough to prove that the map $x \mapsto S_x$ has an injective differential. Let $x \in X$ and $W \subset V$ such that $S_x \oplus W = V$. Choose a coordinate system in a neighborhood of x in X and a small tangent vector $h \in T_xX$. The element $S_{x+h} \in G_r(V)$ is the graph of a small linear map $u = O(|h|) : S_x \to W$. Thus we have

$$S_{x+h} = \{s' = s + t \in V \,;\, s \in S_x,\ t = u(s) \in W,\ s'(x+h) = 0\}.$$

Since $s(x) = 0$ and $|t| = O(|h|)$, we find

$$s'(x+h) = s'(x) + d_xs' \cdot h + O(|s'| \cdot |h|^2) = t(x) + d_xs \cdot h + O(|s| \cdot |h|^2),$$

thus $s'(x+h) = 0$ if and only if $t(x) = -d_xs \cdot h + O(|s| \cdot |h|^2)$. Thanks to the fiber isomorphism $\Psi_V : E_x \longrightarrow V/S_x \simeq W,\ t(x) \longmapsto t \bmod S_x$, we get:

$$u(s) = t = \Psi_V(t(x)) = -\Psi_V\big(d_x s \cdot h + O(|s|\cdot|h|^2)\big).$$

Recall that $T_y G_r(V) = \hom(S_y, Q_y) = \hom(y, V/y)$ (see V-16.5) and use these identifications at $y = S_x$. It follows that

$$\text{(11.11)}\quad (d_x\psi_V)\cdot h = u = \big(S_x \longrightarrow V/S_x,\ s \longmapsto -\Psi_V(d_x s\cdot h)\big),$$

Now hypothesis a) implies that $S_x \ni s \longmapsto d_x s \in \hom(T_x X, E_x)$ is onto, hence $d_x\psi_V$ is injective. □

(11.12) Corollary. *If E is an ample line bundle, then $E > 0$.*

Proof. If E is very ample, diagram (11.4′) shows that E is the pull-back of $\mathcal{O}(1)$ by the embedding ψ_V, hence $i\Theta(E) = \psi_V^\star\big(\mathrm{i}\Theta(\mathcal{O}(1))\big) > 0$ with the induced metric. The ample case follows by extracting roots. □

(11.13) Corollary. *If E is a very ample vector bundle, then E carries a hermitian metric such that $E^\star <_{\mathrm{Nak}} 0$, in particular $E >_{\mathrm{Grif}} 0$.*

Proof. Choose V as in Prop. 11.9 and select an arbitrary hermitian metric on V. Then diagram 11.4 yields $E = \psi_V^\star Q$, hence $\theta_E = \Psi_V^\star\theta_Q$. By formula (V-16.9) we have for every $\xi \in TG_r(V) = \hom(S, Q)$ and $t \in Q$:

$$\theta_Q(\xi\otimes t, \xi\otimes t) = \sum_{j,k,l}\xi_{jk}\overline{\xi}_{lk}t_l\overline{t}_j = \sum_k\Big|\sum_j\overline{t}_j\xi_{jk}\Big|^2 = |\langle\bullet, t\rangle\circ\xi|^2.$$

Let $h \in T_x X$, $t \in E_x$. Thanks to formula (11.11), we get

$$\begin{aligned}\theta_E(h\otimes t, h\otimes t) &= \theta_Q\big((d_x\psi_V\cdot h)\otimes\Psi_V(t), (d_x\psi_V\cdot h)\otimes\Psi_V(t)\big)\\ &= \big|\langle\bullet, \Psi_V(t)\rangle\circ(d_x\psi_V\cdot h)\big|^2 = \big|S_x \ni s \longmapsto \langle\Psi_V(d_x s\cdot h), \Psi_V(t)\rangle\big|^2\\ &= \big|S_x \ni s \longmapsto \langle d_x s\cdot h, t\rangle\big|^2 \ge 0.\end{aligned}$$

As $S_x \ni s \mapsto d_x s \in T^\star X\otimes E$ is surjective, it follows that $\theta_E(h\otimes t, h\otimes t) \ne 0$ when $h \ne 0$, $t \ne 0$. Now, $d_x s$ defines a linear form on $TX\otimes E^\star$ and the above formula for the curvature of E clearly yields

$$\theta_{E^\star}(u,u) = -|S_x \ni s \longmapsto d_x s\cdot u|^2 < 0 \quad \text{if } u \ne 0.$$ □

(11.14) Problem (Griffiths 1969). *If E is an ample vector bundle over a compact manifold X, then is $E >_{\mathrm{Grif}} 0$?*

Griffiths' problem has been solved in the affirmative when X is a curve (Umemura 1973), see also (Campana-Flenner 1990), but the general case is still unclear and seems very deep. The next sections will be concerned with the important result of Kodaira asserting the equivalence between positivity and ampleness for line bundles.

12. Blowing-up along a Submanifold

Here we generalize the blowing-up process already considered in Remark 4.5 to arbitrary manifolds. Let X be a complex n-dimensional manifold and Y a closed submanifold with $\operatorname{codim}_X Y = s$.

(12.1) Notations. *The normal bundle of Y in X is the vector bundle over Y defined as the quotient $NY = (TX)_{\restriction Y}/TY$. The fibers of NY are thus given by $N_yY = T_yX/T_yY$ at every point $y \in Y$. We also consider the projectivized normal bundle $P(NY) \longrightarrow Y$ whose fibers are the projective spaces $P(N_yY)$ associated to the fibers of NY.*

The *blow-up of X with center Y* (to be constructed later) is a complex n-dimensional manifold $\widetilde{X}$ together with a holomorphic map $\sigma : \widetilde{X} \longrightarrow X$ such that:

i) $E := \sigma^{-1}(Y)$ is a smooth *hypersurface* in $\widetilde{X}$, and the restriction $\sigma : E \to Y$ is a holomorphic fiber bundle isomorphic to the projectivized normal bundle $P(NY) \to Y$.

ii) $\sigma : \widetilde{X} \setminus E \longrightarrow X \setminus Y$ is a biholomorphism.

In order to construct $\widetilde{X}$ and σ, we first define the set-theoretic underlying objects as the disjoint sums

$$\begin{aligned} \widetilde{X} &= (X \setminus Y) \amalg E, &&\text{where } E := P(NY),\\ \sigma &= \mathrm{Id}_{X\setminus Y} \amalg \pi, &&\text{where } \pi : E \longrightarrow Y. \end{aligned}$$

This means intuitively that we have replaced each point $y \in Y$ by the projective space of all directions normal to Y. When Y is reduced to a single point, the geometric picture is given by Fig. 1 below. In general, the picture is obtained by slicing X transversally to Y near each point and by blowing-up each slice at the intersection point with Y.

It remains to construct the manifold structure on $\widetilde{X}$ and in particular to describe what are the holomorphic functions near a point of E. Let f, g be holomorphic functions on an open set $U \subset X$ such that $f = g = 0$ on $Y \cap U$. Then df and dg vanish on $TY_{\restriction Y\cap U}$, hence df and dg induce linear forms on $NY_{\restriction Y\cap U}$. The holomorphic function $h(z) = f(z)/g(z)$ on the open set

$$U_g := \big\{z \in U\ ;\ g(z) \neq 0\big\} \subset U \setminus Y$$

can be extended in a natural way to a function $\widetilde{h}$ on the set

$$\widetilde{U}_g = U_g \cup \big\{(z, [\xi]) \in P(NY)_{\restriction Y\cap U}\ ;\ dg_z(\xi) \neq 0\big\} \subset \widetilde{X}$$

by letting

$$\widetilde{h}(z, [\xi]) = \frac{df_z(\xi)}{dg_z(\xi)}, \qquad (z, [\xi]) \in P(NY)_{\restriction Y\cap U}.$$

Fig. 1 Blow-up of one point in X.

Using this observation, we now define the manifold structure on $\widetilde{X}$ by giving explicitly an atlas. Every coordinate chart of $X \setminus Y$ is taken to be also a coordinate chart of $\widetilde{X}$. Furthermore, for every point $y_0 \in Y$, there exists a neighborhood U of y_0 in X and a coordinate chart $\tau(z) = (z_1, \ldots, z_n) : U \to \mathbb{C}^n$ centered at y_0 such that $\tau(U) = B' \times B''$ for some balls $B' \subset \mathbb{C}^s$, $B'' \subset \mathbb{C}^{n-s}$, and such that $Y \cap U = \tau^{-1}(\{0\} \times B'') = \{z_1 = \ldots = z_s = 0\}$. It follows that $(z_{s+1}, \ldots, z_n)$ are local coordinates on $Y \cap U$ and that the vector fields $(\partial/\partial z_1, \ldots, \partial/\partial z_s)$ yield a holomorphic frame of $NY_{\restriction Y \cap U}$. Let us denote by $(\xi_1, \ldots, \xi_s)$ the corresponding coordinates along the fibers of NY. Then $(\xi_1, \ldots, \xi_s, z_{s+1}, \ldots, z_n)$ are coordinates on the total space NY. For every $j = 1, \ldots, s$, we set

$$\widetilde{U}_j = \widetilde{U}_{z_j} = \big\{z \in U \setminus Y \ ; \ z_j \neq 0\big\} \cup \big\{(z, [\xi]) \in P(NY)_{\restriction Y \cap U} \ ; \ \xi_j \neq 0\big\}.$$

Then $(\widetilde{U}_j)_{1 \le j \le s}$ is a covering of $\widetilde{U} = \sigma^{-1}(U)$ and for each j we define a coordinate chart $\widetilde{\tau}_j = (w_1, \ldots, w_n) : \widetilde{U}_j \longrightarrow \mathbb{C}^n$ by

$$w_k := \Big(\frac{z_k}{z_j}\Big)^{\sim} \quad \text{for} \ \ 1 \le k \le s, k \neq j \, ; \qquad w_k := z_k \quad \text{for} \ \ k > s \ \ \text{or} \ \ k = j.$$

For $z \in U \setminus Y$, resp. $(z, [\xi]) \in P(NY)_{\restriction Y \cap U}$, we get

$$\widetilde{\tau}_j(z) = (w_1, \ldots, w_n) = \Big(\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_s}{z_j}, z_{s+1}, \ldots, z_n\Big),$$
$$\widetilde{\tau}_j(z, [\xi]) = (w_1, \ldots, w_n) = \Big(\frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, 0, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_s}{\xi_j}, \xi_{s+1}, \ldots, \xi_n\Big).$$

With respect to the coordinates (w_k) on $\widetilde{U}_j$ and (z_k) on U, the map σ is given by

$$\text{(12.2)} \qquad \begin{aligned} \widetilde{U}_j &\xrightarrow{\ \sigma\ } U \\ w &\xmapsto{\ \sigma_j\ } (w_1 w_j, \ldots, w_{j-1} w_j\ ;\ w_j\ ;\ w_{j+1} w_j, \ldots, w_s w_j\ ;\ w_{s+1}, \ldots, w_n) \end{aligned}$$

where $\sigma_j = \tau \circ \sigma \circ \widetilde{\tau}_j^{-1}$, thus σ is holomorphic. The range of the coordinate chart $\widetilde{\tau}_j$ is $\widetilde{\tau}_j(\widetilde{U}_j) = \sigma_j^{-1}\big(\tau(U)\big)$, so it is actually open in $\mathbb{C}^n$. Furthermore $E \cap \widetilde{U}_j$ is defined by the single equation $w_j = 0$, thus E is a smooth hypersurface in $\widetilde{X}$. It remains only to verify that the coordinate changes $w \longmapsto w'$ associated to any coordinate change $z \longmapsto z'$ on X are holomorphic. For that purpose, it is sufficient to verify that $(f/g)^{\sim}$ is holomorphic in $(w_1, \ldots, w_n)$ on $\widetilde{U}_j \cap \widetilde{U}_g$. As g vanishes on $Y \cap U$, we can write $g(z) = \sum_{1 \le k \le s} z_k A_k(z)$ for some holomorphic functions A_k on U. Therefore

$$\frac{g(z)}{z_j} = A_j(\sigma_j(w)) + \sum_{k \ne j} w_k A_k(\sigma_j(w))$$

has an extension $(g/z_j)^{\sim}$ to $\widetilde{U}_j$ which is a holomorphic function of the variables $(w_1, \ldots, w_n)$. Since $(g/z_j)^{\sim}(z, [\xi]) = dg_z(\xi)/\xi_j$ on $E \cap \widetilde{U}_j$, it is clear that

$$\widetilde{U}_j \cap \widetilde{U}_g = \big\{w \in \widetilde{U}_j\ ;\ (g/z_j)^{\sim}(w) \ne 0\big\}.$$

Hence $\widetilde{U}_j \cap \widetilde{U}_g$ is open in $\widetilde{U}_g$ and $(f/g)^{\sim} = (f/z_j)^{\sim}/(g/z_j)^{\sim}$ is holomorphic on $\widetilde{U}_j \cap \widetilde{U}_g$.

(12.3) Definition. *The map $\sigma : \widetilde{X} \to X$ is called the blow-up of X with center Y and $E = \sigma^{-1}(Y) \simeq P(NY)$ is called the exceptional divisor of $\widetilde{X}$.*

According to (V-13.5), we denote by $\mathcal{O}(E)$ the line bundle on $\widetilde{X}$ associated to the divisor E and by $h \in H^0(\widetilde{X}, \mathcal{O}(E))$ the canonical section such that $\operatorname{div}(h) = [E]$. On the other hand, we denote by $\mathcal{O}_{P(NY)}(-1) \subset \pi^\star(NY)$ the tautological line subbundle over $E = P(NY)$ such that the fiber above the point $(z, [\xi])$ is $\mathbb{C}\xi \subset N_z Y$.

(12.4) Proposition. *$\mathcal{O}(E)$ enjoys the following properties:*

a) $\mathcal{O}(E)_{\restriction E}$ *is isomorphic to* $\mathcal{O}_{P(NY)}(-1)$.

b) *Assume that* X *is compact. For every positive line bundle* L *over* X*, the line bundle* $\mathcal{O}(-E)\otimes\sigma^\star(L^k)$ *over* $\widetilde{X}$ *is positive for* $k>0$ *large enough.*

Proof. a) The canonical section $h\in H^0(\widetilde{X},\mathcal{O}(E))$ vanishes at order 1 along E, hence the kernel of its differential

$$dh:(T\widetilde{X})_{\restriction E}\longrightarrow \mathcal{O}(E)_{\restriction E}$$

is TE. We get therefore an isomorphism $NE\simeq\mathcal{O}(E)_{\restriction E}$. Now, the map $\sigma:\widetilde{X}\to X$ satisfies $\sigma(E)\subset Y$, so its differential $d\sigma:T\widetilde{X}\longrightarrow\sigma^\star(TX)$ is such that $d\sigma(TE)\subset\sigma^\star(TY)$. Therefore $d\sigma$ induces a morphism

$$\text{(12.5)}\quad NE\longrightarrow\sigma^\star(NY)=\pi^\star(NY)$$

of vector bundles over E. The vector field $\partial/\partial w_j$ yields a non vanishing section of NE on $\widetilde{U}_j$, and (12.2) implies

$$d\sigma_j\Big(\frac{\partial}{\partial w_j}\Big)=\frac{\partial}{\partial z_j}+\sum_{1\le k\le s,k\ne j}w_k\frac{\partial}{\partial z_k}\quad /\!/\quad\sum_{1\le k\le s}\xi_k\frac{\partial}{\partial z_k}$$

at every point $(z,[\xi])\in E$. This shows that (12.5) is an isomorphism of NE onto $\mathcal{O}_{P(NY)}(-1)\subset\pi^\star(NY)$, hence

$$\text{(12.6)}\quad \mathcal{O}(E)_{\restriction E}\simeq NE\simeq\mathcal{O}_{P(NY)}(-1).$$

b) Select an arbitrary hermitian metric on TX and consider the induced metrics on NY and on $\mathcal{O}_{P(NY)}(1)\longrightarrow E=P(NY)$. The restriction of $\mathcal{O}_{P(NY)}(1)$ to each fiber $P(N_zY)$ is the standard line bundle $\mathcal{O}(1)$ over $\mathbb{P}^{s-1}$; thus by (V-15.10) this restriction has a positive definite curvature form. Extend now the metric of $\mathcal{O}_{P(NY)}(1)$ on E to a metric of $\mathcal{O}(-E)$ on X in an arbitrary way. If $F=\mathcal{O}(-E)\otimes\sigma^\star(L^k)$, then $\Theta(F)=\Theta(\mathcal{O}(-E))+k\,\sigma^\star\Theta(L)$, thus for every $t\in T\widetilde{X}$ we have

$$\theta_F(t,t)=\theta_{\mathcal{O}(-E)}(t,t)+k\,\theta_L\big(d\sigma(t),d\sigma(t)\big).$$

By the compactness of the unitary tangent bundle to $\widetilde{X}$ and the positivity of θ_L, it is sufficient to verify that $\theta_{\mathcal{O}(-E)}(t,t)>0$ for every unit vector $t\in T_z\widetilde{X}$ such that $d\sigma(t)=0$. However, from the computations of a), this can only happen when $z\in E$ and $t\in TE$, and in that case $d\sigma(t)=d\pi(t)=0$, so t is tangent to the fiber $P(N_zY)$. Therefore

$$\theta_{\mathcal{O}(-E)}(t,t)=\theta_{\mathcal{O}_{P(NY)}(1)}(t,t)>0.\qquad\square$$

(12.7) Proposition. *The canonical line bundle of* $\widetilde{X}$ *is given by*

$$K_{\widetilde{X}} = \mathcal{O}\big((s-1)E\big) \otimes \sigma^\star K_X, \qquad \textit{where} \;\; s = \operatorname{codim}_X Y.$$

Proof. K_X is generated on U by the holomorphic n-form $dz_1 \wedge \ldots \wedge dz_n$. Using (12.2), we see that $\sigma^\star K_X$ is generated on $\widetilde{U}_j$ by

$$\sigma^\star(dz_1 \wedge \ldots \wedge dz_n) = w_j^{s-1}\, dw_1 \wedge \ldots \wedge dw_n.$$

Since the divisor of the section $h \in H^0(\widetilde{X}, \mathcal{O}(E))$ is the hypersurface E defined by the equation $w_j = 0$ in $\widetilde{U}_j$, we have a well defined line bundle isomorphism

$$\sigma^\star K_X \longrightarrow \mathcal{O}\big((1-s)E\big) \otimes K_{\widetilde{X}}, \qquad \alpha \longmapsto h^{1-s}\sigma^\star(\alpha). \qquad \square$$

13. Equivalence of Positivity and Ampleness for Line Bundles

We have seen in section 11 that every ample line bundle carries a hermitian metric of positive curvature. The converse will be a consequence of the following result.

(13.1) Theorem. *Let $L \longrightarrow X$ be a positive line bundle and L^k the k-th tensor power of L. For every N-tuple $(x_1, \ldots, x_N)$ of distinct points of X, there exists a constant $C > 0$ such that the evaluation maps*

$$H^0(X, L^k) \longrightarrow (J^m L^k)_{x_1} \oplus \cdots \oplus (J^m L^k)_{x_N}$$

are surjective for all integers $m \geq 0$, $k \geq C(m+1)$.

(13.2) Lemma. *Let $\sigma : \widetilde{X} \longrightarrow X$ be the blow-up of X with center the finite set $Y = \{x_1, \ldots, x_N\}$, and let $\mathcal{O}(E)$ be the line bundle associated to the exceptional divisor E. Then*

$$H^1(\widetilde{X}, \mathcal{O}(-mE) \otimes \sigma^\star L^k) = 0$$

for $m \geq 1$, $k \geq Cm$ and $C \geq 0$ large enough.

Proof. By Prop. 12.7 we get $K_{\widetilde{X}} = \mathcal{O}\big((n-1)E\big) \otimes \sigma^\star K_X$ and

$$H^1\big(\widetilde{X}, \mathcal{O}(-mE) \otimes \sigma^\star L^k\big) = H^{n,1}\big(\widetilde{X}, K_{\widetilde{X}}^{-1} \otimes \mathcal{O}(-mE) \otimes \sigma^\star L^k\big) = H^{n,1}\big(\widetilde{X}, F\big)$$

where $F = \mathcal{O}\big(-(m+n-1)E\big) \otimes \sigma^\star(K_X^{-1} \otimes L^k)$, so the conclusion will follow from the Kodaira-Nakano vanishing theorem if we can show that $F > 0$ when k is large enough. Fix an arbitrary hermitian metric on K_X. Then

$$\Theta(F) = (m+n-1)\Theta(\mathcal{O}(-E)) + \sigma^\star\big(k\Theta(L) - \Theta(K_X)\big).$$

There is $k_0 \geq 0$ such that $\mathrm{i}\big(k_0\Theta(L) - \Theta(K_X)\big) > 0$ on X, and Prop. 12.4 implies the existence of $C_0 > 0$ such that $\mathrm{i}\big(\Theta(\mathcal{O}(-E)) + C_0\sigma^\star\Theta(L)\big) > 0$ on $\widetilde{X}$. Thus $\mathrm{i}\Theta(F) > 0$ for $m \geq 2 - n$ and $k \geq k_0 + C_0(m + n - 1)$. $\square$

Proof of Theorem 13.1. Let $v_j \in H^0(\Omega_j, L^k)$ be a holomorphic section of L^k in a neighborhood Ω_j of x_j having a prescribed m-jet at x_j. Set

$$v(x) = \sum_j \psi_j(x) v_j(x)$$

where $\psi_j = 1$ in a neighborhood of x_j and ψ_j has compact support in Ω_j. Then $d''v = \sum d''\psi_j \cdot v_j$ vanishes in a neighborhood of $x_1, \ldots, x_N$. Let h be the canonical section of $\mathcal{O}(E)^{-1}$ such that $\mathrm{div}(h) = [E]$. The $(0,1)$-form $\sigma^\star d''v$ vanishes in a neighborhood of $E = h^{-1}(0)$, hence

$$w = h^{-(m+1)}\sigma^\star d''v \in C^\infty_{0,1}\big(\widetilde{X}, \mathcal{O}(-(m+1)E) \otimes \sigma^\star L^k\big).$$

and w is a d''-closed form. By Lemma 13.2 there exists a smooth section $u \in C^\infty_{0,0}\big(\widetilde{X}, \mathcal{O}(-(m+1)E) \otimes \sigma^\star L^k\big)$ such that $d''u = w = h^{-(m+1)}\sigma^\star d''v$. This implies

$$\sigma^\star v - h^{m+1}u \in H^0(\widetilde{X}, \sigma^\star L^k),$$

and since $\sigma^\star L$ is trivial near E, there exists a section $g \in H^0(X, L^k)$ such that $\sigma^\star g = \sigma^\star v - h^{m+1}u$. As h vanishes at order 1 along E, the m-jet of g at x_j must be equal to that of v (or v_j). $\square$

(13.3) Corollary. *For any holomorphic line bundle $L \longrightarrow X$, the following conditions are equivalent:*

a) *L is ample;*

b) *$L > 0$, i.e. L possesses a hermitian metric such that* $\mathrm{i}\Theta(L) > 0$.

Proof. a) $\Longrightarrow$ b) is given by Cor. 11.12, whereas b) $\Longrightarrow$ a) is a consequence of Th. 13.1 for $m = 1$. $\square$

14. Kodaira's Projectivity Criterion

The following fundamental projectivity criterion is due to (Kodaira 1954).

(14.1) Theorem. *Let X be a compact complex manifold,* $\dim_{\mathbb{C}} X = n$. *The following conditions are equivalent.*

a) *X is projective algebraic, i.e. X can be embedded as an algebraic submanifold of the complex projective space $\mathbb{P}^N$ for N large.*

b) *X carries a positive line bundle L.*

c) *X carries a Hodge metric, i.e. a Kähler metric ω with rational cohomology class $\{\omega\} \in H^2(X, \mathbb{Q})$.*

Proof. a) $\Longrightarrow$ b). Take $L = \mathcal{O}(1)_{\restriction X}$.

b) $\Longrightarrow$ c). Take $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$; then $\{\omega\}$ is the image of $c_1(L) \in H^2(X, \mathbb{Z})$.

c) $\Longrightarrow$ b). We can multiply $\{\omega\}$ by a common denominator of its coefficients and suppose that $\{\omega\}$ is in the image of $H^2(X, \mathbb{Z})$. Then Th. V-13.9 b) shows that there exists a hermitian line bundle L such that $\frac{\mathrm{i}}{2\pi}\Theta(L) = \omega > 0$.

b) $\Longrightarrow$ a). Corollary 13.3 shows that $F = L^k$ is very ample for some integer $k > 0$. Then Prop. 11.9 enables us to find a subspace V of $H^0(X, F)$, $\dim V \le 2n + 2$, such that $\psi_V : X \longrightarrow G_1(V) = P(V^\star)$ is an embedding. Thus X can be embedded in $\mathbb{P}^{2n+1}$ and Chow's theorem II-7.10 shows that the image is an algebraic set in $\mathbb{P}^{2n+1}$. □

(14.2) Remark. The above proof shows in particular that every n-dimensional projective manifold X can be embedded in $\mathbb{P}^{2n+1}$. This can be shown directly by using generic projections $\mathbb{P}^N \to \mathbb{P}^{2n+1}$ and Whitney type arguments as in 11.2.

(14.3) Corollary. *Every compact Riemann surface X is isomorphic to an algebraic curve in $\mathbb{P}^3$.*

Proof. Any positive smooth form ω of type $(1,1)$ is Kähler, and ω is in fact a Hodge metric if we normalize its volume so that $\int_X \omega = 1$. □

This example can be somewhat generalized as follows.

(14.4) Corollary. *Every Kähler manifold (X, ω) such that $H^2(X, \mathcal{O}) = 0$ is projective.*

Proof. By hypothesis $H^{0,2}(X, \mathbb{C}) = 0 = H^{2,0}(X, \mathbb{C})$, hence

$$H^2(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C})$$

admits a basis $\{\alpha_1\}, \ldots, \{\alpha_N\} \in H^2(X, \mathbb{Q})$ where $\alpha_1, \ldots, \alpha_N$ are harmonic real $(1,1)$-forms. Since $\{\omega\}$ is real, we have $\{\omega\} = \lambda_1\{\alpha_1\} + \ldots + \lambda_N\{\alpha_N\}$, $\lambda_j \in \mathbb{R}$, thus

$$\omega = \lambda_1\alpha_1 + \ldots + \lambda_N\alpha_N$$

because ω itself is harmonic. If $\mu_1, \ldots, \mu_N$ are rational numbers sufficiently close to $\lambda_1, \ldots, \lambda_N$, then $\widetilde{\omega} := \mu_1\alpha_1 + \cdots \mu_N\alpha_N$ is close to ω, so $\widetilde{\omega}$ is a positive definite d-closed $(1,1)$-form, and $\{\widetilde{\omega}\} \in H^2(X, \mathbb{Q})$. □

We obtain now as a consequence the celebrated Riemann criterion characterizing *abelian varieties* (= projective algebraic complex tori).

(14.5) Corollary. *A complex torus $X = \mathbb{C}^n/\Gamma$ (Γ a lattice of $\mathbb{C}^n$) is an abelian variety if and only if there exists a positive definite hermitian form h on $\mathbb{C}^n$ such that*

$$\operatorname{Im}\big(h(\gamma_1,\gamma_2)\big) \in \mathbb{Z} \quad \text{for all} \quad \gamma_1,\gamma_2 \in \Gamma.$$

Proof (Sufficiency of the condition). Set $\omega = -\operatorname{Im} h$. Then ω defines a constant Kähler metric on $\mathbb{C}^n$, hence also on $X = \mathbb{C}^n/\Gamma$. Let $(a_1,\ldots,a_{2n})$ be an integral basis of the lattice Γ. We denote by T_j, T_{jk} the real 1- and 2-tori

$$T_j = (\mathbb{R}/\mathbb{Z})a_j, \quad 1 \le j \le n, \qquad T_{jk} = T_j \oplus T_k, \quad 1 \le j < k \le 2n.$$

Topologically we have $X \approx T_1 \times \ldots \times T_{2n}$, so the Künneth formula IV-15.7 yields

$$H^\bullet(X,\mathbb{Z}) \simeq \bigotimes_{1\le j\le 2n} \big(H^0(T_j,\mathbb{Z}) \oplus H^1(T_j,\mathbb{Z})\big),$$
$$H^2(X,\mathbb{Z}) \simeq \bigoplus_{1\le j<k\le 2n} H^1(T_j,\mathbb{Z}) \otimes H^1(T_k,\mathbb{Z}) \simeq \bigoplus_{1\le j<k\le 2n} H^2(T_{jk},\mathbb{Z})$$

where the projection $H^2(X,\mathbb{Z}) \longrightarrow H^2(T_{jk},\mathbb{Z})$ is induced by the injection $T_{jk} \subset X$. In the identification $H^2(T_{jk},\mathbb{R}) \simeq \mathbb{R}$, we get

$$(14.6) \quad \{\omega\}_{\restriction T_{jk}} = \int_{T_{jk}} \omega = \omega(a_j,a_k) = -\operatorname{Im} h(a_j,a_k).$$

The assumption on h implies $\{\omega\}_{\restriction T_{jk}} \in H^2(T_{jk},\mathbb{Z})$ for all j,k, therefore $\{\omega\} \in H^2(X,\mathbb{Z})$ and X is projective by Th. (14.1).

Proof (Necessity of the condition). If X is projective, then X admits a Kähler metric ω such that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$. In general, ω is not invariant under the translations $\tau_x(y) = y - x$ of X. Therefore, we replace ω by its "mean value":

$$\widetilde{\omega} = \frac{1}{\operatorname{Vol}(X)} \int_{x\in X} (\tau_x^\star \omega)\, dx,$$

which has the same cohomology class as ω (τ_x is homotopic to the identity). Now $\widetilde{\omega}$ is the imaginary part of a constant positive definite hermitian form h on $\mathbb{C}^n$, and formula (14.6) shows that $\operatorname{Im} h(a_j,a_k) \in \mathbb{Z}$. □

(14.7) Example. Let X be a projective manifold. We shall prove that the Jacobian $\operatorname{Jac}(X)$ and the Albanese variety $\operatorname{Alb}(X)$ (cf. § VI-13 for definitions) are abelian varieties.

In fact, let ω be a Kähler metric on X such that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$ and let h be the hermitian metric on $H^1(X,\mathcal{O}) \simeq H^{0,1}(X,\mathbb{C})$ defined by

$$h(u,v) = \int_X -2\mathrm{i}\, u \wedge \overline{v} \wedge \omega^{n-1}$$

for all closed $(0,1)$-forms u, v. As

$$-2\mathrm{i}\, u \wedge \overline{v} \wedge \omega^{n-1} = \frac{2}{n}\,|u|^2\,\omega^n,$$

we see that h is a positive definite hermitian form on $H^{0,1}(X,\mathbb{C})$. Consider elements $\gamma_j \in H^1(X,\mathbb{Z})$, $j = 1, 2$. If we write $\gamma_j = \gamma'_j + \gamma''_j$ in the decomposition $H^1(X,\mathbb{C}) = H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C})$, we get

$$h(\gamma''_1, \gamma''_2) = \int_X -2\mathrm{i}\,\gamma''_1 \wedge \gamma'_2 \wedge \omega^{n-1},$$

$$\operatorname{Im} h(\gamma''_1, \gamma''_2) = \int_X (\gamma'_1 \wedge \gamma''_2 + \gamma''_1 \wedge \gamma'_2) \wedge \omega^{n-1} = \int_X \gamma_1 \wedge \gamma_2 \wedge \omega^{n-1} \in \mathbb{Z}.$$

Therefore $\mathrm{Jac}(X)$ is an abelian variety.

Now, we observe that $H^{n-1,n}(X,\mathbb{C})$ is the anti-dual of $H^{0,1}(X,\mathbb{C})$ by Serre duality. We select on $H^{n-1,n}(X,\mathbb{C})$ the dual hermitian metric $h^\star$. Since the Poincaré bilinear pairing yields a unimodular bilinear map

$$H^1(X,\mathbb{Z}) \times H^{2n-1}(X,\mathbb{Z}) \longrightarrow \mathbb{Z},$$

we easily conclude that $\operatorname{Im} h^\star(\gamma''_1, \gamma''_2) \in \mathbb{Q}$ for all $\gamma_1, \gamma_2 \in H^{2n-1}(X,\mathbb{Z})$. Therefore $\mathrm{Alb}(X)$ is also an abelian variety.