Chapter VIII
L^2 Estimates on Pseudoconvex Manifolds

The main goal of this chapter is to show that the differential geometric technique that has been used in order to prove vanishing theorems also yields very precise L^2 estimates for the solutions of equations $d''u = v$ on pseudoconvex manifolds. The central idea, due to (Hörmander 1965), is to introduce weights of the type $e^{-\varphi}$ where φ is a function satisfying suitable convexity conditions. This method leads to generalizations of many standard vanishing theorems to weakly pseudoconvex manifolds. As a special case, we obtain the original Hörmander estimates for pseudoconvex domains of $\mathbb{C}^n$, and give some applications to algebraic geometry (Hörmander-Bombieri-Skoda theorem, properties of zero sets of polynomials in $\mathbb{C}^n$). We also derive the Ohsawa-Takegoshi extension theorem for L^2 holomorphic functions and Skoda's L^2 estimates for surjective bundle morphisms (Skoda 1972a, 1978, Demailly 1982c). Skoda's estimates can be used to obtain a quick solution of the Levi problem, and have important applications to local algebra and Nullstellensatz theorems. Finally, L^2 estimates are used to prove the Newlander-Nirenberg theorem on the analyticity of almost complex structures. We apply it to establish Kuranishi's theorem on deformation theory of compact complex manifolds.

1. Non Bounded Operators on Hilbert Spaces

A few preliminaries of functional analysis will be needed here. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be complex Hilbert spaces. We consider a linear operator T defined on a subspace $\operatorname{Dom} T \subset \mathcal{H}_1$ (called the domain of T) into $\mathcal{H}_2$. The operator T is said to be *densely defined* if $\operatorname{Dom} T$ is dense in $\mathcal{H}_1$, and *closed* if its graph

$$\operatorname{Gr} T = \big\{(x, Tx)\ ;\ x \in \operatorname{Dom} T\big\}$$

is closed in $\mathcal{H}_1 \times \mathcal{H}_2$.

Assume now that T is closed and densely defined. The adjoint $T^\star$ of T (in Von Neumann's sense) is constructed as follows: $\operatorname{Dom} T^\star$ is the set of $y \in \mathcal{H}_2$ such that the linear form

$$\operatorname{Dom} T \ni x \longmapsto \langle Tx, y\rangle_2$$

is bounded in $\mathcal{H}_1$-norm. Since $\operatorname{Dom} T$ is dense, there exists for every y in $\operatorname{Dom} T^\star$ a unique element $T^\star y \in \mathcal{H}_1$ such that $\langle Tx, y\rangle_2 = \langle x, T^\star y\rangle_1$ for all $x \in \operatorname{Dom} T^\star$. It is immediate to verify that $\operatorname{Gr} T^\star = \big(\operatorname{Gr}(-T)\big)^\perp$ in $\mathcal{H}_1 \times \mathcal{H}_2$.

It follows that $T^\star$ is closed and that every pair $(u,v) \in \mathcal{H}_1 \times \mathcal{H}_2$ can be written

$$(u,v) = (x, -Tx) + (T^\star y, y), \qquad x \in \mathrm{Dom}\, T, \quad y \in \mathrm{Dom}\, T^\star.$$

Take in particular $u = 0$. Then

$$x + T^\star y = 0, \qquad v = y - Tx = y + TT^\star y, \qquad \langle v, y\rangle_2 = \|y\|_2^2 + \|T^\star y\|_1^2.$$

If $v \in (\mathrm{Dom}\, T^\star)^\perp$ we get $\langle v, y\rangle_2 = 0$, thus $y = 0$ and $v = 0$. Therefore $T^\star$ is densely defined and our discussion implies:

(1.1) Theorem (Von Neumann 19??). *If $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a closed and densely defined operator, then its adjoint $T^\star$ is also closed and densely defined and $(T^\star)^\star = T$. Furthermore, we have the relation $\mathrm{Ker}\, T^\star = (\mathrm{Im}\, T)^\perp$ and its dual $(\mathrm{Ker}\, T)^\perp = \overline{\mathrm{Im}\, T^\star}$.* □

Consider now two closed and densely defined operators T, S :

$$\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3$$

such that $S \circ T = 0$. By this, we mean that the range $T(\mathrm{Dom}\, T)$ is contained in $\mathrm{Ker}\, S \subset \mathrm{Dom}\, S$, in such a way that there is no problem for defining the composition $S \circ T$. The starting point of all L^2 estimates is the following abstract existence theorem.

(1.2) Theorem. *There are orthogonal decompositions*

$$\mathcal{H}_2 = (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^\star) \oplus \overline{\mathrm{Im}\, T} \oplus \overline{\mathrm{Im}\, S^\star},$$
$$\mathrm{Ker}\, S = (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^\star) \oplus \overline{\mathrm{Im}\, T}.$$

In order that $\mathrm{Im}\, T = \mathrm{Ker}\, S$, it suffices that

$$\|T^\star x\|_1^2 + \|Sx\|_3^2 \ge C\|x\|_2^2, \qquad \forall x \in \mathrm{Dom}\, S \cap \mathrm{Dom}\, T^\star \tag{1.3}$$

for some constant $C > 0$. In that case, for every $v \in \mathcal{H}_2$ such that $Sv = 0$, there exists $u \in \mathcal{H}_1$ such that $Tu = v$ and

$$\|u\|_1^2 \le \frac{1}{C}\|v\|_2^2.$$

In particular

$$\overline{\mathrm{Im}\, T} = \mathrm{Im}\, T = \mathrm{Ker}\, S, \qquad \overline{\mathrm{Im}\, S^\star} = \mathrm{Im}\, S^\star = \mathrm{Ker}\, T^\star.$$

Proof. Since S is closed, the kernel $\mathrm{Ker}\, S$ is closed in $\mathcal{H}_2$. The relation $(\mathrm{Ker}\, S)^\perp = \overline{\mathrm{Im}\, S^\star}$ implies

$$\mathcal{H}_2 = \mathrm{Ker}\, S \oplus \overline{\mathrm{Im}\, S^\star} \tag{1.4}$$

and similarly $\mathcal{H}_2 = \operatorname{Ker} T^\star \oplus \overline{\operatorname{Im} T}$. However, the assumption $S \circ T = 0$ shows that $\overline{\operatorname{Im} T} \subset \operatorname{Ker} S$, therefore

$$(1.5) \qquad \operatorname{Ker} S = (\operatorname{Ker} S \cap \operatorname{Ker} T^\star) \oplus \overline{\operatorname{Im} T}.$$

The first two equalities in Th. 1.2 are then equivalent to the conjunction of (1.4) and (1.5).

Now, under assumption (1.3), we are going to show that the equation $Tu = v$ is always solvable if $Sv = 0$. Let $x \in \operatorname{Dom} T^\star$. One can write

$$x = x' + x'' \quad \text{where} \quad x' \in \operatorname{Ker} S \quad \text{and} \quad x'' \in (\operatorname{Ker} S)^\perp \subset (\operatorname{Im} T)^\perp = \operatorname{Ker} T^\star.$$

Since $x, x'' \in \operatorname{Dom} T^\star$, we have also $x' \in \operatorname{Dom} T^\star$. We get

$$\langle v, x\rangle_2 = \langle v, x'\rangle_2 + \langle v, x''\rangle_2 = \langle v, x'\rangle_2$$

because $v \in \operatorname{Ker} S$ and $x'' \in (\operatorname{Ker} S)^\perp$. As $Sx' = 0$ and $T^\star x'' = 0$, the Cauchy-Schwarz inequality combined with (1.3) implies

$$|\langle v, x\rangle_2|^2 \le \|v\|_2^2\ \|x'\|_2^2 \le \frac{1}{C}\|v\|_2^2\ \|T^\star x'\|_1^2 = \frac{1}{C}\|v\|_2^2\ \|T^\star x\|_1^2.$$

This shows that the linear form $T_X^\star \ni x \longmapsto \langle x, v\rangle_2$ is continuous on $\operatorname{Im} T^\star \subset \mathcal{H}_1$ with norm $\le C^{-1/2}\|v\|_2$. By the Hahn-Banach theorem, this form can be extended to a continuous linear form on $\mathcal{H}_1$ of norm $\le C^{-1/2}\|v\|_2$, i.e. we can find $u \in \mathcal{H}_1$ such that $\|u\|_1 \le C^{-1/2}\|v\|_2$ and

$$\langle x, v\rangle_2 = \langle T^\star x, u\rangle_1, \qquad \forall x \in \operatorname{Dom} T^\star.$$

This means that $u \in \operatorname{Dom}(T^\star)^\star = \operatorname{Dom} T$ and $v = Tu$. We have thus shown that $\operatorname{Im} T = \operatorname{Ker} S$, in particular $\operatorname{Im} T$ is closed. The dual equality $\operatorname{Im} S^\star = \operatorname{Ker} T^\star$ follows by considering the dual pair $(S^\star, T^\star)$. □

2. Complete Riemannian Manifolds

Let (M, g) be a riemannian manifold of dimension m, with metric

$$g(x) = \sum g_{jk}(x)\, dx_j \otimes dx_k, \qquad 1 \le j, k \le m.$$

The length of a path $\gamma : [a, b] \longrightarrow M$ is by definition

$$\ell(\gamma) = \int_a^b |\gamma'(t)|_g dt = \int_a^b \Big(\sum_{j,k} g_{jk}\big(\gamma(t)\big)\, \gamma_j'(t)\gamma_k'(t)\Big)^{1/2} dt.$$

The geodesic distance of two points $x, y \in M$ is

$$\delta(x, y) = \inf_\gamma \ell(\gamma) \qquad \text{over paths } \gamma \text{ with } \ \gamma(a) = x, \ \ \gamma(b) = y,$$

if x,y are in the same connected component of M, $\delta(x,y)=+\infty$ otherwise. It is easy to check that δ satisfies the usual axioms of distances: for the separation axiom, use the fact that if y is outside some closed coordinate ball $\overline{B}$ of radius r centered at x and if $g\geq c|dx|^2$ on $\overline{B}$, then $\delta(x,y)\geq c^{1/2}r$. In addition, δ satisfies the axiom:

$$\text{(2.1)}\quad \text{for every } x,y\in M,\qquad \inf_{z\in M}\max\{\delta(x,z),\delta(y,z)\}=\frac{1}{2}\delta(x,y).$$

In fact for every $\varepsilon>0$ there is a path γ such that $\gamma(a)=x$, $\gamma(b)=y$, $\ell(\gamma)<\delta(x,y)+\varepsilon$ and we can take z to be at mid-distance between x and y along γ. A metric space E with a distance δ satisfying the additional axiom (2.1) will be called a *geodesic* metric space. It is then easy to see by dichotomy that any two points $x,y\in E$ can be joined by a chain of points $x=x_0,\ x_1,\ldots,x_N=y$ such that $\delta(x_j,x_{j+1})<\varepsilon$ and $\sum\delta(x_j,x_{j+1})<\delta(x,y)+\varepsilon$.

(2.2) Lemma (Hopf-Rinow)**.** *Let (E,δ) be a geodesic metric space. Then the following properties are equivalent:*

a) *E is locally compact and complete;*

b) *all closed geodesic balls $\overline{B}(x_0,r)$ are compact.*

Proof. Since any Cauchy sequence is bounded, it is immediate that b) implies a). We now check that a) $\Longrightarrow$ b). Fix x_0 and define R to be the supremum of all $r>0$ such that $\overline{B}(x_0,r)$ is compact. Since E is locally compact, we have $R>0$. Suppose that $R<+\infty$. Then $\overline{B}(x_0,r)$ is compact for every $r<R$. Let y_ν be a sequence of points in $\overline{B}(x_0,R)$. Fix an integer p. As $\delta(x_0,y_\nu)\leq R$, axiom (2.1) shows that we can find points $z_\nu\in M$ such that $\delta(x_0,z_\nu)\leq(1-2^{-p})R$ and $\delta(z_\nu,y_\nu)\leq 2^{1-p}R$. Since $\overline{B}(x_0,(1-2^{-p})R)$ is compact, there is a subsequence $(z_{\nu(p,q)})_{q\in\mathbb{N}}$ converging to a limit point w_p with $\delta(z_{\nu(p,q)},w_p)\leq 2^{-q}$. We proceed by induction on p and take $\nu(p+1,q)$ to be a subsequence of $\nu(p,q)$. Then

$$\delta(y_{\nu(p,q)},w_p)\leq\delta(y_{\nu(p,q)},z_{\nu(p,q)})+\delta(z_{\nu(p,q)},w_p)\leq 2^{1-p}R+2^{-q}.$$

Since $(y_{\nu(p+1,q)})$ is a subsequence of $(y_{\nu(p,q)})$, we infer from this that $\delta(w_p,w_{p+1})\leq 3\,2^{-p}R$ by letting q tend to $+\infty$. By the completeness hypothesis, the Cauchy sequence (w_p) converges to a limit point $w\in M$, and the above inequalities show that $(y_{\nu(p,p)})$ converges to $w\in\overline{B}(x_0,R)$. Therefore $\overline{B}(x_0,R)$ is compact. Now, each point $y\in\overline{B}(x_0,R)$ can be covered by a compact ball $\overline{B}(y,\varepsilon_y)$, and the compact set $\overline{B}(x_0,R)$ admits a finite covering by concentric balls $B(y_j,\varepsilon_{y_j}/2)$. Set $\varepsilon=\min\varepsilon_{y_j}$. Every point $z\in\overline{B}(x_0,R+\varepsilon/2)$ is at distance $\leq\varepsilon/2$ of some point $y\in\overline{B}(x_0,R)$, hence at distance $\leq$ $\varepsilon/2+\varepsilon_{y_j}/2$ of some point y_j, in particular $\overline{B}(x_0,R+\varepsilon/2)\subset\bigcup\overline{B}(y_j,\varepsilon_{y_j})$ is compact. This is a contradiction, so $R=+\infty$. $\square$

The following standard definitions and properties will be useful in order to deal with the completeness of the metric.

(2.3) Definitions.

a) *A riemannian manifold* (M, g) *is said to be complete if* (M, δ) *is complete as a metric space.*

b) *A continuous function* $\psi : M \to \mathbb{R}$ *is said to be exhaustive if for every* $c \in \mathbb{R}$ *the sublevel set* $M_c = \{x \in M \;;\; \psi(x) < c\}$ *is relatively compact in* M.

c) *A sequence* $(K_\nu)_{\nu \in \mathbb{N}}$ *of compact subsets of* M *is said to be exhaustive if* $M = \bigcup K_\nu$ *and if* K_ν *is contained in the interior of* $K_{\nu+1}$ *for all* ν *(so that every compact subset of* M *is contained in some* K_ν*).*

(2.4) Lemma. *The following properties are equivalent:*

a) (M, g) *is complete;*

b) *there exists an exhaustive function* $\psi \in C^\infty(M, \mathbb{R})$ *such that* $|d\psi|_g \le 1$;

c) *there exists an exhaustive sequence* $(K_\nu)_{\nu \in \mathbb{N}}$ *of compact subsets of* M *and functions* $\psi_\nu \in C^\infty(M, \mathbb{R})$ *such that*

$$\psi_\nu = 1 \quad \text{in a neighborhood of } K_\nu, \qquad \operatorname{Supp} \psi_\nu \subset K^\circ_{\nu+1},$$
$$0 \le \psi_\nu \le 1 \quad \text{and} \quad |d\psi_\nu|_g \le 2^{-\nu}.$$

Proof. a) $\Longrightarrow$ b). Without loss of generality, we may assume that M is connected. Select a point $x_0 \in M$ and set $\psi_0(x) = \frac{1}{2}\delta(x_0, x)$. Then ψ_0 is a Lipschitz function with constant $\frac{1}{2}$, thus ψ_0 is differentiable almost everywhere on M and $|d\psi_0|_g \le \frac{1}{2}$. We can find a smoothing ψ of ψ_0 such that $|d\psi|_g \le 1$ and $|\psi - \psi_0| \le 1$. Then ψ is an exhaustion function of M.

b) $\Longrightarrow$ c). Choose ψ as in a) and a function $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\rho = 1$ on $]-\infty, 1.1]$, $\rho = 0$ on $[1.9, +\infty[$ and $0 \le \rho' \le 2$ on $[1, 2]$. Then

$$K_\nu = \{x \in M \;;\; \psi(x) \le 2^{\nu+1}\}, \qquad \psi_\nu(x) = \rho\big(2^{-\nu-1}\psi(x)\big)$$

satisfy our requirements.

c) $\Longrightarrow$ b). Set $\psi = \sum 2^\nu (1 - \psi_\nu)$.

b) $\Longrightarrow$ a). The inequality $|d\psi|_g \le 1$ implies $|\psi(x) - \psi(y)| \le \delta(x, y)$ for all $x, y \in M$, so all δ-balls must be relatively compact in M. □

3. L^2 Hodge Theory on Complete Riemannian Manifolds

Let (M,g) be a riemannian manifold and let F_1, F_2 be hermitian C^∞ vector bundles over M. If $P : C^\infty(M,F_1) \longrightarrow C^\infty(M,F_2)$ is a differential operator with smooth coefficients, then P induces a non bounded operator

$$\widetilde{P} : L^2(M,F_1) \longrightarrow L^2(M,F_2)$$

as follows: if $u \in L^2(M,F_1)$, we compute $\widetilde{P}u$ in the sense of distribution theory and we say that $u \in \operatorname{Dom}\widetilde{P}$ if $\widetilde{P}u \in L^2(M,F_2)$. It follows that $\widetilde{P}$ is densely defined, since $\operatorname{Dom} P$ contains the set $\mathcal{D}(M,F_1)$ of compactly supported sections of $C^\infty(M,F_1)$, which is dense in $L^2(M,F_1)$. Furthermore $\operatorname{Gr}\widetilde{P}$ is closed: if $u_\nu \to u$ in $L^2(M,F_1)$ and $\widetilde{P}u_\nu \to v$ in $L^2(M,F_2)$ then $\widetilde{P}u_\nu \to \widetilde{P}u$ in the weak topology of distributions, thus we must have $\widetilde{P}u = v$ and $(u,v) \in \operatorname{Gr}\widetilde{P}$. By the general results of § 1, we see that $\widetilde{P}$ has a closed and densely defined Von Neumann adjoint $\big(\widetilde{P}\big)^\star$. We want to stress, however, that $\big(\widetilde{P}\big)^\star$ *does not always* coincide with the extension $(P^\star)^\sim$ of the formal adjoint $P^\star : C^\infty(M,F_2) \longrightarrow C^\infty(M,F_1)$, computed in the sense of distribution theory. In fact $u \in \operatorname{Dom}(\widetilde{P})^\star$, resp. $u \in \operatorname{Dom}(P^\star)^\sim$, if and only if there is an element $v \in L^2(M,F_1)$ such that $\langle u, \widetilde{P}f\rangle = \langle v, f\rangle$ for all $f \in \operatorname{Dom}\widetilde{P}$, resp. for all $f \in \mathcal{D}(M,F_1)$. Therefore we always have $\operatorname{Dom}(\widetilde{P})^\star \subset \operatorname{Dom}(P^\star)^\sim$ and the inclusion may be strict because the integration by parts to perform may involve boundary integrals for $(\widetilde{P})^\star$.

(3.1) Example. Consider

$$P = \frac{d}{dx} : L^2\big(]0,1[\big) \longrightarrow L^2\big(]0,1[\big)$$

where the L^2 space is taken with respect to the Lebesgue measure dx. Then $\operatorname{Dom}\widetilde{P}$ consists of all L^2 functions with L^2 derivatives on $]0,1[$. Such functions have a continuous extension to the interval $[0,1]$. An integration by parts shows that

$$\int_0^1 u\overline{\frac{df}{dx}}\,dx = \int_0^1 -\frac{du}{dx}\overline{f}\,dx$$

for all $f \in \mathcal{D}(]0,1[)$, thus $P^\star = -d/dx = -P$. However for $f \in \operatorname{Dom}\widetilde{P}$ the integration by parts involves the extra term $u(1)\overline{f}(1) - u(0)\overline{f}(0)$ in the right hand side, which is thus continuous in f with respect to the L^2 topology if and only if $du/dx \in L^2$ and $u(0) = u(1) = 0$. Therefore $\operatorname{Dom}(\widetilde{P})^\star$ consists of all $u \in \operatorname{Dom}(P^\star)^\sim = \operatorname{Dom}\widetilde{P}$ satisfying the additional boundary condition $u(0) = u(1) = 0$. □

Let $E \to M$ be a differentiable hermitian bundle. In what follows, we still denote by D, δ, Δ the differential operators of § VI-2 extended in the sense of distribution theory (as explained above). These operators are thus closed and densely defined operators on $L^2_\bullet(M,E) = \bigoplus_p L^2_p(M,E)$. We also introduce the space $\mathcal{D}_p(M,E)$ of compactly supported forms in $C^\infty_p(M,E)$. The theory relies heavily on the following important result.

(3.2) Theorem. *Assume that (M,g) is complete. Then*

a) *$\mathcal{D}_\bullet(M,E)$ is dense in $\operatorname{Dom} D$, $\operatorname{Dom}\delta$ and $\operatorname{Dom} D \cap \operatorname{Dom}\delta$ respectively for the graph norms*

$$u \mapsto \|u\| + \|Du\|, \qquad u \mapsto \|u\| + \|\delta u\|, \qquad u \mapsto \|u\| + \|Du\| + \|\delta u\|.$$

b) *$D^\star = \delta$, $\delta^\star = D$ as adjoint operators in Von Neumann's sense.*

c) *One has $\langle u, \Delta u\rangle = \|Du\|^2 + \|\delta u\|^2$ for every $u \in \operatorname{Dom}\Delta$. In particular*

$$\operatorname{Dom}\Delta \subset \operatorname{Dom} D \cap \operatorname{Dom}\delta, \qquad \operatorname{Ker}\Delta = \operatorname{Ker} D \cap \operatorname{Ker}\delta,$$

and Δ is self-adjoint.

d) *If $D^2 = 0$, there are orthogonal decompositions*

$$L^2_\bullet(M,E) = \mathcal{H}^\bullet(M,E) \oplus \overline{\operatorname{Im} D} \oplus \overline{\operatorname{Im}\delta},$$
$$\operatorname{Ker} D = \mathcal{H}^\bullet(M,E) \oplus \overline{\operatorname{Im} D},$$

where $\mathcal{H}^\bullet(M,E) = \big\{u \in L^2_\bullet(M,E)\,;\ \Delta u = 0\big\} \subset C^\infty_\bullet(M,E)$ is the space of L^2 harmonic forms.

Proof. a) We show that every element $u \in \operatorname{Dom} D$ can be approximated in the graph norm of D by smooth and compactly supported forms. By hypothesis, u and Du belong to $L^2_\bullet(M,E)$. Let (ψ_ν) be a sequence of functions as in Lemma 2.4 c). Then $\psi_\nu u \to u$ in $L^2_\bullet(M,E)$ and $D(\psi_\nu u) = \psi_\nu Du + d\psi_\nu \wedge u$ where

$$|d\psi_\nu \wedge u| \le |d\psi_\nu|\ |u| \le 2^{-\nu}|u|.$$

Therefore $d\psi_\nu \wedge u \to 0$ and $D(\psi_\nu u) \to Du$. After replacing u by $\psi_\nu u$, we may assume that u has compact support, and by using a finite partition of unity on a neighborhood of $\operatorname{Supp} u$ we may also assume that $\operatorname{Supp} u$ is contained in a coordinate chart of M on which E is trivial. Let A be the connection form of D on this chart and (ρ_ε) a family of smoothing kernels. Then $u \star \rho_\varepsilon \in \mathcal{D}_\bullet(M,E)$ converges to u in $L^2(M,E)$ and

$$D(u\star\rho_\varepsilon) - (Du)\star\rho_\varepsilon = A \wedge (u\star\rho_\varepsilon) - (A\wedge u)\star\rho_\varepsilon$$

because d commutes with convolution (as any differential operator with constant coefficients). Moreover $(Du)\star\rho_\varepsilon$ converges to Du in $L^2(M,E)$ and $A\wedge(u\star\rho_\varepsilon)$, $(A\wedge u)\star\rho_\varepsilon$ both converge to $A\wedge u$ since $A\wedge\bullet$ acts continuously on L^2. Thus $D(u\star\rho_\varepsilon)$ converges to Du and the density of $\mathcal{D}_\bullet(M,E)$ in $\operatorname{Dom} D$ follows. The proof for $\operatorname{Dom}\delta$ and $\operatorname{Dom} D\cap\operatorname{Dom}\delta$ is similar, except that the principal part of δ no longer has constant coefficients in general. The convolution technique requires in this case the following lemma due to K.O. Friedrichs (see e.g. Hörmander 1963).

(3.3) Lemma. *Let $Pf = \sum a_k \partial f/\partial x_k + bf$ be a differential operator of order 1 on an open set $\Omega \subset \mathbb{R}^n$, with coefficients $a_k \in C^1(\Omega)$, $b \in C^0(\Omega)$. Then for any $v \in L^2(\mathbb{R}^n)$ with compact support in Ω we have*

$$\lim_{\varepsilon \to 0} ||P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon||_{L^2} = 0.$$

Proof. It is enough to consider the case when $P = a\partial/\partial x_k$. As the result is obvious if $v \in C^1$, we only have to show that

$$||P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon||_{L^2} \le C||v||_{L^2}$$

and to use a density argument. A computation of $w_\varepsilon = P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon$ by means of an integration by parts gives

$$\begin{aligned} w_\varepsilon(x) &= \int_{\mathbb{R}^n} \Big(a(x) \frac{\partial v}{\partial x_k}(x - \varepsilon y)\rho(y) - a(x - \varepsilon y)\frac{\partial v}{\partial x_k}(x - \varepsilon y)\rho(y)\Big) dy \\ &= \int_{\mathbb{R}^n} \Big(\big(a(x) - a(x - \varepsilon y)\big) v(x - \varepsilon y) \frac{1}{\varepsilon} \partial_k \rho(y) \\ &\qquad\qquad + \partial_k a(x - \varepsilon y) v(x - \varepsilon y)\rho(y)\Big) dy. \end{aligned}$$

If C is a bound for $|da|$ in a neighborhood of Supp v, we get

$$|w_\varepsilon(x)| \le C \int_{\mathbb{R}^n} |v(x - \varepsilon y)| \big(|y|\, |\partial_k \rho(y)| + |\rho(y)|\big) dy,$$

so Minkowski's inequality $||f \star g||_{L^p} \le ||f||_{L^1} ||g||_{L^p}$ gives

$$||w_\varepsilon||_{L^2} \le C\Big(\int_{\mathbb{R}^n} \big(|y|\, |\partial_k \rho(y)| + |\rho(y)|\big) dy\Big) ||v||_{L^2}. \qquad \square$$

Proof (end). b) is equivalent to the fact that

$$\langle\!\langle Du, v \rangle\!\rangle = \langle\!\langle u, \delta v \rangle\!\rangle, \qquad \forall u \in \mathrm{Dom}\, D, \quad \forall v \in \mathrm{Dom}\, \delta.$$

By a), we can find $u_\nu,\ v_\nu \in \mathcal{D}_\bullet(M, E)$ such that

$$u_\nu \to u, \qquad v_\nu \to v, \qquad Du_\nu \to Du \quad \text{and} \quad \delta v_\nu \to \delta v \quad \text{in} \quad L^2_\bullet(M, E),$$

and the required equality is the limit of the equalities $\langle\!\langle Du_\nu, v_\nu \rangle\!\rangle = \langle\!\langle u_\nu, \delta v_\nu \rangle\!\rangle$.

c) Let $u \in \mathrm{Dom}\, \Delta$. As Δ is an elliptic operator of order 2, u must be in $W^2_\bullet(M, E, \mathrm{loc})$ by Gårding's inequality. In particular $Du,\ \delta u \in L^2(M, E, \mathrm{loc})$

and we can perform all integrations by parts that we want if the forms are multiplied by compactly supported functions ψ_ν. Let us compute

$$\begin{aligned}
&\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 = \\
&= \langle\langle \psi_\nu^2 Du, Du\rangle\rangle + \langle\langle u, D(\psi_\nu^2 \delta u)\rangle\rangle \\
&= \langle\langle D(\psi_\nu^2 u), Du\rangle\rangle + \langle\langle u, \psi_\nu^2 D\delta u\rangle\rangle - 2\langle\langle \psi_\nu d\psi_\nu \wedge u, Du\rangle\rangle + 2\langle\langle u, \psi_\nu d\psi_\nu \wedge \delta u\rangle\rangle \\
&= \langle\langle \psi_\nu^2 u, \Delta u\rangle\rangle - 2\langle\langle d\psi_\nu \wedge u, \psi_\nu Du\rangle\rangle + 2\langle\langle u, d\psi_\nu \wedge (\psi_\nu \delta u)\rangle\rangle \\
&\le \langle\langle \psi_\nu^2 u, \Delta u\rangle\rangle + 2^{-\nu}\big(2\|\psi_\nu Du\|\,\|u\| + 2\|\psi_\nu \delta u\|\,\|u\|\big) \\
&\le \langle\langle \psi_\nu^2 u, \Delta u\rangle\rangle + 2^{-\nu}\big(\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 + 2\|u\|^2\big).
\end{aligned}$$

We get therefore

$$\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 \le \frac{1}{1-2^{-\nu}}\big(\langle\langle \psi_\nu^2 u, \Delta u\rangle\rangle + 2^{1-\nu}\|u\|^2\big).$$

By letting ν tend to $+\infty$, we obtain $\|Du\|^2 + \|\delta u\|^2 \le \langle\langle u, \Delta u\rangle\rangle$, in particular Du, δu are in $L^2_\bullet(M,E)$. This implies

$$\langle\langle u, \Delta v\rangle\rangle = \langle\langle Du, Dv\rangle\rangle + \langle\langle \delta u, \delta v\rangle\rangle, \qquad \forall u, v \in \operatorname{Dom}\Delta,$$

because the equality holds for $\psi_\nu u$ and v, and because we have $\psi_\nu u \to u$, $D(\psi_\nu u) \to Du$ and $\delta(\psi_\nu u) \to \delta u$ in L^2. Therefore Δ is self-adjoint.

d) is an immediate consequence of b), c) and Th. 1.2. □

On a complete hermitian manifold (X,ω), there are of course similar results for the operators $D', D'', \delta', \delta'', \Delta', \Delta''$ attached to a hermitian vector bundle E.

4. General Estimate for d'' on Hermitian Manifolds

Let (X,ω) be a *complete* hermitian manifold and E a hermitian holomorphic vector bundle of rank r over X. Assume that the hermitian operator

(4.1) $\quad A_{E,\omega} = [\mathrm{i}\Theta(E), \Lambda] + T_\omega$

is *semi-positive* on $\Lambda^{p,q}T_X^\star \otimes E$. Then for every form $u \in \operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$ of bidegree (p,q) we have

(4.2) $\quad \|D''u\|^2 + \|\delta'' u\|^2 \ge \int_X \langle A_{E,\omega} u, u\rangle\, dV.$

In fact (4.2) is true for all $u \in \mathcal{D}_{p,q}(X,E)$ in view of the Bochner-Kodaira-Nakano identity VII-2.3, and this result is easily extended to every u in $\operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$ by density of $\mathcal{D}_{p,q}(X,E)$ (Th. 3.2 a)).

Assume now that a form $g \in L^2_{p,q}(X,E)$ is given such that

$$(4.3)\quad D''g = 0,$$

and that for almost every $x \in X$ there exists $\alpha \in [0, +\infty[$ such that

$$|\langle g(x), u\rangle|^2 \le \alpha\,\langle A_{E,\omega}u, u\rangle$$

for every $u \in (\Lambda^{p,q}T^\star_X \otimes E)_x$. If the operator $A_{E,\omega}$ is invertible, the minimal such number α is $|A_{E,\omega}^{-1/2}g(x)|^2 = \langle A_{E,\omega}^{-1}g(x), g(x)\rangle$, so we shall always denote it in this way even when $A_{E,\omega}$ is no longer invertible. Assume furthermore that

$$(4.4)\quad \int_X \langle A_{E,\omega}^{-1}g, g\rangle\, dV < +\infty.$$

The basic result of L^2 theory can be stated as follows.

(4.5) Theorem. *If (X,ω) is complete and $A_{E,\omega} \ge 0$ in bidegree (p,q), then for any $g \in L^2_{p,q}(X,E)$ satisfying* (4.4) *such that $D''g = 0$ there exists $f \in L^2_{p,q-1}(X,E)$ such that $D''f = g$ and*

$$\|f\|^2 \le \int_X \langle A_{E,\omega}^{-1}g, g\rangle\, dV.$$

Proof. For every $u \in \operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$ we have

$$\begin{aligned}\big|\langle\!\langle u, g\rangle\!\rangle\big|^2 = \Big|\int_X \langle u, g\rangle\, dV\Big|^2 &\le \Big(\int_X \langle A_{E,\omega}u, u\rangle^{1/2}\langle A_{E,\omega}^{-1}g, g\rangle^{1/2}\, dV\Big)^2\\ &\le \int_X \langle A_{E,\omega}^{-1}g, g\rangle\, dV \cdot \int_X \langle A_{E,\omega}u, u\rangle\, dV\end{aligned}$$

by means of the Cauchy-Schwarz inequality. The a priori estimate (4.2) implies

$$\big|\langle\!\langle u, g\rangle\!\rangle\big|^2 \le C\big(\|D''u\|^2 + \|\delta''u\|^2\big), \qquad \forall u \in \operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$$

where C is the integral (4.4). Now we just have to repeat the proof of the existence part of Th. 1.2. For any $u \in \operatorname{Dom}\delta''$, let us write

$$u = u_1 + u_2, \qquad u_1 \in \operatorname{Ker} D'', \qquad u_2 \in (\operatorname{Ker} D'')^\perp = \overline{\operatorname{Im}\delta''}.$$

Then $D''u_1 = 0$ and $\delta''u_2 = 0$. Since $g \in \operatorname{Ker} D''$, we get

$$\big|\langle\!\langle u, g\rangle\!\rangle\big|^2 = \big|\langle\!\langle u_1, g\rangle\!\rangle\big|^2 \le C\|\delta''u_1\|^2 = C\|\delta''u\|^2.$$

The Hahn-Banach theorem shows that the continuous linear form

$$L^2_{p,q-1}(X,E) \ni \delta''u \longmapsto \langle\!\langle u, g\rangle\!\rangle$$

can be extended to a linear form $v \longmapsto \langle\langle v, f\rangle\rangle$, $f \in L^2_{p,q-1}(X,E)$, of norm $\|f\| \le C^{1/2}$. This means that

$$\langle\langle u, g\rangle\rangle = \langle\langle \delta'' u, f\rangle\rangle, \qquad \forall u \in \operatorname{Dom} \delta'',$$

i.e. that $D'' f = g$. The theorem is proved. □

(4.6) Remark. One can always find a solution $f \in (\operatorname{Ker} D'')^\perp$: otherwise replace f by its orthogonal projection on $(\operatorname{Ker} D'')^\perp$. This solution is clearly unique and is precisely the solution of minimal L^2 norm of the equation $D'' f = g$. We have $f \in \overline{\operatorname{Im} \delta''}$, thus f satifies the additional equation

$$(4.7) \quad \delta'' f = 0.$$

Consequently $\Delta'' f = \delta'' D'' f = \delta'' g$. If $g \in C^\infty_{p,q}(X,E)$, the ellipticity of Δ'' shows that $f \in C^\infty_{p,q-1}(X,E)$.

(4.8) Remark. If $A_{E,\omega}$ is positive definite, let $\lambda(x) > 0$ be the smallest eigenvalue of this operator at $x \in X$. Then λ is continuous on X and we have

$$\int_X \langle A^{-1}_{E,\omega} g, g\rangle \, dV \le \int_X \lambda(x)^{-1} |g(x)|^2 \, dV.$$

The above situation occurs for example if ω is complete Kähler, $E >_m 0$ and $p = n$, $q \ge 1$, $m \ge \min\{n - q + 1, r\}$ (apply Lemma VII-7.2).

5. Estimates on Weakly Pseudoconvex Manifolds

We first introduce a large class of complex manifolds on which the L^2 estimates will be easily tractable.

(5.1) Definition. *A complex manifold X is said to be weakly pseudoconvex if there exists an exhaustion function $\psi \in C^\infty(X, \mathbb{R})$ such that $\mathrm{i} d'd''\psi \ge 0$ on X, i.e. ψ is plurisubharmonic.*

For domains $\Omega \subset \mathbb{C}^n$, the above weak pseudoconvexity notion is equivalent to pseudoconvexity (cf. Th. I-4.14). Note that every compact manifold is also weakly pseudoconvex (take $\psi \equiv 0$). Other examples that will appear later are Stein manifolds, or the total space of a Griffiths semi-negative vector bundle over a compact manifold (cf. Prop. IX-?.?).

(5.2) Theorem. *Every weakly pseudoconvex Kähler manifold (X, ω) carries a complete Kähler metric $\widehat{\omega}$.*

Proof. Let $\psi \in C^\infty(X,\mathbb{R})$ be an exhaustive plurisubharmonic function on X. After addition of a constant to ψ, we can assume $\psi \geq 0$. Then $\widehat{\omega} = \omega + \mathrm{i}d'd''(\psi^2)$ is Kähler and

$$\widehat{\omega} = \omega + 2i\psi d'd''\psi + 2\mathrm{i}d'\psi \wedge d''\psi \geq \omega + 2\mathrm{i}d'\psi \wedge d''\psi.$$

Since $d\psi = d'\psi + d''\psi$, we get $|d\psi|_{\widehat{\omega}} = \sqrt{2}|d'\psi|_{\widehat{\omega}} \leq 1$ and Lemma 2.4 shows that $\widehat{\omega}$ is complete. $\square$

Observe that we could have set more generally $\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi \circ \psi)$ where χ is a convex increasing function. Then

$$\begin{aligned} \widehat{\omega} &= \omega + i(\chi' \circ \psi)d'd''\psi + i(\chi'' \circ \psi)d'\psi \wedge d''\psi \\ &\geq \omega + \mathrm{i}d'(\rho \circ \psi) \wedge d''(\rho \circ \psi) \end{aligned} \tag{5.3}$$

where $\rho(t) = \int_0^t \sqrt{\chi''(u)}\,du$. We thus have $|d'(\rho \circ \psi)|_{\widehat{\omega}} \leq 1$ and $\widehat{\omega}$ will be complete as soon as $\lim_{t\to+\infty} \rho(t) = +\infty$, i.e.

$$\int_0^{+\infty} \sqrt{\chi''(u)}\,du = +\infty. \tag{5.4}$$

One can take for example $\chi(t) = t - \log(t)$ for $t \geq 1$.

It follows from the above considerations that almost all vanishing theorems for positive vector bundles over compact manifolds are also valid on weakly pseudoconvex manifolds. Let us mention here the analogues of some results proved in Chapter 7.

(5.5) Theorem. *For any m-positive vector bundle of rank r over a weakly pseudoconvex manifold X, we have $H^{n,q}(X,E) = 0$ for all $q \geq 1$ and $m \geq \min\{n-q+1, r\}$.*

Proof. The curvature form $\mathrm{i}\Theta(\det E)$ is a Kähler metric on X, hence X possesses a complete Kähler metric ω. Let $\psi \in C^\infty(X,\mathbb{R})$ be an exhaustive plurisubharmonic function. For any convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$, we denote by E_χ the holomorphic vector bundle E together with the modified metric $|u|_\chi^2 = |u|^2 \exp\big(-\chi \circ \psi(x)\big)$, $u \in E_x$. We get

$$\mathrm{i}\Theta(E_\chi) = \mathrm{i}\Theta(E) + \mathrm{i}d'd''(\chi \circ \psi) \otimes \mathrm{Id}_E \geq_m \mathrm{i}\Theta(E),$$

thus $A_{E_\chi,\omega} \geq A_{E,\omega} > 0$ in bidegree (n,q). Let g be a given form of bidegree (n,q) with L^2_{loc} coefficients, such that $D''g = 0$. The integrals

$$\int_X \langle A_{E_\chi,\omega}^{-1} g, g\rangle_\chi\,dV \leq \int_X \langle A_{E,\omega}^{-1} g, g\rangle\, e^{-\chi\circ\psi}\,dV, \qquad \int_X |g|^2\, e^{-\chi\circ\psi}\,dV$$

become convergent if χ grows fast enough. We can thus apply Th. 4.5 to (X, E_χ, ω) and find a $(n, q-1)$ form f such that $D''f = g$. If g is smooth, Remark 4.6 shows that f can also be chosen smooth. $\square$

(5.6) Theorem. *If E is a positive line bundle over a weakly pseudoconvex manifold X, then $H^{p,q}(X,E)=0$ for $p+q\ge n+1$.*

Proof. The proof is similar to that of Th. 5.5, except that we use here the Kähler metric

$$\omega_\chi = \mathrm{i}\Theta(E_\chi) = \omega + \mathrm{i}d'd''(\chi\circ\psi), \qquad \omega = \mathrm{i}\Theta(E),$$

which depends on χ. By (5.4) ω_χ is complete as soon as χ is a convex increasing function that grows fast enough. Apply now Th. 4.5 to (X, E_χ, ω_χ) and observe that $A_{E_\chi,\omega_\chi} = [\mathrm{i}\Theta(E_\chi), \Lambda_\chi] = (p+q-n)\ \mathrm{Id}$ in bidegree (p,q) in virtue of Cor. VI-8.4 It remains to show that for every form $g\in C^\infty_{p,q}(X,E)$ there exists a choice of χ such that $g\in L^2_{p,q}(X,E_\chi,\omega_\chi)$. By (5.3) the norm of a scalar form with respect to ω_χ is less than its norm with respect to ω, hence $|g|^2_\chi \le |g|^2\exp(-\chi\circ\psi)$. On the other hand

$$dV_\chi \le C\big(1+\chi'\circ\psi+\chi''\circ\psi\big)^n\,dV$$

where C is a positive continuous function on X. The following lemma implies that we can always choose χ in order that the integral of $|g|^2_\chi\,dV_\chi$ converges on X.

(5.7) Lemma. *For any positive function $\lambda\in C^\infty\big([0,+\infty[,\mathbb{R}\big)$, there exists a smooth convex function $\chi\in C^\infty\big([0,+\infty[,\mathbb{R}\big)$ such that $\chi,\chi',\chi''\ge\lambda$ and $(1+\chi'+\chi'')^n e^{-\chi}\le 1/\lambda$.*

Proof. We shall construct χ such that $\chi''\ge\chi'\ge\chi\ge\lambda$ and $\chi''/\chi^2\le C$ for some constant C. Then χ satisties the conclusion of the lemma after addition of a constant. Without loss of generality, we may assume that λ is increasing and $\lambda\ge 1$. We define χ as a power series

$$\chi(t)=\sum_{k=0}^{+\infty} a_0a_1\ldots a_k\,t^k,$$

where $a_k>0$ is a decreasing sequence converging to 0 very slowly. Then χ is real analytic on $\mathbb{R}$ and the inequalities $\chi''\ge\chi'\ge\chi$ are realized if we choose $a_k\ge 1/k$, $k\ge 1$. Select a strictly increasing sequence of integers $(N_p)_{p\ge1}$ so large that $\frac{1}{p}\lambda(p+1)^{1/N_p}\in[1/p, 1/(p-1)[$. We set

$$a_0=\ldots=a_{N_1-1}=e\,\lambda(2),$$
$$a_k=\frac{1}{p}\,\lambda(p+1)^{1/N_p}\,e^{1/\sqrt{k}}, \qquad N_p\le k<N_{p+1}.$$

Then (a_k) is decreasing. For $t\in[0,1]$ we have $\chi(t)\ge a_0\ge\lambda(t)$ and for $t\in[1,+\infty[$ the choice $k=N_p$ where $p=[t]$ is the integer part of t gives

$$\chi(t)\ge\chi(p)\ge(a_0a_1\ldots a_k)p^k\ge(a_kp)^k\ge\lambda(p+1)\ge\lambda(t).$$

Furthermore, we have

$$\chi(t)^2 \geq \sum_{k\geq 0} (a_0a_1\dots a_k)^2\, t^{2k},$$
$$\chi''(t) = \sum_{k\geq 0} (k+1)(k+2)\, a_0a_1\dots a_{k+2}\, t^k,$$

thus we will get $\chi''(t) \leq C\chi(t)^2$ if we can prove that

$$m^2\, a_0a_1\dots a_{2m} \leq C'(a_0a_1\dots a_m)^2, \qquad m\geq 0.$$

However, as $\frac{1}{p}\lambda(p+1)^{1/N_p}$ is decreasing, we find

$$\begin{aligned}\frac{a_0a_1\dots a_{2m}}{(a_0a_1\dots a_m)^2} &= \frac{a_{m+1}\dots a_{2m}}{a_0a_1\dots a_m}\\ &\leq \exp\Big(\frac{1}{\sqrt{m+1}}+\dots+\frac{1}{\sqrt{2m}}-\frac{1}{\sqrt{1}}-\dots-\frac{1}{\sqrt{m}}+O(1)\Big)\\ &\leq \exp\big(2\sqrt{2m}-4\sqrt{m}+O(1)\big)\leq C'm^{-2}.\end{aligned}$$ □

As a last application, we generalize the Girbau vanishing theorem in the case of weakly pseudoconvex manifolds. This result is due to (Abdelkader 1980) and (Ohsawa 1981). We present here a simplified proof which appeared in (Demailly 1985).

(5.8) Theorem. *Let (X,ω) be a weakly pseudoconvex Kähler manifold. If E is a semi-positive line bundle such that* $\mathrm{i}\Theta(E)$ *has at least $n-s+1$ positive eigenvalues at every point, then*

$$H^{p,q}(X,E)=0 \qquad \text{for}\quad p+q\geq n+s.$$

Proof. Let $\chi,\rho\in C^\infty(\mathbb{R},\mathbb{R})$ be convex increasing functions to be specified later. We use here the *hermitian* metric

$$\begin{aligned}\alpha &= \mathrm{i}\Theta(E_\chi)+\exp(-\rho\circ\psi)\,\omega\\ &= \mathrm{i}\Theta(E)+\mathrm{i}d'd''(\chi\circ\psi)+\exp(-\rho\circ\psi)\,\omega.\end{aligned}$$

Although ω is Kähler, the metric α is not so. Denote by $\gamma_j^{\chi,\omega}$ (resp. $\gamma_j^{\chi,\alpha}$), $1\leq j\leq n$, the eigenvalues of $\mathrm{i}\Theta(E_\chi)$ with respect to ω (resp. α), rearranged in increasing order. The minimax principle implies $\gamma_j^{\chi,\omega}\geq\gamma_j^{0,\omega}$, and the hypothesis yields $0<\gamma_s^{0,\omega}\leq\gamma_{s+1}^{0,\omega}\leq\dots\leq\gamma_n^{0,\omega}$ on X. By means of a diagonalization of $\mathrm{i}\Theta(E_\chi)$ with respect to ω, we find

$$1\geq\gamma_j^{\chi,\alpha}=\frac{\gamma_j^{\chi,\omega}}{\gamma_j^{\chi,\omega}+\exp(-\rho\circ\psi)}\geq\frac{\gamma_j^{0,\omega}}{\gamma_j^{0,\omega}+\exp(-\rho\circ\psi)}.$$

Let $\varepsilon > 0$ be small. Select ρ such that $\exp(-\rho\circ\psi(x)) \le \varepsilon\gamma_s^{0,\omega}(x)$ at every point. Then for $j \ge s$ we get

$$\gamma_j^{\chi,\alpha} \ge \frac{\gamma_j^{0,\omega}}{\gamma_j^{0,\omega} + \varepsilon\gamma_j^{0,\omega}} = \frac{1}{1+\varepsilon} \ge 1-\varepsilon,$$

and Th. VI-8.3 implies

$$\begin{aligned}\langle [\mathrm{i}\Theta(E_\chi), \Lambda_\alpha]u, u\rangle_\alpha &\ge \big(\gamma_1^{\chi,\alpha} + \cdots + \gamma_p^{\chi,\alpha} - \gamma_{q+1}^{\chi,\alpha} - \ldots - \gamma_n^{\chi,\alpha}\big)|u|^2\\ &\ge \big((p-s+1)(1-\varepsilon) - (n-q)\big)|u|^2\\ &\ge \big(1-(p-s+1)\varepsilon\big)|u|^2.\end{aligned}$$

It remains however to control the torsion term T_α. As ω is Kähler, trivial computations yield

$$\begin{aligned} d'\alpha &= -\rho'\circ\psi\ \exp(-\rho\circ\psi)\ d'\psi\wedge\omega,\\ d'd''\alpha &= \exp(-\rho\circ\psi)\ \big[\big((\rho'\circ\psi)^2 - \rho''\circ\psi\big)d'\psi\wedge d''\psi - \rho'\circ\psi\ d'd''\psi\big]\wedge\omega.\end{aligned}$$

Since

$$\alpha \ge i(\chi'\circ\psi\ d'd''\psi + \chi''\circ\psi\ d'\psi\wedge d''\psi) + \exp(-\rho\circ\psi)\omega,$$

we get the upper bounds

$$\begin{aligned} |d'\alpha|_\alpha &\le \rho'\circ\psi\ |d'\psi|_\alpha\ |\exp(-\rho\circ\psi)\omega|_\alpha \le \rho'\circ\psi\ (\chi''\circ\psi)^{-\frac{1}{2}}\\ |d'd''\alpha|_\alpha &\le \frac{(\rho'\circ\psi)^2 + \rho''\circ\psi}{\chi''\circ\psi} + \frac{\rho'\circ\psi}{\chi'\circ\psi}.\end{aligned}$$

It is then clear that we can choose χ growing sufficiently fast in order that $|T_\alpha|_\alpha \le \varepsilon$. If ε is chosen sufficiently small, we get $A_{E_\chi,\alpha} \ge \frac{1}{2}\,\mathrm{Id}$, and the conclusion is obtained in the same way as for Th. 5.6. □

6. Hörmander's Estimates for non Complete Kähler Metrics

Our aim here is to derive also estimates for a non complete Kähler metric, for example the standard metric of $\mathbb{C}^n$ on a bounded domain $\Omega \subset\subset \mathbb{C}^n$. A result of this type can be obtained in the situation described at the end of Remark 4.8. The underlying idea is due to (Hörmander 1966), although we do not apply his so called "three weights" technique, but use instead an approximation of the given metric ω by complete Kähler metrics.

(6.1) Theorem. *Let $(X,\widehat{\omega})$ be a complete Kähler manifold, ω another Kähler metric, possibly non complete, and $E \longrightarrow X$ a m-semi-positive vector bundle. Let $g \in L^2_{n,q}(X,E)$ be such that $D''g = 0$ and*

$$\int_X \langle A_q^{-1} g, g\rangle \, dV < +\infty$$

with respect to ω, *where* A_q *stands for the operator* $\mathrm{i}\Theta(E) \wedge \Lambda$ *in bidegree* (n,q) *and* $q \geq 1$, $m \geq \min\{n-q+1, r\}$. *Then there exists* $f \in L^2_{n,q-1}(X,E)$ *such that* $D'' f = g$ *and*

$$\|f\|^2 \leq \int_X \langle A_q^{-1} g, g\rangle \, dV.$$

Proof. For every $\varepsilon > 0$, the Kähler metric

$$\omega_\varepsilon = \omega + \varepsilon \widehat{\omega}$$

is complete. The idea of the proof is to apply the L^2 estimates to ω_ε and to let ε tend to zero. Let us put an index ε to all objects depending on ω_ε. It follows from Lemma 6.3 below that

$$(6.2) \quad |u|^2_\varepsilon \, dV_\varepsilon \leq |u|^2 \, dV, \qquad \langle A_{q,\varepsilon}^{-1} u, u\rangle_\varepsilon \, dV_\varepsilon \leq \langle A_q^{-1} u, u\rangle \, dV$$

for every $u \in \Lambda^{n,q} T^\star_X \otimes E$. If these estimates are taken for granted, Th. 4.5 applied to ω_ε yields a section $f_\varepsilon \in L^2_{n,q-1}(X,E)$ such that $D'' f_\varepsilon = g$ and

$$\int_X |f_\varepsilon|^2_\varepsilon \, dV_\varepsilon \leq \int_X \langle A_{q,\varepsilon}^{-1} g, g\rangle_\varepsilon \, dV_\varepsilon \leq \int_X \langle A_q^{-1} g, g\rangle \, dV.$$

This implies that the family (f_ε) is bounded in L^2 norm on every compact subset of X. We can thus find a weakly convergent subsequence (f_{ε_ν}) in L^2_{loc}. The weak limit f is the solution we are looking for. □

(6.3) Lemma. *Let* ω, γ *be hermitian metrics on* X *such that* $\gamma \geq \omega$. *For every* $u \in \Lambda^{n,q} T^\star_X \otimes E$, $q \geq 1$, *we have*

$$|u|^2_\gamma \, dV_\gamma \leq |u|^2 \, dV, \qquad \langle A_{q,\gamma}^{-1} u, u\rangle_\gamma \, dV_\gamma \leq \langle A_q^{-1} u, u\rangle \, dV$$

where an index γ *means that the corresponding term is computed in terms of* γ *instead of* ω.

Proof. Let $x_0 \in X$ be a given point and $(z_1, \ldots, z_n)$ coordinates such that

$$\omega = \mathrm{i} \sum_{1 \leq j \leq n} dz_j \wedge d\overline{z}_j, \qquad \gamma = \mathrm{i} \sum_{1 \leq j \leq n} \gamma_j \, dz_j \wedge d\overline{z}_j \quad \text{at} \quad x_0,$$

where $\gamma_1 \leq \ldots \leq \gamma_n$ are the eigenvalues of γ with respect to ω (thus $\gamma_j \geq 1$). We have $|dz_j|^2_\gamma = \gamma_j^{-1}$ and $|dz_K|^2_\gamma = \gamma_K^{-1}$ for any multi-index K, with the notation $\gamma_K = \prod_{j \in K} \gamma_j$. For every $u = \sum u_{K,\lambda} dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_K \otimes e_\lambda$, $|K| = q$, $1 \leq \lambda \leq r$, the computations of § VII-7 yield

$$
\begin{aligned}
|u|^2_\gamma &= \sum_{K,\lambda} (\gamma_1\ldots\gamma_n)^{-1}\gamma_K^{-1}\,|u_{K,\lambda}|^2, \qquad dV_\gamma = \gamma_1\ldots\gamma_n\,dV,\\
|u|^2_\gamma\,dV_\gamma &= \sum_{K,\lambda} \gamma_K^{-1}\,|u_{K,\lambda}|^2\,dV \le |u|^2\,dV,\\
\Lambda_\gamma u &= \sum_{|I|=q-1}\ \sum_{j,\lambda} \mathrm{i}(-1)^{n+j-1}\gamma_j^{-1}\,u_{jI,\lambda}\,(\widehat{dz_j})\wedge d\overline{z}_I\otimes e_\lambda,
\end{aligned}
$$

where $(\widehat{dz_j})$ means $dz_1\wedge\ldots\widehat{dz_j}\ldots\wedge dz_n$,

$$
\begin{aligned}
A_{q,\gamma}u &= \sum_{|I|=q-1}\ \sum_{j,k,\lambda,\mu} \gamma_j^{-1}\,c_{jk\lambda\mu}\,u_{jI,\lambda}\,dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z}_{kI}\otimes e_\mu,\\
\langle A_{q,\gamma}u,u\rangle_\gamma &= (\gamma_1\ldots\gamma_n)^{-1}\sum_{|I|=q-1}\gamma_I^{-1}\sum_{j,k,\lambda,\mu}\gamma_j^{-1}\gamma_k^{-1}c_{jk\lambda\mu}\,u_{jI,\lambda}\overline{u}_{kI,\mu}\\
&\ge (\gamma_1\ldots\gamma_n)^{-1}\sum_{|I|=q-1}\gamma_I^{-2}\sum_{j,k,\lambda,\mu}\gamma_j^{-1}\gamma_k^{-1}c_{jk\lambda\mu}\,u_{jI,\lambda}\overline{u}_{kI,\mu}\\
&= \gamma_1\ldots\gamma_n\,\langle A_q S_\gamma u, S_\gamma u\rangle
\end{aligned}
$$

where S_γ is the operator defined by

$$
S_\gamma u = \sum_K (\gamma_1\ldots\gamma_n\gamma_K)^{-1}\,u_{K,\lambda}\,dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z}_K\otimes e_\lambda.
$$

We get therefore

$$
\begin{aligned}
|\langle u,v\rangle_\gamma|^2 = |\langle u, S_\gamma v\rangle|^2 &\le \langle A_q^{-1}u,u\rangle\langle A_q S_\gamma v, S_\gamma v\rangle\\
&\le (\gamma_1\ldots\gamma_n)^{-1}\langle A_q^{-1}u,u\rangle\langle A_{q,\gamma}v,v\rangle_\gamma,
\end{aligned}
$$

and the choice $v = A_{q,\gamma}^{-1}u$ implies

$$
\langle A_{q,\gamma}^{-1}u,u\rangle_\gamma \le (\gamma_1\ldots\gamma_n)^{-1}\,\langle A_q^{-1}u,u\rangle\,;
$$

this relation is equivalent to the last one in the lemma. □

An important special case is that of a semi-positive line bundle E. If we let $0\le\lambda_1(x)\le\ldots\le\lambda_n(x)$ be the eigenvalues of $\mathrm{i}\Theta(E)_x$ with respect to ω_x for all $x\in X$, formula VI-8.3 implies

$$
\begin{aligned}
&\langle A_q u,u\rangle \ge (\lambda_1+\cdots+\lambda_q)|u|^2,\\
(6.4)\qquad &\int_X \langle A_q^{-1}g,g\rangle\,dV \le \int_X \frac{1}{\lambda_1+\cdots+\lambda_q}\,|g|^2\,dV.
\end{aligned}
$$

A typical situation where these estimates can be applied is the case when E is the trivial line bundle $X\times\mathbb{C}$ with metric given by a weight $e^{-\varphi}$. One can assume for example that φ is plurisubharmonic and that $\mathrm{i}d'd''\varphi$ has at least $n-q+1$ positive eigenvalues at every point, i.e. $\lambda_q>0$ on X. This situation leads to very important L^2 estimates, which are precisely those given by (Hörmander 1965, 1966). We state here a slightly more general result.

(6.5) Theorem. *Let (X, ω) be a weakly pseudoconvex Kähler manifold, E a hermitian line bundle on X, $\varphi \in C^\infty(X, \mathbb{R})$ a weight function such that the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ of $\mathrm{i}\Theta(E) + \mathrm{i}d'd''\varphi$ are ≥ 0. Then for every form g of type (n, q), $q \geq 1$, with L^2_{loc} (resp. C^∞) coefficients such that $D''g = 0$ and*

$$\int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi} \, dV < +\infty,$$

we can find a L^2_{loc} (resp. C^∞) form f of type $(n, q-1)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi} \, dV \leq \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi} \, dV.$$

Proof. Apply the general estimates to the bundle E_φ deduced from E by multiplication of the metric by $e^{-\varphi}$; we have $\mathrm{i}\Theta(E_\varphi) = \mathrm{i}\Theta(E) + \mathrm{i}d'd''\varphi$. It is not necessary here to assume in addition that $g \in L^2_{n,q}(X, E_\varphi)$. In fact, g is in L^2_{loc} and we can exhaust X by the relatively compact weakly pseudoconvex domains

$$X_c = \big\{ x \in X \ ; \ \psi(x) < c \big\}$$

where $\psi \in C^\infty(X, \mathbb{R})$ is a plurisubharmonic exhaustion function (note that $-\log(c - \psi)$ is also such a function on X_c). We get therefore solutions f_c on X_c with uniform L^2 bounds; any weak limit f gives the desired solution. □

If estimates for (p, q)-forms instead of (n, q)-forms are needed, one can invoke the isomorphism $\Lambda^p T^\star_X \simeq \Lambda^{n-p} T_X \otimes \Lambda^n T^\star_X$ (obtained through contraction of n-forms by $(n-p)$-vectors) to get

$$\Lambda^{p,q} T^\star_X \otimes E \simeq \Lambda^{n,q} T^\star_X \otimes F, \qquad F = E \otimes \Lambda^{n-p} T_X.$$

Let us look more carefully to the case $p = 0$. The $(1, 1)$-curvature form of $\Lambda^n T_X$ with respect to a hermitian metric ω on T_X is called the *Ricci curvature* of ω. We denote:

(6.6) Definition. $\mathrm{Ricci}(\omega) = \mathrm{i}\Theta(\Lambda^n T_X) = \mathrm{i}\ \mathrm{Tr}\ \Theta(T_X)$.

For any local coordinate system $(z_1, \ldots, z_n)$, the holomorphic n-form $dz_1 \wedge \ldots \wedge dz_n$ is a section of $\Lambda^n T^\star_X$, hence Formula V-13.3 implies

(6.7) $\quad \mathrm{Ricci}(\omega) = \mathrm{i}d'd'' \log |dz_1 \wedge \ldots \wedge dz_n|^2_\omega = -\mathrm{i}d'd'' \log \det(\omega_{jk}).$

The estimates of Th. 6.5 can therefore be applied to any $(0,q)$-form g, but $\lambda_1 \leq \ldots \leq \lambda_n$ must be replaced by the eigenvalues of the $(1,1)$-form

$$\mathrm{i}\Theta(E) + \mathrm{Ricci}(\omega) + \mathrm{i}d'd''\varphi \qquad (\text{supposed } \geq 0). \tag{6.8}$$

We consider now domains $\Omega \subset \mathbb{C}^n$ equipped with the euclidean metric of $\mathbb{C}^n$, and the trivial bundle $E = \Omega \times \mathbb{C}$. The following result is especially convenient because it requires only weak plurisubharmonicity and avoids to compute the curvature eigenvalues.

(6.9) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex open subset and φ an upper semi-continuous plurisubharmonic function on Ω. For every $\varepsilon \in\,]0,1]$ and every $g \in L^2_{p,q}(\Omega, \mathrm{loc})$ such that $d''g = 0$ and*

$$\int_\Omega \big(1+|z|^2\big)|g|^2 e^{-\varphi} dV < +\infty,$$

we can find a L^2_{loc} form f of type $(p, q-1)$ such that $d''f = g$ and

$$\int_\Omega \big(1+|z|^2\big)^{-\varepsilon} |f|^2 e^{-\varphi}\, dV \leq \frac{4}{q\varepsilon^2} \int_\Omega \big(1+|z|^2\big)|g|^2 e^{-\varphi}\, dV < +\infty.$$

Moreover f can be chosen smooth if g and φ are smooth.

Proof. Since $\Lambda^p T\Omega$ is a trivial bundle with trivial metric, the proof is immediately reduced to the case $p = 0$ (or equivalently $p = n$). Let us first suppose that φ is smooth. We replace φ by $\Phi = \varphi + \tau$ where

$$\tau(z) = \log\big(1 + (1+|z|^2)^\varepsilon\big).$$

(6.10) Lemma. *The smallest eigenvalue $\lambda_1(z)$ of $\mathrm{i}d'd''\tau(z)$ satisfies*

$$\lambda_1(z) \geq \frac{\varepsilon^2}{2(1+|z|^2)\big(1+(1+|z|^2)^\varepsilon\big)}.$$

In fact a brute force computation of the complex hessian $H\tau_z(\xi)$ and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&H\tau_z(\xi) = \\
&= \frac{\varepsilon(1+|z|^2)^{\varepsilon-1}|\xi|^2}{1+(1+|z|^2)^\varepsilon} + \frac{\varepsilon(\varepsilon-1)(1+|z|^2)^{\varepsilon-2}|\langle\xi,z\rangle|^2}{1+(1+|z|^2)^\varepsilon} - \frac{\varepsilon^2(1+|z|^2)^{2\varepsilon-2}|\langle\xi,z\rangle|^2}{\big(1+(1+|z|^2)^\varepsilon\big)^2} \\
&\geq \varepsilon\left(\frac{(1+|z|^2)^{\varepsilon-1}}{1+(1+|z|^2)^\varepsilon} - \frac{(1-\varepsilon)(1+|z|^2)^{\varepsilon-2}|z|^2}{1+(1+|z|^2)^\varepsilon} - \frac{\varepsilon(1+|z|^2)^{2\varepsilon-2}|z|^2}{\big(1+(1+|z|^2)^\varepsilon\big)^2}\right)|\xi|^2 \\
&= \varepsilon\,\frac{1+\varepsilon|z|^2+(1+|z|^2)^\varepsilon}{(1+|z|^2)^{2-\varepsilon}\big(1+(1+|z|^2)^\varepsilon\big)^2}\,|\xi|^2 \geq \frac{\varepsilon^2|\xi|^2}{(1+|z|^2)^{1-\varepsilon}\big(1+(1+|z|^2)^\varepsilon\big)^2} \\
&\geq \frac{\varepsilon^2}{2(1+|z|^2)\big(1+(1+|z|^2)^\varepsilon\big)}\,|\xi|^2. \qquad \square
\end{aligned}$$

The Lemma implies $e^{-\tau}/\lambda_1 \le 2(1+|z|^2)/\varepsilon^2$, thus Cor. 6.5 provides an f such that

$$\int_\Omega \big(1+(1+|z|^2)^\varepsilon\big)^{-1}|f|^2 e^{-\varphi}\,dV \le \frac{2}{q\varepsilon^2}\int_\Omega \big(1+|z|^2\big)|g|^2 e^{-\varphi}\,dV < +\infty,$$

and the required estimate follows. If φ is not smooth, apply the result to a sequence of regularized weights $\rho_\varepsilon \star \varphi \ge \varphi$ on an increasing sequence of domains $\Omega_c \subset\subset \Omega$, and extract a weakly convergent subsequence of solutions. □

7. Extension of Holomorphic Functions from Subvarieties

The existence theorems for solutions of the d'' operator easily lead to an extension theorem for sections of a holomorphic line bundle defined in a neighborhood of an analytic subset. The following result (Demailly 1982) is an improvement and a generalization of Jennane's extension theorem (Jennane 1976).

(7.1) Theorem. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, L a hermitian line bundle and E a hermitian vector bundle over X. Let Y be an analytic subset of X such that $Y=\sigma^{-1}(0)$ for some section σ of E, and p the maximal codimension of the irreducible components of Y. Let f be a holomorphic section of $K_X\otimes L$ defined in the open set $U\supset Y$ of points $x\in X$ such that $|\sigma(x)|<1$. If $\int_U |f|^2 dV < +\infty$ and if the curvature form of L satisfies*

$$\mathrm{i}\Theta(L) \ge \Big(\frac{p}{|\sigma|^2}+\frac{\varepsilon}{1+|\sigma|^2}\Big)\{\mathrm{i}\Theta(E)\sigma,\sigma\}$$

for some $\varepsilon>0$, there is a section $F\in H^0(X,K_X\otimes L)$ such that $F_{\restriction Y}=f_{\restriction Y}$ and

$$\int_X \frac{|F|^2}{(1+|\sigma|^2)^{p+\varepsilon}}\,dV \le \Big(1+\frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\,dV.$$

The proof will involve a weight with logarithmic singularities along Y. We must therefore apply the existence theorem over $X\smallsetminus Y$. This requires to know whether $X\smallsetminus Y$ has a complete Kähler metric.

(7.2) Lemma. *Let (X,ω) be a Kähler manifold, and $Y=\sigma^{-1}(0)$ an analytic subset defined by a section of a hermitian vector bundle $E\to X$. If X is weakly pseudoconvex and exhausted by $X_c=\{x\in X\ ;\ \psi(x)<c\}$, then*

$X_c \smallsetminus Y$ has a complete Kähler metric for all $c \in \mathbb{R}$. The same conclusion holds for $X \smallsetminus Y$ if (X, ω) is complete and if for some constant $C \ge 0$ we have $\Theta_E \le_{\mathrm{Grif}} C\,\omega \otimes \langle\ ,\ \rangle_E$ on X.

Proof. Set $\tau = \log|\sigma|^2$. Then $d'\tau = \{D'\sigma, \sigma\}/|\sigma|^2$ and $D''D'\sigma = D^2\sigma = \Theta(E)\sigma$, thus

$$\mathrm{i}d'd''\tau = \mathrm{i}\frac{\{D'\sigma, D'\sigma\}}{|\sigma|^2} - \mathrm{i}\frac{\{D'\sigma,\sigma\}\wedge\{\sigma, D'\sigma\}}{|\sigma|^4} - \frac{\{\mathrm{i}\Theta(E)\sigma,\sigma\}}{|\sigma|^2}.$$

For every $\xi \in T_X$, we find therefore

$$\begin{aligned} H\tau(\xi) &= \frac{|\sigma|^2\,|D'\sigma\cdot\xi|^2 - |\langle D'\sigma\cdot\xi, \sigma\rangle|^2}{|\sigma|^4} - \frac{\Theta_E(\xi\otimes\sigma, \xi\otimes\sigma)}{|\sigma|^2} \\ &\ge -\frac{\Theta_E(\xi\otimes\sigma,\xi\otimes\sigma)}{|\sigma|^2} \end{aligned}$$

by the Cauchy-Schwarz inequality. If C is a bound for the coefficients of Θ_E on the compact subset $\overline{X}_c$, we get $\mathrm{i}d'd''\tau \ge -C\omega$ on X_c. Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be a convex increasing function. We set

$$\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi\circ\tau).$$

Formula 5.3 shows that $\widehat{\omega}$ is positive definite if $\chi' \le 1/2C$ and that $\widehat{\omega}$ is complete near $Y = \tau^{-1}(-\infty)$ as soon as

$$\int_{-\infty}^0 \sqrt{\chi''(t)}\,dt = +\infty.$$

One can choose for example χ such that $\chi(t) = \frac{1}{5C}(t - \log|t|)$ for $t \le -1$. In order to obtain a complete Kähler metric on $X_c \smallsetminus Y$, we need also that the metric be complete near ∂X_c. Such a metric is given by

$$\begin{aligned} \widetilde{\omega} &= \widehat{\omega} + \mathrm{i}d'd''\log(c-\psi)^{-1} = \widehat{\omega} + \frac{\mathrm{i}d'd''\psi}{c-\psi} + \frac{\mathrm{i}d'\psi\wedge d''\psi}{(c-\psi)^2} \\ &\ge \mathrm{i}d'\log(c-\psi)^{-1}\wedge d''\log(c-\psi)^{-1}\ ; \end{aligned}$$

$\widetilde{\omega}$ is complete on $X_c \smallsetminus \Omega$ because $\log(c-\psi)^{-1}$ tends to $+\infty$ on ∂X_c. □

Proof of Theorem 7.1. When we replace σ by $(1+\eta)\sigma$ for some small $\eta > 0$ and let η tend to 0, we see that we can assume f defined in a neighborhood of $\overline{U}$. Let h be the continuous section of L such that $h = (1-|\sigma|^{p+1})f$ on $U = \{|\sigma| < 1\}$ and $h = 0$ on $X \smallsetminus U$. We have $h_{\restriction Y} = f_{\restriction Y}$ and

$$d''h = -\frac{p+1}{2}\,|\sigma|^{p-1}\,\{\sigma, D'\sigma\}\,f \quad \text{on } U, \qquad d''h = 0 \quad \text{on } X\smallsetminus U.$$

We consider $g = d''h$ as a $(n,1)$-form with values in the hermitian line bundle $L_\varphi = L$, endowed with the weight $e^{-\varphi}$ given by

$$\varphi = p\log|\sigma|^2 + \varepsilon\log(1+|\sigma|^2).$$

Notice that φ is singular along Y. The Cauchy-Schwarz inequality implies $\mathrm{i}\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\} \le \mathrm{i}\{D'\sigma,D'\sigma\}$ as in Lemma 7.2, and we find

$$\begin{aligned}\mathrm{i}d'd''\log(1+|\sigma|^2) &= \frac{(1+|\sigma|^2)\mathrm{i}\{D'\sigma,D'\sigma\} - \mathrm{i}\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\}}{(1+|\sigma|^2)^2}\\ &\quad - \frac{\{\mathrm{i}\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2} \ge \frac{\mathrm{i}\{D'\sigma,D'\sigma\}}{(1+|\sigma|^2)^2} - \frac{\{\mathrm{i}\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2}.\end{aligned}$$

The inequality $\mathrm{i}d'd''\log|\sigma|^2 \ge -\{\mathrm{i}\Theta(E)\sigma,\sigma\}/|\sigma|^2$ obtained in Lemma 7.2 and the above one imply

$$\begin{aligned}\mathrm{i}\Theta(L_\varphi) &= \mathrm{i}\Theta(L) + p\,\mathrm{i}d'd''\log|\sigma|^2 + \varepsilon\,\mathrm{i}d'd''\log(1+|\sigma|^2)\\ &\ge \mathrm{i}\Theta(L) - \Big(\frac{p}{|\sigma|^2} + \frac{\varepsilon}{1+|\sigma|^2}\Big)\{\mathrm{i}\Theta(E)\sigma,\sigma\} + \varepsilon\frac{\mathrm{i}\{D'\sigma,D'\sigma\}}{(1+|\sigma|^2)^2}\\ &\ge \varepsilon\frac{\mathrm{i}\,\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\}}{|\sigma|^2\,(1+|\sigma|^2)^2},\end{aligned}$$

thanks to the hypothesis on the curvature of L and the Cauchy-Schwarz inequality. Set $\xi = (p+1)/2\,|\sigma|^{p-1}\{D'\sigma,\sigma\} = \sum\xi_j\,dz_j$ in an ω-orthonormal basis $\partial/\partial z_j$, and let $\widehat{\xi} = \sum\xi_j\partial/\partial\overline{z}_j$ be the dual $(0,1)$-vector field. For every $(n,1)$-form v with values in L_φ, we find

$$\begin{aligned}\big|\langle d''h,v\rangle\big| &= \big|\langle\overline{\xi}\wedge f,v\rangle\big| = \big|\langle f,\widehat{\xi}\lrcorner\, v\rangle\big| \le |f|\,|\widehat{\xi}\lrcorner\, v|,\\ \widehat{\xi}\lrcorner\, v &= \sum -\mathrm{i}\xi_j\,dz_j\wedge\Lambda v = -\mathrm{i}\xi\wedge\Lambda v,\\ |\langle d''h,v\rangle|^2 &\le |f|^2\,|\widehat{\xi}\lrcorner\, v|^2 = |f|^2\langle -\mathrm{i}\xi\wedge\Lambda v,\widehat{\xi}\lrcorner\, v\rangle\\ &= |f|^2\langle -\mathrm{i}\overline{\xi}\wedge\xi\wedge\Lambda v,v\rangle = |f|^2\langle[\mathrm{i}\xi\wedge\overline{\xi},\Lambda]v,v\rangle\\ &\le \frac{(p+1)^2}{4\varepsilon}\,|\sigma|^{2p}\,(1+|\sigma|^2)^2\,|f|^2\,\langle[\mathrm{i}\Theta(L_\varphi),\Lambda]v,v\rangle.\end{aligned}$$

Thus, in the notations of Th. 6.1, the form $g = d''h$ satisfies

$$\langle A_1^{-1}g,g\rangle \le \frac{(p+1)^2}{4\varepsilon}\,|\sigma|^{2p}(1+|\sigma|^2)^2\,|f|^2 \le \frac{(p+1)^2}{\varepsilon}\,|f|^2\,e^{\varphi},$$

where the last equality results from the fact that $(1+|\sigma|^2)^2 \le 4$ on the support of g. Lemma 7.2 shows that the existence theorem 6.1 can be applied on each set $X_c\setminus Y$. Letting c tend to infinity, we infer the existence of a $(n,0)$-form u with values in L such that $d''u = g$ on $X\setminus Y$ and

$$\int_{X\setminus Y}|u|^2\,e^{-\varphi}\,dV \le \int_{X\setminus Y}\langle A_1^{-1}g,g\rangle e^{-\varphi},\qquad\text{thus}$$

$$\int_{X\setminus Y}\frac{|u|^2}{|\sigma|^{2p}(1+|\sigma|^2)^{\varepsilon}}\,dV \le \frac{(p+1)^2}{\varepsilon}\int_U|f|^2\,dV.$$

This estimate implies in particular that u is locally L^2 near Y. As g is continuous over X, Lemma 7.3 below shows that the equality $d''u = g = d''h$ extends to X, thus $F = h - u$ is holomorphic everywhere. Hence $u = h - F$ is continuous on X. As $|\sigma(x)| \le C\, d(x,Y)$ in a neighborhood of every point of Y, we see that $|\sigma|^{-2p}$ is non integrable at every point $x_0 \in Y_{\mathrm{reg}}$, because $\operatorname{codim} Y \le p$. It follows that $u = 0$ on Y, so

$$F_{\restriction Y} = h_{\restriction Y} = f_{\restriction Y}.$$

The final L^2-estimate of Th. 7.1 follows from the inequality

$$|F|^2 = |h-u|^2 \le (1+|\sigma|^{-2p})\,|u|^2 + (1+|\sigma|^{2p})\,|f|^2$$

which implies

$$\frac{|F|^2}{(1+|\sigma|^2)^p} \le \frac{|u|^2}{|\sigma|^{2p}} + |f|^2. \qquad \square$$

(7.3) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega \smallsetminus Y$ (in the sense of distribution theory). Then $d''v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it is sufficient to prove the result in a neighborhood of a regular point $a \in Y$. By using a local analytic isomorphism, the proof is reduced to the case where Y is contained in the hyperplane $z_1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that $\lambda(t) = 0$ for $t \le \frac{1}{2}$ and $\lambda(t) = 1$ for $t \ge 1$. We must show that

$$\text{(7.4)} \qquad \int_\Omega w \wedge \alpha = (-1)^{p+q} \int_\Omega v \wedge d''\alpha$$

for all $\alpha \in \mathcal{D}_{n-p,n-q}(\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(|z_1|/\varepsilon)$ and replace α in the integral by $\lambda_\varepsilon\alpha$. Then $\lambda_\varepsilon\alpha \in \mathcal{D}_{n-p,n-q}(\Omega \smallsetminus Y)$ and the hypotheses imply

$$\int_\Omega w \wedge \lambda_\varepsilon\alpha = (-1)^{p+q} \int_\Omega v \wedge d''(\lambda_\varepsilon\alpha) = (-1)^{p+q} \int_\Omega v \wedge (d''\lambda_\varepsilon \wedge \alpha + \lambda_\varepsilon d''\alpha).$$

As w and v have L^1_{loc} coefficients on Ω, the integrals of $w \wedge \lambda_\varepsilon\alpha$ and $v \wedge \lambda_\varepsilon d''\alpha$ converge respectively to the integrals of $w \wedge \alpha$ and $v \wedge d''\alpha$ as ε tends to 0. The remaining term can be estimated by means of the Cauchy-Schwarz inequality:

$$\Big|\int_\Omega v \wedge d''\lambda_\varepsilon \wedge \alpha\Big|^2 \le \int_{|z_1|\le\varepsilon} |v \wedge \alpha|^2\, dV. \int_{\operatorname{Supp}\alpha} |d''\lambda_\varepsilon|^2\, dV\,;$$

as $v \in L^2_{\mathrm{loc}}(\Omega)$, the integral $\int_{|z_1|\le\varepsilon} |v\wedge\alpha|^2\, dV$ converges to 0 with ε, whereas

$$\int_{\mathrm{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV \le \frac{C}{\varepsilon^2}\,\mathrm{Vol}\big(\mathrm{Supp}\ \alpha \cap \{|z_1| \le \varepsilon\}\big) \le C'.$$

Equality (7.4) follows when ε tends to 0. □

(7.5) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and let φ, ψ be plurisubharmonic functions on Ω, where ψ is supposed to be finite and continuous. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be a family of holomorphic functions on Ω, let $Y = \sigma^{-1}(0)$, $p =$ maximal codimension of Y and set*

a) $U = \{z \in \Omega\,;\ |\sigma(z)|^2 < e^{-\psi(z)}\}$, *resp.*

b) $U' = \{z \in \Omega\,;\ |\sigma(z)|^2 < e^{\psi(z)}\}$.

For every $\varepsilon > 0$ and every holomorphic function f on U, there exists a holomorphic function F on Ω such that $F_{\restriction Y} = f_{\restriction Y}$ and

a) $$\int_\Omega \frac{|F|^2\, e^{-\varphi+p\psi}}{(1+|\sigma|^2 e^{\psi})^{p+\varepsilon}}\,dV \le \Big(1 + \frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\, e^{-\varphi+p\psi}\,dV, \qquad \textit{resp.}$$

b) $$\int_\Omega \frac{|F|^2\, e^{-\varphi}}{(e^{\psi}+|\sigma|^2)^{p+\varepsilon}}\,dV \le \Big(1 + \frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\, e^{-\varphi-(p+\varepsilon)\psi}\,dV.$$

Proof. After taking convolutions with smooth kernels on pseudoconvex subdomains $\Omega_c \subset\subset \Omega$, we may assume φ, ψ smooth. In either case a) or b), apply Th. 7.1 to

a) $E = \Omega \times \mathbb{C}^r$ with the weight e^{ψ}, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi+p\psi}$, and $U = \{|\sigma|^2 e^{\psi} < 1\}$. Then

$$\mathrm{i}\Theta(E) = -\mathrm{i}d'd''\psi \otimes \mathrm{Id}_E \le 0, \qquad \mathrm{i}\Theta(L) = \mathrm{i}d'd''\varphi - p\,\mathrm{i}d'd''\psi \ge p\,\mathrm{i}\Theta(E).$$

b) $E = \Omega \times \mathbb{C}^r$ with the weight $e^{-\psi}$, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi-(p+\varepsilon)\psi}$, and $U = \{|\sigma|^2 e^{-\psi} < 1\}$. Then

$$\mathrm{i}\Theta(E) = \mathrm{i}d'd''\psi \otimes \mathrm{Id}_E \ge 0, \qquad \mathrm{i}\Theta(L) = \mathrm{i}d'd''\varphi + (p+\varepsilon)\,\mathrm{i}d'd''\psi \ge (p+\varepsilon)\,\mathrm{i}\Theta(E).$$

The condition on $\Theta(L)$ is satisfied in both cases and K_Ω is trivial. □

(7.6) Hörmander-Bombieri-Skoda theorem. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and φ a plurisubharmonic function on Ω. For every $\varepsilon > 0$ and every point $z_0 \in \Omega$ such that $e^{-\varphi}$ is integrable in a neighborhood of z_0, there exists a holomorphic function F on Ω such that $F(z_0) = 1$ and*

$$\int_\Omega \frac{|F(z)|^2\, e^{-\varphi(z)}}{(1+|z|^2)^{n+\varepsilon}}\,dV < +\infty.$$

(Bombieri 1970) originally stated the theorem with the exponent $3n$ instead of $n+\varepsilon$; the improved exponent $n+\varepsilon$ is due to (Skoda 1975). The example $\Omega = \mathbb{C}^n$, $\varphi(z) = 0$ shows that one cannot replace ε by 0.

Proof. Apply Cor. 7.5 b) to $f \equiv 1$, $\sigma(z) = z - z_0$, $p = n$ and $\psi \equiv \log r^2$ where $U = B(z_0, r)$ is a ball such that $\int_U e^{-\varphi}\, dV < +\infty$. □

(7.7) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Let A be the set of points $z \in X$ such that $e^{-\varphi}$ is not locally integrable in a neighborhood of z. Then A is an analytic subset of X.*

Proof. Let $\Omega \subset X$ be an open coordinate patch isomorphic to a ball of $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$. Define $E \subset H^0(\Omega, \mathcal{O})$ to be the Hilbert space of holomorphic functions f on Ω such that

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)}\, dV(z) < +\infty.$$

Then $A \cap \Omega = \bigcap_{f \in E} f^{-1}(0)$. In fact, every f in E must obviously vanish on A; conversely, if $z_0 \notin A$, Th. 7.6 shows that there exists $f \in E$ such that $f(z_0) \neq 0$. By Th. II-5.5, we conclude that A is analytic. □

8. Applications to Hypersurface Singularities

We first give some basic definitions and results concerning multiplicities of divisors on a complex manifold.

(8.1) Proposition. *Let X be a complex manifold and $\Delta = \sum \lambda_j [Z_j]$ a divisor on X with real coefficients $\lambda_j \geq 0$. Let $x \in X$ and $f_j = 0$, $1 \leq j \leq N$, irreducible equations of Z_j on a neighborhood U of x.*

a) *The multiplicity of Δ at x is defined by*

$$\mu(\Delta, x) = \sum \lambda_j \operatorname{ord}_x f_j.$$

b) *Δ is said to have normal crossings at a point $x \in \operatorname{Supp} \Delta$ if all hypersurfaces Z_j containing x are smooth at x and intersect transversally, i.e. if the linear forms df_j defining the corresponding tangent spaces $T_x Z_j$ are linearly independent at x. The set* $\operatorname{nnc}(\Delta)$ *of non normal crossing points is an analytic subset of X.*

c) *The non-integrability locus* $\operatorname{nil}(\Delta)$ *is defined as the set of points $x \in X$ such that $\prod |f_j|^{-2\lambda_j}$ is non integrable near x. Then* $\operatorname{nil}(\Delta)$ *is an analytic subset of X and there are inclusions*

$$\{x \in X\,;\ \mu(\Delta, x) \geq n\} \subset \operatorname{nil}(\Delta) \subset \{x \in X\,;\ \mu(\Delta, x) \geq 1\}.$$

Moreover $\operatorname{nil}(\Delta) \subset \operatorname{nnc}(\Delta)$ *if all coefficients of Δ satisfy $\lambda_j < 1$.*

Proof. b) The set $\operatorname{nnc}(\Delta) \cap U$ is the union of the analytic sets

$$f_{j_1} = \ldots = f_{j_p} = 0, \qquad df_{j_1} \wedge \ldots \wedge df_{j_p} = 0,$$

for each subset $\{j_1, \ldots, j_p\}$ of the index set $\{1, \ldots, N\}$. Thus $\mathrm{nnc}(\Delta)$ is analytic.

c) The analyticity of $\mathrm{nil}(\Delta)$ follows from Cor. 7.7 applied to the plurisubharmonic function $\varphi = \sum 2\lambda_j \log|f_j|$. Assume first that $\lambda_j < 1$ and that Δ has normal crossings at x. Let $f_{j_1}(x) = \ldots = f_{j_s}(x) = 0$ and $f_j(x) \neq 0$ for $j \neq j_l$. Then, we can choose local coordinates $(w_1, \ldots, w_n)$ on U such that $w_1 = f_{j_1}(z)$, …, $w_s = f_{j_s}(z)$, and we have

$$\int_U \frac{d\lambda(z)}{\prod |f_j(z)|^{2\lambda_j}} \leq \int_U \frac{C\, d\lambda(w)}{|w_1|^{2\lambda_1} \ldots |w_s|^{2\lambda_s}} < +\infty.$$

It follows that $\mathrm{nil}(\Delta) \subset \mathrm{nnc}(\Delta)$. Let us prove now the statement relating $\mathrm{nil}(\Delta)$ with multiplicity sets. Near any point x, we have $|f_j(z)| \leq C_j|z-x|^{m_j}$ with $m_j = \mathrm{ord}_x f_j$, thus

$$\prod |f_j|^{-2\lambda_j} \geq C\,|z-x|^{-2\mu(\Delta,x)}.$$

It follows that $x \in \mathrm{nil}(\Delta)$ as soon as $\mu(\Delta,x) \geq n$. On the other hand, we are going to prove that $\mu(\Delta,x) < 1$ implies $x \notin \mathrm{nil}(\Delta)$, i.e. $\prod |f_j|^{-2\lambda_j}$ integrable near x. We may assume λ_j rational; otherwise replace each λ_j by a slightly larger rational number in such a way that $\mu(\Delta,x) < 1$ is still true. Set $f = \prod f_j^{k\lambda_j}$ where k is a common denominator. The result is then a consequence of the following lemma. □

(8.2) Lemma. *If $f \in \mathcal{O}_{X,x}$ is not identically 0, there exists a neighborhood U of x such that $\int_U |f|^{-2\lambda}\, dV$ converges for all $\lambda < 1/m$, $m = \mathrm{ord}_x f$.*

Proof. One can assume that f is a Weierstrass polynomial

$$f(z) = z_n^m + a_1(z')z_n^{m-1} + \cdots + a_m(z'), \qquad a_j(z') \in \mathcal{O}_{n-1}, \qquad a_j(0) = 0,$$

with respect to some coordinates $(z_1, \ldots, z_n)$ centered at x. Let $v_j(z')$, $1 \leq j \leq m$, denote the roots z_n of $f(z) = 0$. On a small neighborhood U of x we have $|v_j(z')| \leq 1$. The inequality between arithmetic and geometric mean implies

$$\begin{aligned}\int_{\{|z_n|\leq 1\}} |f(z)|^{-2\lambda}\, dx_n dy_n &= \int_{\{|z_n|\leq 1\}} \prod_{1\leq j\leq m} |z_n - v_j(z')|^{-2\lambda}\, dx_n dy_n \\ &\leq \frac{1}{m} \int_{\{|z_n|\leq 1\}} \sum_{1\leq j\leq m} |z_n - v_j(z')|^{-2m\lambda}\, dx_n dy_n \\ &\leq \int_{\{|z_n|\leq 2\}} \frac{dx_n dy_n}{|z_n|^{2m\lambda}},\end{aligned}$$

so the Lemma follows from the Fubini theorem. □

Another interesting application concerns the study of multiplicities of singular points for algebraic hypersurfaces in $\mathbb{P}^n$. Following (Waldschmidt 1975), we introduce the following definition.

(8.3) Definition. *Let S be a finite subset of $\mathbb{P}^n$. For any integer $t \ge 1$, we define $\omega_t(S)$ as the minimum of the degrees of non zero homogeneous polynomials $P \in \mathbb{C}[z_0, \ldots, z_n]$ which vanish at order t at every point of S, i.e. $D^\alpha P(w) = 0$ for every $w \in S$ and every multi-index $\alpha = (\alpha_0, \ldots, \alpha_n)$ of length $|\alpha| < t$.*

It is clear that $t \longmapsto \omega_t(S)$ is a non-decreasing and subadditive function, i.e. for all integers $t_1, t_2 \ge 1$ we have $\omega_{t_1+t_2}(S) \le \omega_{t_1}(S) + \omega_{t_2}(S)$. One defines

$$(8.4) \quad \Omega(S) = \inf_{t \ge 1} \frac{\omega_t(S)}{t}.$$

For all integers $t, t' \ge 1$, the monotonicity and subadditivity of $\omega_t(S)$ show that

$$\omega_t(S) \le ([t/t'] + 1)\, \omega_{t'}(S), \qquad \text{hence} \quad \Omega(S) \le \frac{\omega_t(S)}{t} \le \Big(\frac{1}{t'} + \frac{1}{t}\Big)\, \omega_{t'}(S).$$

We find therefore

$$(8.5) \quad \Omega(S) = \lim_{t \to +\infty} \frac{\omega_t(S)}{t}.$$

Our goal is to find a lower bound of $\Omega(S)$ in terms of $\omega_t(S)$. For $n = 1$, it is obvious that $\Omega(S) = \omega_t(S)/t = \operatorname{card} S$ for all t. From now on, we assume that $n \ge 2$.

(8.6) Theorem. *Let $t_1, t_2 \ge 1$ be integers, let P be a homogeneous polynomial of degree $\omega_{t_2}(S)$ vanishing at order $\ge t_2$ at every point of S. If $P = P_1^{k_1} \ldots P_N^{k_N}$ is the decomposition of P in irreducible factors and $Z_j = P_j^{-1}(0)$, we set*

$$\alpha = \frac{t_1 + n - 1}{t_2}, \qquad \Delta = \sum (k_j \alpha - [k_j \alpha])\, [Z_j], \qquad a = \dim\big(\mathrm{nil}(\Delta)\big).$$

Then we have the inequality

$$\frac{\omega_{t_1}(S) + n - a - 1}{t_1 + n - 1} \le \frac{\omega_{t_2}(S)}{t_2}.$$

Let us first make a few comments before giving the proof. If we let t_2 tend to infinity and observe that $\mathrm{nil}(\Delta) \subset \mathrm{nnc}(\Delta)$ by Prop. 8.1 c), we get $a \le 2$ and

$$(8.7) \quad \frac{\omega_{t_1}(S) + 1}{t_1 + n - 1} \le \Omega(S) \le \frac{\omega_{t_2}(S)}{t_2}.$$

Such a result was first obtained by (Waldschmidt 1975, 1979) with the lower bound $\omega_{t_1}(S)/(t_1+n-1)$, as a consequence of the Hörmander-Bombieri-Skoda theorem. The above improved inequalities were then found by (Esnault-Viehweg 1983), who used rather deep tools of algebraic geometry. Our proof will consist in a refinement of the Bombieri-Waldschmidt method due to (Azhari 1990). It has been conjectured by (Chudnovsky 1979) that $\Omega(S) \ge (\omega_1(S)+n-1)/n$. Chudnovsky's conjecture is true for $n=2$ (as shown by (8.7)); this case was first verified independently by (Chudnovsky 1979) and (Demailly 1982). The conjecture can also be verified in case S is a complete polytope, and the lower bound of the conjecture is then optimal (see Demailly 1982a and ??.?.?). More generally, it is natural to ask whether the inequality

$$(8.8) \qquad \frac{\omega_{t_1}(S)+n-1}{t_1+n-1} \le \Omega(S) \le \frac{\omega_{t_2}(S)}{t_2}$$

always holds; this is the case if there are infinitely many t_2 for which P can be chosen in such a way that $\mathrm{nil}(\Delta)$ has dimension $a=0$.

(8.9) Bertini's lemma. *If $E \subset \mathbb{P}^n$ is an analytic subset of dimension a, there exists a dense subset in the grassmannian of k-codimensional linear subspaces Y of $\mathbb{P}^n$ such that* $\dim(E\cap Y) \le a-k$ *(when $k>a$ this means that $E\cap Y=\emptyset$).*

Proof. By induction on n, it suffices to show that $\dim(E\cap H) \le a-1$ for a generic hyperplane $H \subset \mathbb{P}^n$. Let E_j be the (finite) family of irreducible components of E, and $w_j \in E_j$ an arbitrary point. Then $E\cap H = \bigcup E_j \cap H$ and we have $\dim E_j \cap H < \dim E_j \le a$ as soon as H avoids all points w_j. □

Proof of Theorem 8.6. By Bertini's lemma, there exists a linear subspace $Y \subset \mathbb{P}^n$ of codimension $a+1$ such that $\mathrm{nil}(\Delta)\cap Y=\emptyset$. We consider P as a section of the line bundle $\mathcal{O}(D)$ over $\mathbb{P}^n$, where $D=\deg P$ (cf. Th. V-15.5). There are sections $\sigma_1,\ldots,\sigma_{a+1}$ of $\mathcal{O}(1)$ such that $Y=\sigma^{-1}(0)$. We shall apply Th. 7.1 to $E=\mathcal{O}(1)$ with its standard hermitian metric, and to $L=\mathcal{O}(k)$ equipped with the additional weight $\varphi=\alpha\log|P|^2$. We may assume that the open set $U=\{|\sigma|<1\}$ is such that $\mathrm{nil}(\Delta)\cap\overline{U}=\emptyset$, otherwise it suffices to multiply σ by a large constant. This implies that the polynomial $Q=\prod P_j^{[k_j\alpha]}$ satisfies

$$\int_U |Q|^2 e^{-\varphi}\,dV = \int_U \prod |P_j|^{-2(k_j\alpha-[k_j\alpha])}\,dV < +\infty.$$

Set $\omega = ic\big(\mathcal{O}(1)\big)$. We have $\mathrm{i}d'd''\log|P|^2 \ge -ic\big(\mathcal{O}(D)\big) = -D\omega$ by the Lelong-Poincaré equation, thus $\mathrm{i}\Theta(L_\varphi) \ge (k-\alpha D)\omega$. The desired curvature inequality $\mathrm{i}\Theta(L_\varphi) \ge (a+1+\varepsilon)\mathrm{i}\Theta(E)$ is satisfied if $k-\alpha D \ge (a+1+\varepsilon)$. We thus take

$$k = [\alpha D] + a + 2.$$

The section $f \in H^0(U, K_{\mathbb{P}^n} \otimes L) = H^0\big(U, \mathcal{O}(k-n-1)\big)$ is taken to be a multiple of Q by some polynomial. This is possible provided that

$$k - n - 1 \geq \deg\, Q \quad \iff \quad \alpha D + a + 2 - n - 1 \geq \sum [k_j\alpha] \deg\, P_j,$$

or equivalently, as $D = \sum k_j \deg\, P_j$,

$$\text{(8.10)} \quad \sum (k_j\alpha - [k_j\alpha]) \deg\, P_j \geq n - a - 1.$$

Then we get $f \in H^0(U, K_{\mathbb{P}^n} \otimes L)$ such that $\int_U |f|^2 e^{-\varphi}\, dV < +\infty$. Theorem 7.1 implies the existence of $F \in H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes L)$, i.e. of a polynomial F of degree $k - n - 1$, such that

$$\int_{\mathbb{P}^n} |F|^2 e^{-\varphi}\, dV = \int_{\mathbb{P}^n} \frac{|F|^2}{|P|^{2\alpha}}\, dV < +\infty\,;$$

observe that $|\sigma|$ is bounded, for we are on a compact manifold. Near any $w \in S$, we have $|P(z)| \leq C|z - w|^{t_2}$, thus $|P(z)|^{2\alpha} \leq C|z - w|^{2(t_1+n-1)}$. This implies that the above integral can converge only if F vanishes at order $\geq t_1$ at each point $w \in S$. Therefore

$$\omega_{t_1}(S) \leq \deg\, F = k - n - 1 = [\alpha D] + a + 1 - n \leq \alpha\omega_{t_2}(S) + a + 1 - n,$$

which is the desired inequality.

However, the above proof only works under the additional assumption (8.10). Assume on the contrary that

$$\beta = \sum (k_j\alpha - [k_j\alpha]) \deg\, P_j < n - a - 1.$$

Then the polynomial Q has degree

$$\sum [k_j\alpha] \deg\, P_j = \alpha \deg\, P - \beta = \alpha D - \beta,$$

and Q vanishes at every point $w \in S$ with order

$$\begin{aligned} \mathrm{ord}_w Q &\geq \sum [k_j\alpha]\, \mathrm{ord}_w P_j = \alpha \sum k_j\, \mathrm{ord}_w P_j - \sum (k_j\alpha - [k_j\alpha])\, \mathrm{ord}_w P_j \\ &\geq \alpha\ \mathrm{ord}_w P - \beta \geq \alpha t_2 - \beta = t_1 - (\beta - n + 1). \end{aligned}$$

This implies $\mathrm{ord}_w Q \geq t_1 - [\beta - n + 1]$. As $[\beta - n + 1] < n - a - 1 - n + 1 = -a \leq 0$, we can take a derivative of order $-[\beta - n + 1]$ of Q to get a polynomial F with

$$\deg\, F = \alpha D - \beta + [\beta - n + 1] \leq \alpha D - n + 1,$$

which vanishes at order t_1 on S. In this case, we obtain therefore

$$\omega_{t_1}(S) \leq \alpha D - n + 1 = \frac{t_1 + n - 1}{t_2}\, \omega_{t_2}(S) - n + 1$$

and the proof of Th. 8.6 is complete. □

9. Skoda's L^2 Estimates for Surjective Bundle Morphisms

Let (X,ω) be a Kähler manifold, $\dim X = n$, and $g : E \longrightarrow Q$ a holomorphic morphism of hermitian vector bundles over X. Assume in the first instance that g is *surjective*. We are interested in conditions insuring for example that the induced morphism $g : H^k(X, K_X \otimes E) \longrightarrow H^k(X, K_X \otimes Q)$ is also surjective. For that purpose, it is natural to consider the subbundle $S = \operatorname{Ker} g \subset E$ and the exact sequence

$$(9.1)\quad 0 \longrightarrow S \longrightarrow E \xrightarrow{\ g\ } Q \longrightarrow 0.$$

Assume for the moment that S and Q are endowed with the metrics induced by that of E. Let L be a line bundle over X. We consider the tensor product of sequence (9.1) by L :

$$(9.2)\quad 0 \longrightarrow S\otimes L \longrightarrow E\otimes L \xrightarrow{\ g\ } Q\otimes L \longrightarrow 0.$$

(9.3) Theorem. *Let k be an integer such that $0 \le k \le n$. Set $r = \operatorname{rk} E$, $q = \mathrm{rkQ}$, $s = \operatorname{rk} S = r - q$ and*

$$m = \min\{n-k, s\} = \min\{n-k, r-q\}.$$

Assume that (X,ω) possesses also a complete Kähler metric $\widehat{\omega}$, that $E \ge_m 0$, and that $L \longrightarrow X$ is a hermitian line bundle such that

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \ge 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q\otimes L$ such that $\|f\| < +\infty$, there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g\cdot h$ and

$$\|h\|^2 \le (1 + m/\varepsilon)\,\|f\|^2.$$

The idea of the proof is essentially due to (Skoda 1978), who actually proved the special case $k = 0$. The general case appeared in (Demailly 1982c).

Proof. Let $j : S \to E$ be the inclusion morphism, $g^\star : Q \to E$ and $j^\star : E \to S$ the adjoints of g, j, and

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix}, \quad \beta \in C^\infty_{1,0}\big(X, \hom(S,Q)\big), \quad \beta^\star \in C^\infty_{0,1}\big(X, \hom(Q,S)\big),$$

the matrix of D_E with respect to the orthogonal splitting $E \simeq S \oplus Q$ (cf. §V-14). Then $g^\star f$ is a lifting of f in $E\otimes L$. We shall try to find h under the form

$$h = g^\star f + ju, \qquad u \in L^2_{n,k}(X, S \otimes L).$$

As the images of S and Q in E are orthogonal, we have $|h|^2 = |f|^2 + |u|^2$ at every point of X. On the other hand $D''_{Q\otimes L} f = 0$ by hypothesis and $D'' g^\star = -j \circ \beta^\star$ by V-14.3 d), hence

$$D''_{E\otimes L} h = -j(\beta^\star \wedge f) + j\, D''_{S\otimes L} = j(D''_{S\otimes L} - \beta^\star \wedge f).$$

We are thus led to solve the equation

$$(9.4) \quad D''_{S\otimes L} u = \beta^\star \wedge f,$$

and for that, we apply Th. 4.5 to the $(n, k+1)$-form $\beta^\star \wedge f$. One observes now that the curvature of $S \otimes L$ can be expressed in terms of β. This remark will be used to prove:

(9.5) Lemma. $\langle A_k^{-1}(\beta^\star \wedge f), (\beta^\star \wedge f)\rangle \le (m/\varepsilon)\, |f|^2$.

If the Lemma is taken for granted, Th. 4.5 yields a solution u of (9.4) in $L^2_{n,q}(X, S \otimes L)$ such that $\|u\|^2 \le (m/\varepsilon)\, \|f\|^2$. As $\|h\|^2 = \|f\|^2 + \|u\|^2$, the proof of Th. 9.3 is complete. □

Proof of Lemma 9.5. Exactly as in the proof of Th. VII-10.3, formulas (V-14.6) and (V-14.7) yield

$$\mathrm{i}\Theta(S) \ge_m \mathrm{i}\beta^\star \wedge \beta, \qquad \mathrm{i}\Theta(\det Q) \ge \mathrm{Tr}_Q(\mathrm{i}\beta \wedge \beta^\star) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta).$$

Since $C^\infty_{1,1}(X, \mathrm{Herm}\, S) \ni \Theta := -\mathrm{i}\beta^\star \wedge \beta \ge_{\mathrm{Grif}} 0$, Prop. VII-10.1 implies

$$m\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\beta^\star \wedge \beta \ge_m 0.$$

From the hypothesis on the curvature of L we get

$$\begin{aligned}
\mathrm{i}\Theta(S \otimes L) &\ge_m \mathrm{i}\Theta(S) \otimes \mathrm{Id}_L + (m+\varepsilon)\, \mathrm{i}\Theta(\det Q) \otimes \mathrm{Id}_{S\otimes L} \\
&\ge_m \big(\mathrm{i}\beta^\star \wedge \beta + (m+\varepsilon)\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S\,\big) \otimes \mathrm{Id}_L \\
&\ge_m (\varepsilon/m)\, (-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S \otimes \mathrm{Id}_L\,.
\end{aligned}$$

For any $v \in \Lambda^{n,k+1} T^\star_X \otimes S \otimes L$, Lemma VII-7.2 implies

$$(9.6) \quad \langle A_{k,S\otimes L} v, v\rangle \ge (\varepsilon/m)\, \langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle,$$

because $\mathrm{rk}(S \otimes L) = s$ and $m = \min\{n-k, s\}$. Let $(dz_1, \ldots, dz_n)$ be an orthonormal basis of $T^\star_X$ at a given point $x_0 \in X$ and set

$$\beta = \sum_{1\le j\le n} dz_j \otimes \beta_j, \qquad \beta_j \in \hom(S, Q).$$

The adjoint of the operator $\beta^\star \wedge \bullet = \sum d\overline{z}_j \wedge \beta^\star_j \bullet$ is the contraction $\beta \,\lrcorner\, \bullet$ defined by

$$\beta \lrcorner v = \sum \frac{\partial}{\partial \overline{z}_j} \lrcorner (\beta_j v) = \sum -\mathrm{i}dz_j \wedge \Lambda(\beta_j v) = -\mathrm{i}\beta \wedge \Lambda v.$$

We get consequently $\langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle = |\beta \lrcorner v|^2$ and (9.6) implies

$$|\langle \beta^\star \wedge f, v\rangle|^2 = |\langle f, \beta \lrcorner v\rangle|^2 \le |f|^2\,|\beta \lrcorner v|^2 \le (m/\varepsilon)\langle A_{k,S\otimes L}v, v\rangle\,|f|^2. \quad \square$$

If X has a plurisubharmonic exhaustion function ψ, we can select a convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ and multiply the metric of L by the weight $\exp(-\chi\circ\psi)$ in order to make the L^2 norm of f converge. Theorem 9.3 implies therefore:

(9.7) Corollary. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, let $g : E \to Q$ be a surjective bundle morphism with $r = \mathrm{rk}\, E$, $q = \mathrm{rk}\, Q$, let $m = \min\{n-k, r-q\}$ and let $L \to X$ be a hermitian line bundle. Suppose that $E \ge_m 0$ and*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\,\mathrm{i}\Theta(\det Q) \ge 0$$

for some $\varepsilon > 0$. Then g induces a surjective map

$$H^k(X, K_X \otimes E \otimes L) \longrightarrow H^k(X, K_X \otimes Q \otimes L).$$

The most remarkable feature of this result is that it does not require any strict positivity assumption on the curvature (for instance E can be a flat bundle). A careful examination of the proof shows that it amounts to verify that the image of the coboundary morphism

$$-\beta^\star \wedge \bullet \;:\; H^k(X, K_X \otimes Q \otimes L) \longrightarrow H^{k+1}(X, K_X \otimes S \otimes L)$$

vanishes; however the cohomology group $H^{k+1}(X, K_X \otimes S \otimes L)$ itself does not vanish in general as it would do under a strict positivity assumption (cf. Th. VII-9.4).

We want now to get also estimates when Q is endowed with a metric given a priori, that can be distinct from the quotient metric of E by g. Then the map $g^\star(gg^\star)^{-1} : Q \longrightarrow E$ is the lifting of Q orthogonal to $S = \mathrm{Ker}\, g$. The quotient metric $|\bullet|'$ on Q is therefore defined in terms of the original metric $|\bullet|$ by

$$|v|'^2 = |g^\star(gg^\star)^{-1}v|^2 = \langle (gg^\star)^{-1}v, v\rangle = \det(gg^\star)^{-1}\,\langle \widetilde{gg^\star} v, v\rangle$$

where $\widetilde{gg^\star} \in \mathrm{End}(Q)$ denotes the endomorphism of Q whose matrix is the transposed of the comatrix of $gg^\star$. For every $w \in \det Q$, we find

$$|w|'^2 = \det(gg^\star)^{-1}\,|w|^2.$$

If Q' denotes the bundle Q with the quotient metric, we get

$$\mathrm{i}\Theta(\det Q') = \mathrm{i}\Theta(\det Q) + \mathrm{i}d'd'' \log\det(gg^\star).$$

In order that the hypotheses of Th. 9.3 be satisfied, we are led to define a new metric $|\bullet|'$ on L by $|u|'^2 = |u|^2\left(\det(gg^\star)\right)^{-m-\varepsilon}$. Then

$$\mathrm{i}\Theta(L') = \mathrm{i}\Theta(L) + (m+\varepsilon)\,\mathrm{i}d'd''\log\det(gg^\star) \geq (m+\varepsilon)\,\mathrm{i}\Theta(\det Q').$$

Theorem 9.3 applied to (E, Q', L') can now be reformulated:

(9.8) Theorem. *Let X be a complete Kähler manifold equipped with a Kähler metric ω on X, let $E \to Q$ be a surjective morphism of hermitian vector bundles and let $L \to X$ be a hermitian line bundle. Set $r = \mathrm{rk}\,E$, $q = \mathrm{rk}\,Q$ and $m = \min\{n-k, r-q\}$ and suppose $E \geq_m 0$,*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that

$$I = \int_X \langle \widetilde{gg^\star} f, f\rangle\,(\det gg^\star)^{-m-1-\varepsilon}\,dV < +\infty,$$

there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g \cdot h$ and

$$\int_X |h|^2\,(\det gg^\star)^{-m-\varepsilon}\,dV \leq (1 + m/\varepsilon)\,I. \qquad \square$$

Our next goal is to extend Th. 9.8 in the case when $g : E \longrightarrow Q$ is only *generically* surjective; this means that the analytic set

$$Y = \{x \in X\ ;\ g_x\ :\ E_x \longrightarrow Q_x \text{ is not surjective }\}$$

defined by the equation $\Lambda^q g = 0$ is nowhere dense in X. Here $\Lambda^q g$ is a section of the bundle $\mathrm{hom}(\Lambda^q E, \det Q)$.

(9.9) Theorem. *The existence statement and the estimates of Th. 9.8 remain true for a generically surjective morphism $g : E \to Q$ provided that X is weakly pseudoconvex.*

Proof. Apply Th. 9.8 to each relatively compact domain $X_c \smallsetminus Y$ (these domains are complete Kähler by Lemma 7.2). From a sequence of solutions on $X_c \smallsetminus Y$ we can extract a subsequence converging weakly on $X \smallsetminus Y$ as c tends to $+\infty$. One gets a form h satisfying the estimates, such that $D''h = 0$ on $X \smallsetminus Y$ and $f = g \cdot h$. In order to see that $D''h = 0$ on X, it suffices to apply Lemma 7.3 and to observe that h has L^2_{loc} coefficients on X by our estimates. $\square$

A very special but interesting case is obtained for the trivial bundles $E = \Omega \times \mathbb{C}^r$, $Q = \Omega \times \mathbb{C}$ over a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Then the morphism g is given by a r-tuple $(g_1, \ldots, g_r)$ of holomorphic functions on Ω. Let us take $k = 0$ and $L = \Omega \times \mathbb{C}$ with the metric given by a weight $e^{-\varphi}$. If we observe that $\widetilde{gg^\star} = \mathrm{Id}$ when $\mathrm{rk}\, Q = 1$, Th. 9.8 applied on $X = \Omega \smallsetminus g^{-1}(0)$ and Lemmas 7.2, 7.3 give:

(9.10) Theorem (Skoda 1978)**.** *Let Ω be a complete Kähler open subset of $\mathbb{C}^n$ and φ a plurisubharmonic function on Ω. Set $m = \min\{n, r-1\}$. Then for every holomorphic function f on Ω such that*

$$I = \int_{\Omega \smallsetminus Z} |f|^2 \, |g|^{-2(m+1+\varepsilon)} e^{-\varphi} \, dV < +\infty,$$

where $Z = g^{-1}(0)$, there exist holomorphic functions $(h_1, \ldots, h_r)$ on Ω such that $f = \sum g_j h_j$ and

$$\int_{\Omega \smallsetminus Y} |h|^2 \, |g|^{-2(m+\varepsilon)} e^{-\varphi} \, dV \le (1 + m/\varepsilon) I. \qquad \square$$

This last theorem can be used in order to obtain a quick solution of the Levi problem mentioned in §I-4. It can be used also to prove a result of (Diederich-Pflug 1981), relating the pseudoconvexity property and the existence of complete Kähler metrics for domains of $\mathbb{C}^n$.

(9.11) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. Then:*

a) *Ω is a domain of holomorphy if and only if Ω is pseudoconvex ;*

b) *If $(\overline{\Omega})^\circ = \Omega$ and if Ω has a complete Kähler metric $\widehat{\omega}$, then Ω is pseudoconvex.*

Note that statement b) can be false if the assumption $(\overline{\Omega})^\circ = \Omega$ is omitted: in fact $\mathbb{C}^n \smallsetminus \{0\}$ is complete Kähler by Lemma 7.2, but it is not pseudoconvex if $n \ge 2$.

Proof. b) By Th. I-4.12, it is enough to verify that Ω is a domain of holomorphy, i.e. that for every connected open subset U such that $U \cap \partial\Omega \ne \emptyset$ and every connected component W of $U \cap \Omega$ there exists a holomorphic function h on Ω such that $h_{\restriction W}$ cannot be continued to U. Since $(\overline{\Omega})^\circ = \Omega$, the set $U \smallsetminus \overline{\Omega}$ is not empty. We select $a \in U \smallsetminus \overline{\Omega}$. Then the integral

$$\int_\Omega |z - a|^{-2(n+\varepsilon)} \, dV(z)$$

converges. By Th. 9.10 applied to $f(z) = 1$, $g_j(z) = z_j - a_j$ and $\varphi = 0$, there exist holomorphic functions h_j on Ω such that $\sum (z_j - a_j)\, h_j(z) = 1$. This shows that at least one of the functions h_j cannot be analytically continued at $a \in U$.

a) Assume that Ω is pseudoconvex. Given any open connected set U such that $U \cap \partial\Omega \neq \emptyset$, choose $a \in U \cap \partial\Omega$. By Th. I-4.14 c) the function

$$\varphi(z) = (n+\varepsilon)\big(\log(1+|z|^2) - 2\log d(z, \complement\Omega)\big)$$

is plurisubharmonic on Ω. Then the integral

$$\int_\Omega |z-a|^{-2(n+\varepsilon)}\, e^{-\varphi(z)}\, dV(z) \le \int_\Omega (1+|z|^2)^{-n-\varepsilon}\, dV(z)$$

converges, and we conclude as for b). $\square$

10. Application of Skoda's L^2 Estimates to Local Algebra

We apply here Th. 9.10 to the study of ideals in the ring $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ of germs of holomorphic functions on $(\mathbb{C}^n, 0)$. Let $\mathcal{I} = (g_1, \ldots, g_r) \neq (0)$ be an ideal of $\mathcal{O}_n$.

(10.1) Definition. *Let $k \in \mathbb{R}_+$. We associate to $\mathcal{I}$ the following ideals:*

a) *the ideal $\overline{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that $|u| \le C|g|^k$ for some constant $C \ge 0$, where $|g|^2 = |g_1|^2 + \cdots + |g_r|^2$.*

b) *the ideal $\widehat{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that*

$$\int_\Omega |u|^2\, |g|^{-2(k+\varepsilon)}\, dV < +\infty$$

on a small ball Ω centered at 0, if $\varepsilon > 0$ is small enough.

(10.2) Proposition. *For all $k, l \in \mathbb{R}_+$ we have*

a) $\overline{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k)}$;

b) $\mathcal{I}^k \subset \overline{\mathcal{I}}^{(k)}$ *if $k \in \mathbb{N}$;*

c) $\overline{\mathcal{I}}^{(k)}.\overline{\mathcal{I}}^{(l)} \subset \overline{\mathcal{I}}^{(k+l)}$;

d) $\overline{\mathcal{I}}^{(k)}.\widehat{\mathcal{I}}^{(l)} \subset \widehat{\mathcal{I}}^{(k+l)}$.

All properties are immediate from the definitions except a) which is a consequence of Lemma 8.2. Before stating the main result, we need a simple lemma.

(10.3) Lemma. *If $\mathcal{I} = (g_1, \ldots, g_r)$ and $r > n$, we can find elements $\widetilde{g}_1, \ldots, \widetilde{g}_n \in \mathcal{I}$ such that $C^{-1}|g| \le |\widetilde{g}| \le C|g|$ on a neighborhood of 0. Each $\widetilde{g}_j$ can be taken to be a linear combination*

$$\widetilde{g}_j = a_j \cdot g = \sum_{1 \le k \le r} a_{jk} g_k, \qquad a_j \in \mathbb{C}^r \smallsetminus \{0\}$$

where the coefficients $([a_1], \ldots, [a_n])$ are chosen in the complement of a proper analytic subset of $(\mathbb{P}^{r-1})^n$.

It follows from the Lemma that the ideal $\mathcal{J} = (\widetilde{g}_1, \ldots, \widetilde{g}_n) \subset \mathcal{I}$ satisfies $\overline{\mathcal{J}}^{(k)} = \overline{\mathcal{I}}^{(k)}$ and $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$ for all k.

Proof. Assume that $g \in \mathcal{O}(\Omega)^r$. Consider the analytic subsets in $\Omega \times (\mathbb{P}^{r-1})^n$ defined by

$$A = \big\{(z, [w_1], \ldots, [w_n])\,;\, w_j \cdot g(z) = 0\big\},$$
$$A^\star = \bigcup \text{irreducible components of } A \text{ not contained in } g^{-1}(0) \times (\mathbb{P}^{r-1})^n.$$

For $z \notin g^{-1}(0)$ the fiber $A_z = \{([w_1], \ldots, [w_n])\,;\, w_j \cdot g(z) = 0\} = A^\star_z$ is a product of n hyperplanes in $\mathbb{P}^{r-1}$, hence $A \cap (\Omega \smallsetminus g^{-1}(0)) \times (\mathbb{P}^{r-1})^n$ is a fiber bundle with base $\Omega \smallsetminus g^{-1}(0)$ and fiber $(\mathbb{P}^{r-2})^n$. As $A^\star$ is the closure of this set in $\Omega \times (\mathbb{P}^{r-1})^n$, we have

$$\dim A^\star = n + n(r-2) = n(r-1) = \dim(\mathbb{P}^{r-1})^n.$$

It follows that the zero fiber

$$A^\star_0 = A^\star \cap \big(\{0\} \times (\mathbb{P}^{r-1})^n\big)$$

is a proper subset of $\{0\} \times (\mathbb{P}^{r-1})^n$. Choose $(a_1, \ldots, a_n) \in (\mathbb{C}^r \smallsetminus \{0\})^n$ such that $(0, [a_1], \ldots, [a_n])$ is not in $A^\star_0$. By an easy compactness argument the set $A^\star \cap \big(\overline{B}(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big)$ is disjoint from the neighborhood $B(0,\varepsilon) \times \prod [B(a_j, \varepsilon)]$ of $(0, [a_1], \ldots, [a_n])$ for ε small enough. For $z \in B(0,\varepsilon)$ we have $|a_j \cdot g(z)| \ge \varepsilon |g(z)|$ for some j, otherwise the inequality $|a_j \cdot g(z)| < \varepsilon |g(z)|$ would imply the existence of $h_j \in \mathbb{C}^r$ with $|h_j| < \varepsilon$ and $a_j \cdot g(z) = h_j \cdot g(z)$. Since $g(z) \ne 0$, we would have

$$(z, [a_1 - h_1], \ldots, [a_n - h_n]) \in A^\star \cap \big(B(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big),$$

a contradiction. We obtain therefore

$$\varepsilon |g(z)| \le \max |a_j \cdot g(z)| \le (\max |a_j|)\, |g(z)| \quad \text{on } B(0,\varepsilon). \qquad \square$$

(10.4) Theorem (Briançon-Skoda 1974)**.** *Set $p = \min\{n-1, r-1\}$. Then*

a) $\widehat{\mathcal{I}}^{(k+1)} = \mathcal{I}\,\widehat{\mathcal{I}}^{(k)} = \overline{\mathcal{I}}\,\widehat{\mathcal{I}}^{(k)}$ *for $k \ge p$.*

b) $\overline{\mathcal{J}}^{(k+p)} \subset \widehat{\mathcal{J}}^{(k+p)} \subset \mathcal{J}^k$ *for all* $k \in \mathbb{N}$.

Proof. a) The inclusions $\mathcal{J}\widehat{\mathcal{J}}^{(k)} \subset \overline{\mathcal{J}}\widehat{\mathcal{J}}^{(k)} \subset \widehat{\mathcal{J}}^{(k+1)}$ are obvious thanks to Prop. 10.2, so we only have to prove that $\widehat{\mathcal{J}}^{(k+1)} \subset \mathcal{J}\widehat{\mathcal{J}}^{(k)}$. Assume first that $r \le n$. Let $f \in \widehat{\mathcal{J}}^{(k+1)}$ be such that

$$\int_\Omega |f|^2\,|g|^{-2(k+1+\varepsilon)}\,dV < +\infty.$$

For $k \ge p-1$, we can apply Th. 9.10 with $m = r-1$ and with the weight $\varphi = (k-m)\log|g|^2$. Hence f can be written $f = \sum g_j h_j$ with

$$\int_\Omega |h|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty,$$

thus $h_j \in \widehat{\mathcal{J}}^{(k)}$ and $f \in \mathcal{J}\widehat{\mathcal{J}}^{(k)}$. When $r > n$, Lemma 10.3 shows that there is an ideal $\mathcal{G} \subset \mathcal{J}$ with n generators such that $\widehat{\mathcal{G}}^{(k)} = \widehat{\mathcal{J}}^{(k)}$. We find

$$\widehat{\mathcal{J}}^{(k+1)} = \widehat{\mathcal{G}}^{(k+1)} \subset \mathcal{G}\,\widehat{\mathcal{G}}^{(k)} \subset \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} \qquad \text{for} \quad k \ge n-1.$$

b) Property a) implies inductively $\widehat{\mathcal{J}}^{(k+p)} = \mathcal{J}^k\,\widehat{\mathcal{J}}^{(p)}$ for all $k \in \mathbb{N}$. This gives in particular $\widehat{\mathcal{J}}^{(k+p)} \subset \mathcal{J}^k$. □

(10.5) Corollary.

a) *The ideal* $\overline{\mathcal{J}}$ *is the integral closure of* $\mathcal{J}$, *i.e. by definition the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{J}^s, \qquad 1 \le s \le d.$$

b) *Similarly,* $\overline{\mathcal{J}}^{(k)}$ *is the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{J}^{]ks[}, \qquad 1 \le s \le d,$$

where $]t[$ *denotes the smallest integer* $\ge t$.

As the ideal $\overline{\mathcal{J}}^{(k)}$ is finitely generated, property b) shows that there always exists a rational number $l \ge k$ such that $\overline{\mathcal{J}}^{(l)} = \overline{\mathcal{J}}^{(k)}$.

Proof. a) If $u \in \mathcal{O}_n$ satisfies a polynomial equation with coefficients $a_s \in \mathcal{J}^s$, then clearly $|a_s| \le C_s\,|g|^s$ and Lemma II-4.10 implies $|u| \le C\,|g|$.

Conversely, assume that $u \in \overline{\mathcal{J}}$. The ring $\mathcal{O}_n$ is Noetherian, so the ideal $\widehat{\mathcal{J}}^{(p)}$ has a finite number of generators $v_1, \ldots, v_N$. For every j we have $uv_j \in \overline{\mathcal{J}}\,\widehat{\mathcal{J}}^{(p)} = \mathcal{J}\,\widehat{\mathcal{J}}^{(p)}$, hence there exist elements $b_{jk} \in \mathcal{J}$ such that

$$uv_j = \sum_{1 \le k \le N} b_{jk} v_k.$$

The matrix $(u\delta_{jk} - b_{jk})$ has the non zero vector (v_j) in its kernel, thus u satisfies the equation $\det(u\delta_{jk} - b_{jk}) = 0$, which is of the required type.

b) Observe that $v_1, \ldots, v_N$ satisfy simultaneously some integrability condition $\int_\Omega |v_j|^{-2(p+\varepsilon)} < +\infty$, thus $\widehat{\mathcal{I}}^{(p)} = \widehat{\mathcal{I}}^{(p+\eta)}$ for $\eta \in [0, \varepsilon[$. Let $u \in \overline{\mathcal{I}}^{(k)}$. For every integer $m \in \mathbb{N}$ we have

$$u^m v_j \in \overline{\mathcal{I}}^{(km)}\,\widehat{\mathcal{I}}^{(p+\eta)} \subset \widehat{\mathcal{I}}^{(km+\eta+p)}.$$

If $k \notin \mathbb{Q}$, we can find m such that $d(km + \varepsilon/2, \mathbb{Z}) < \varepsilon/2$, thus $km + \eta \in \mathbb{N}$ for some $\eta \in]0, \varepsilon[$. If $k \in \mathbb{Q}$, we take m such that $km \in \mathbb{N}$ and $\eta = 0$. Then

$$u^m v_j \in \widehat{\mathcal{I}}^{(N+p)} = \mathcal{I}^N\,\widehat{\mathcal{I}}^{(p)} \quad \text{with} \quad N = km + \eta \in \mathbb{N},$$

and the reasoning made in a) gives $\det(u^m\delta_{jk} - b_{jk}) = 0$ for some $b_{jk} \in \mathcal{I}^N$. This is an equation of the type described in b), where the coefficients a_s vanish when s is not a multiple of m and $a_{ms} \in \mathcal{I}^{Ns} \subset \mathcal{I}^{]kms[}$. □

Let us mention that Briançon and Skoda's result 10.4 b) is optimal for $k = 1$. Take for example $\mathcal{I} = (g_1, \ldots, g_r)$ with $g_j(z) = z_j^r$, $1 \le j \le r$, and $f(z) = z_1 \ldots z_r$. Then $|f| \le C|g|$ and 10.4 b) yields $f^r \in \mathcal{I}$; however, it is easy to verify that $f^{r-1} \notin \mathcal{I}$. The theorem also gives an answer to the following conjecture made by J. Mather.

(10.6) Corollary. *Let $f \in \mathcal{O}_n$ and $\mathcal{I}_f = (z_1\partial f/\partial z_1, \ldots, z_n\partial f/\partial z_n)$. Then $f \in \overline{\mathcal{I}}_f$, and for every integer $k \ge 0$, $f^{k+n-1} \in \mathcal{I}_f^k$.*

The Corollary is also optimal for $k = 1$: for example, one can verify that the function $f(z) = (z_1 \ldots z_n)^3 + z_1^{3n-1} + \ldots + z_n^{3n-1}$ is such that $f^{n-1} \notin \mathcal{I}_f$.

Proof. Set $g_j(z) = z_j\,\partial f/\partial z_j$, $1 \le j \le n$. By 10.4 b), it suffices to show that $|f| \le C|g|$. For every germ of analytic curve $\mathbb{C} \ni t \longmapsto \gamma(t)$, $\gamma \not\equiv 0$, the vanishing order of $f \circ \gamma(t)$ at $t = 0$ is the same as that of

$$t\,\frac{d(f\circ\gamma)}{dt} = \sum_{1\le j\le n} t\,\gamma_j'(t)\,\frac{\partial f}{\partial z_j}\big(\gamma(t)\big).$$

We thus obtain

$$|f\circ\gamma(t)| \le C_1\,|t|\,\left|\frac{d(f\circ\gamma)}{dt}\right| \le C_2 \sum_{1\le j\le n} |t\,\gamma_j'(t)|\,\left|\frac{\partial f}{\partial z_j}\big(\gamma(t)\big)\right| \le C_3\,|g\circ\gamma(t)|$$

and conclude by the following elementary lemma. □

(10.7) Lemma. *Let $f, g_1, \ldots, g_r \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at 0. Then we have $|f| \le C|g|$ for some constant C if and only if* *for every germ of analytic curve* γ *through* 0 *there exists a constant* C_γ *such that* $|f\circ\gamma| \le C_\gamma |g\circ\gamma|$.

Proof. If the inequality $|f| \le C|g|$ does not hold on any neighborhood of 0, the germ of analytic set $(A,0) \subset (\mathbb{C}^{n+r},0)$ defined by

$$g_j(z) = f(z)z_{n+j}, \qquad 1 \le j \le r,$$

contains a sequence of points $\big(z_\nu, g_j(z_\nu)/f(z_\nu)\big)$ converging to 0 as ν tends to $+\infty$, with $f(z_\nu) \ne 0$. Hence $(A,0)$ contains an irreducible component on which $f \not\equiv 0$ and there is a germ of curve $\widetilde\gamma = (\gamma, \gamma_{n+j}) : (\mathbb{C},0) \to (\mathbb{C}^{n+r},0)$ contained in $(A,0)$ such that $f\circ\gamma \not\equiv 0$. We get $g_j\circ\gamma = (f\circ\gamma)\gamma_{n+j}$, hence $|g\circ\gamma(t)| \le C|t|\,|f\circ\gamma(t)|$ and the inequality $|f\circ\gamma| \le C_\gamma|g\circ\gamma|$ does not hold. □

11. Integrability of Almost Complex Structures

Let M be a C^∞ manifold of real dimension $m = 2n$. An *almost complex structure* on M is by definition an endomorphism $J \in \mathrm{End}(TM)$ of class C^∞ such that $J^2 = -\mathrm{Id}$. Then TM becomes a complex vector bundle for which the scalar multiplication by i is given by J. The pair (M,J) is said to be an *almost complex manifold*. For such a manifold, the complexified tangent space $T_{\mathbb{C}}M = \mathbb{C}\otimes_{\mathbb{R}} TM$ splits into conjugate complex subspaces

$$(11.1)\quad T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \qquad \dim_{\mathbb{C}} T^{1,0}M = \dim_{\mathbb{C}} T^{0,1}M = n,$$

where $T^{1,0}M$, $T^{0,1}M \subset T_{\mathbb{C}}M$ are the eigenspaces of $\mathrm{Id}\otimes J$ corresponding to the eigenvalues i and $-i$. The complexified exterior algebra $\mathbb{C}\otimes_{\mathbb{R}} \Lambda^\bullet T^\star M = \Lambda^\bullet T^\star_{\mathbb{C}}M$ has a corresponding splitting

$$(11.2)\quad \Lambda^k T^\star_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^{p,q}T^\star_{\mathbb{C}}M$$

where we denote by definition

$$(11.3)\quad \Lambda^{p,q}T^\star_{\mathbb{C}}M = \Lambda^p(T^{1,0}M)^\star \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}M)^\star.$$

As for complex manifolds, we let $C^s_{p,q}(M,E)$ be the space of differential forms of class C^s and bidegree (p,q) on M with values in a complex vector bundle E. There is a natural antisymmetric bilinear map

$$\theta \;:\; C^\infty(M,T^{1,0}M)\times C^\infty(M,T^{1,0}M) \longrightarrow C^\infty(M,T^{0,1}M)$$

which associates to a pair (ξ,η) of $(1,0)$-vector fields the $(0,1)$-component of the Lie bracket $[\xi,\eta]$. Since

$$[\xi, f\eta] = f[\xi,\eta] + (\xi.f)\,\eta, \qquad \forall f\in C^\infty(M,\mathbb{C})$$

we see that $\theta(\xi, f\eta) = f\,\theta(\xi,\eta)$. It follows that θ is in fact a $(2,0)$-form on M with values in $T^{0,1}M$.

If M is a *complex analytic manifold* and J its natural almost complex structure, we have in fact $\theta = 0$, because $[\partial/\partial z_j, \partial/\partial z_k] = 0$, $1 \le j,k \le n$, for any holomorphic local coordinate system $(z_1,\ldots,z_n)$.

(11.4) Definition. *The form $\theta \in C^\infty_{2,0}(M, T^{0,1}M)$ is called the torsion form of J. The almost complex structure J is said to be integrable if $\theta = 0$.*

(11.5) Example. If M is of real dimension $m = 2$, every almost complex structure is integrable, because $n = 1$ and alternate $(2,0)$-forms must be zero. Assume that M is a smooth oriented surface. To any Riemannian metric g we can associate the endomorphism $J \in \mathrm{End}(TM)$ equal to the rotation of $+\pi/2$. A change of orientation changes J into the conjugate structure $-J$. Conversely, if J is given, TM is a complex line bundle, so M is oriented, and a Riemannian metric g is associated to J if and only if g is J-hermitian. As a consequence, there is a one-to-one correspondence between conformal classes of Riemannian metrics on M and almost complex structures corresponding to a given orientation. □

If (M,J) is an almost complex manifold and $u \in C^\infty_{p,q}(M,\mathbb{C})$, we let $d'u$, $d''u$ be the components of type $(p+1,q)$ and $(p,q+1)$ in the exterior derivative du. Let $(\xi_1,\ldots,\xi_n)$ be a frame of $T^{1,0}M_{\restriction\Omega}$. The torsion form θ can be written

$$\theta = \sum_{1\le j\le n} \alpha_j \otimes \overline{\xi}_j, \qquad \alpha_j \in C^\infty_{2,0}(\Omega,\mathbb{C}).$$

Then θ yields conjugate operators θ', θ'' on $\Lambda^\bullet T^\star_{\mathbb{C}}M$ such that

$$(11.6)\quad \theta' u = \sum_{1\le j\le n} \alpha_j \wedge (\overline{\xi}_j \mathbin{\lrcorner} u), \qquad \theta'' u = \sum_{1\le j\le n} \overline{\alpha}_j \wedge (\xi_j \mathbin{\lrcorner} u).$$

If u is of bidegree (p,q), then $\theta' u$ and $\theta'' u$ are of bidegree $(p+2, q-1)$ and $(p-1, q+2)$. It is clear that θ', θ'' are derivations, i.e.

$$\theta'(u\wedge v) = (\theta' u)\wedge v + (-1)^{\deg u} u \wedge (\theta' v)$$

for all smooth forms u, v, and similarly for θ''.

(11.7) Proposition. *We have* $d = d' + d'' - \theta' - \theta''$.

Proof. Since all operators occuring in the formula are derivations, it is sufficient to check the formula for forms of degree 0 or 1. If u is of degree 0, the result is obvious because $\theta' u = \theta'' u = 0$ and du can only have components of types $(1,0)$ or $(0,1)$. If u is a 1-form and ξ, η are complex vector fields, we have

$$du(\xi,\eta) = \xi.u(\eta) - \eta.du(\xi) - u([\xi,\eta]).$$

When u is of type $(0,1)$ and ξ, η of type $(1,0)$, we find

$$(du)^{2,0}(\xi,\eta) = -u\big(\theta(\xi,\eta)\big)$$

thus $(du)^{2,0} = -\theta' u$, and of course $(du)^{1,1} = d'u$, $(du)^{0,2} = d''u$, $\theta'' u = 0$ by definition. The case of a $(1,0)$-form u follows by conjugation. □

Proposition 11.7 shows that J is integrable if and only if $d = d' + d''$. In this case, we infer immediately

$$d'^2 = 0, \qquad d'd'' + d''d' = 0, \qquad d''^2 = 0.$$

For an integrable almost complex structure, we thus have the same formalism as for a complex analytic structure, and indeed we shall prove:

(11.8) Newlander-Nirenberg theorem (1957). *Every integrable almost complex structure J on M is defined by a unique analytic structure.*

The proof we shall give follows rather closely that of (Hörmander 1966), which was itself based on previous ideas of (Kohn 1963, 1964). A function $f \in C^1(\Omega, \mathbb{C})$, $\Omega \subset M$, is said to be J-holomorphic if $d''f = 0$. Let $f_1, \ldots, f_p \in C^1(\Omega, \mathbb{C})$ and let h be a function of class C^1 on an open subset of $\mathbb{C}^p$ containing the range of $f = (f_1, \ldots, f_p)$. An easy computation gives

$$(11.9) \quad d''(h \circ f) = \sum_{1 \le j \le p} \Big(\frac{\partial h}{\partial z_j} \circ f\Big) d'' f_j + \Big(\frac{\partial h}{\partial \overline{z}_j} \circ f\Big) \overline{d' f_j},$$

in particular $h \circ f$ is J-holomorphic as soon as $f_1, \ldots, f_p$ are J-holomorphic and h holomorphic in the usual sense.

Constructing a complex analytic structure on M amounts to show the existence of J-holomorphic complex coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of every point $a \in M$. Formula (11.9) then shows that all coordinate changes $h : (z_k) \mapsto (w_k)$ are holomorphic in the usual sense, so that M is furnished with a complex analytic atlas. The uniqueness of the analytic structure associated to J is clear, since the holomorphic functions are characterized by the condition $d''f = 0$. In order to show the existence, we need a lemma.

(11.10) Lemma. *For every point $a \in M$ and every integer $s \ge 1$, there exist C^∞ complex coordinates $(z_1, \ldots, z_n)$ centered at a such that*

$$d'' z_j = O(|z|^s), \qquad 1 \le j \le n.$$

Proof. By induction on s. Let $(\xi_1^\star, \ldots, \xi_n^\star)$ be a basis of $\Lambda^{1,0} T^\star_{\mathbb{C}} M$. One can find complex functions z_j such that $dz_j(a) = \xi_j^\star$, i.e.

$$d'z_j(a) = \xi_j^\star, \qquad d''z_j(a) = 0.$$

Then $(z_1, \ldots, z_n)$ satisfy the conclusions of the Lemma for $s = 1$. If $(z_1, \ldots, z_n)$ are already constructed for the integer s, we have a Taylor expansion

$$d''z_j = \sum_{1 \le k \le n} P_{jk}(z, \overline{z})\, \overline{d'z_k} + \mathrm{O}(|z|^{s+1})$$

where $P_{jk}(z, w)$ is a homogeneous polynomial in $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ of total degree s. As J is integrable, we have

$$\begin{aligned} 0 = d''^2 z_j &= \sum_{1 \le k, l \le n} \frac{\partial P_{jk}}{\partial z_l}\, d''z_l \wedge \overline{d'z_k} + \frac{\partial P_{jk}}{\partial \overline{z}_l}\, \overline{d'z_l} \wedge \overline{d'z_k} + \mathrm{O}(|z|^s) \\ &= \sum_{1 \le k < l \le n} \Big[\frac{\partial P_{jk}}{\partial \overline{z}_l} - \frac{\partial P_{jl}}{\partial \overline{z}_k} \Big]\, \overline{d'z_l} \wedge \overline{d'z_k} + \mathrm{O}(|z|^s) \end{aligned}$$

because $\partial P_{jk} / \partial z_l$ is of degree $s-1$ and $d''z_l = \mathrm{O}(|z|^s)$. Since the polynomial between brackets is of degree $s-1$, we must have

$$\frac{\partial P_{jk}}{\partial \overline{z}_l} - \frac{\partial P_{jl}}{\partial \overline{z}_k} = 0, \qquad \forall j, k, l.$$

We define polynomials Q_j of degree $s+1$

$$Q_j(z, \overline{z}) = \int_0^1 \sum_{1 \le l \le n} \overline{z}_l\, P_{jl}(z, t\overline{z})\, dt.$$

Trivial computations show that

$$\begin{aligned} \frac{\partial Q_j}{\partial \overline{z}_k} &= \int_0^1 \Big(P_{jk} + \sum_{1 \le l \le n} \overline{z}_l\, \frac{\partial P_{jl}}{\partial \overline{z}_k} \Big)(z, t\overline{z})\, dt \\ &= \int_0^1 \frac{d}{dt} \Big[t\, P_{jk}(z, t\overline{z}) \Big]\, dt = P_{jk}(z, \overline{z}), \\ d''\big(z_j - Q_j(z, \overline{z})\big) &= d''z_j - \sum_{1 \le k \le n} \frac{\partial Q_j}{\partial \overline{z}_k}\, \overline{d'z_k} - \sum_{1 \le k \le n} \frac{\partial Q_j}{\partial z_k}\, d''z_k \\ &= - \sum_{1 \le k \le n} \frac{\partial Q_j}{\partial z_k}\, d''z_k + \mathrm{O}(|z|^{s+1}) = \mathrm{O}(|z|^{s+1}) \end{aligned}$$

because $\partial Q_j / \partial z_k$ is of degree s and $d''z_l = \mathrm{O}(|z|)$. The new coordinates

$$\widetilde{z}_j = z_j - Q_j(z, \overline{z}), \qquad 1 \le j \le n$$

fulfill the Lemma at step $s+1$. □

All usual notions defined on complex analytic manifolds can be extended to integrable almost complex manifolds. For example, a smooth function φ is said to be strictly plurisubharmonic if $\mathrm{i}d'd''\varphi$ is a positive definite $(1,1)$-form. Then $\omega = \mathrm{i}d'd''\varphi$ is a Kähler metric on (M,J).

In this context, all L^2 estimates proved in the previous paragraphs still apply to an integrable almost complex manifold; remember that the proof of the Bochner-Kodaira-Nakano identity used only Taylor developments of order ≤ 2, and the coordinates given by Lemma 11.10 work perfectly well for that purpose. In particular, Th. 6.5 is still valid.

(11.11) Lemma. *Let $(z_1,\ldots,z_n)$ be coordinates centered at a point $a\in M$ with $d''z_j = \mathrm{O}(|z|^s)$, $s\ge 3$. Then the functions*

$$\psi(z) = |z|^2, \qquad \varphi_\varepsilon(z) = |z|^2 + \log(|z|^2+\varepsilon^2), \quad \varepsilon\in\,]0,1]$$

are strictly plurisubharmonic on a small ball $|z|<r_0$.

Proof. We have

$$\mathrm{i}d'd''\psi = \mathrm{i}\sum_{1\le j\le n} d'z_j\wedge\overline{d'z_j} + d'\overline{z}_j\wedge d''z_j + z_j\,d'd''\overline{z}_j + \overline{z}_j\,d'd''z_j.$$

The last three terms are $\mathrm{O}(|z|^s)$ and the first one is positive definite at $z=0$, so the result is clear for ψ. Moreover

$$\begin{aligned}\mathrm{i}d'd''\varphi_\varepsilon = \mathrm{i}d'd''\psi + i\frac{(|z|^2+\varepsilon^2)\sum d'z_j\wedge\overline{d'z_j} - \sum\overline{z}_j d'z_j\wedge\overline{\sum\overline{z}_j d'z_j}}{(|z|^2+\varepsilon^2)^2}\\ + \frac{\mathrm{O}(|z|^s)}{|z|^2+\varepsilon^2} + \frac{\mathrm{O}(|z|^{s+2})}{(|z|^2+\varepsilon^2)^2}.\end{aligned}$$

We observe that the first two terms are positive definite, whereas the remainder is $\mathrm{O}(|z|)$ uniformly in ε.

Proof of theorem 11.8. With the notations of the previous lemmas, consider the pseudoconvex open set

$$\Omega = \{|z|<r\} = \{\psi(z)-r^2<0\}, \qquad r<r_0,$$

endowed with the Kähler metric $\omega = \mathrm{i}d'd''\psi$. Let $h\in\mathcal{D}(\Omega)$ be a cut-off function with $0\le h\le 1$ and $h=1$ on a neighborhood of $z=0$. We apply Th. 6.5 to the $(0,1)$-forms

$$g_j = d''\big(z_j h(z)\big)\in C^\infty_{0,1}(\Omega,\mathbb{C})$$

for the weight

$$\varphi(z) = A|z|^2 + (n+1)\log|z|^2 = \lim_{\varepsilon\to 0} A|z|^2 + (n+1)\log(|z|^2+\varepsilon^2).$$

Lemma 11.11 shows that φ is plurisubharmonic for $A \geq n+1$, and for A large enough we obtain

$$\mathrm{i}d'd''\varphi + \mathrm{Ricci}(\omega) \geq \omega \quad \text{on } \Omega.$$

By Remark (6.8) we get a function f_j such that $d''f_j = g_j$ and

$$\int_\Omega |f_j|^2 e^{-\varphi} dV \leq \int_\Omega |g_j|^2 e^{-\varphi} dV.$$

As $g_j = d''z_j = \mathrm{O}(|z|^s)$ and $e^{-\varphi} = \mathrm{O}(|z|^{-2n-2})$ near $z = 0$, the integral of g_j converges provided that $s \geq 2$. Then $\int |f_j(z)|^2 |z|^{-2n-2} dV$ converges also at $z = 0$. Since the solution f_j is smooth, we must have $f_j(0) = df_j(0) = 0$. We set

$$\widetilde{z}_j = z_j h(z) - f_j, \qquad 1 \leq j \leq n.$$

Then $\widetilde{z}_j$ is J-holomorphic and $d\widetilde{z}_j(0) = dz_j(0)$, so $(z_1, \ldots, z_n)$ is a J-holomorphic coordinate system at $z = 0$. □

In particular, any Riemannian metric on an oriented 2-dimensional real manifold defines a unique analytic structure. This fact will be used in order to obtain a simple proof of the well-known:

(11.12) Uniformization theorem. *Every simply connected Riemann surface X is biholomorphic either to $\mathbb{P}^1$, $\mathbb{C}$ or the unit disk Δ.*

Proof. We will merely use the fact that $H^1(X, \mathbb{R}) = 0$. If X is compact, then X is a complex curve of genus 0, so $X \simeq \mathbb{P}^1$ by Th. VI-14.16. On the other hand, the elementary Riemann mapping theorem says that an open set $\Omega \subset \mathbb{C}$ with $H^1(\Omega, \mathbb{R}) = 0$ is either equal to $\mathbb{C}$ or biholomorphic to the unit disk. Thus, all we have to show is that a non compact Riemann surface X with $H^1(X, \mathbb{R}) = 0$ can be embedded in the complex plane $\mathbb{C}$.

Let Ω_ν be an exhausting sequence of relatively compact connected open sets with smooth boundary in X. We may assume that $X \smallsetminus \Omega_\nu$ has no relatively compact connected components, otherwise we "fill the holes" of Ω_ν by taking the union with all such components. We let Y_ν be the double of the manifold with boundary $(\overline{\Omega}_\nu, \partial\Omega_\nu)$, i.e. the union of two copies of $\overline{\Omega}_\nu$ with opposite orientations and the boundaries identified. Then Y_ν is a compact oriented surface without boundary.

(11.13) Lemma. *We have $H^1(Y_\nu, \mathbb{R}) = 0$.*

Proof. Let us first compute $H^1_c(\Omega_\nu, \mathbb{R})$. Let u be a closed 1-form with compact support in Ω_ν. By Poincaré duality $H^1_c(X, \mathbb{R}) = 0$, so $u = df$ for some function $f \in \mathcal{D}(X)$. As $df = 0$ on a neighborhood of $X \smallsetminus \Omega_\nu$ and as all connected components of this set are non compact, f must be equal to the constant zero near $X \smallsetminus \Omega_\nu$. Hence $u = df$ is the zero class in $H^1_c(\Omega_\nu, \mathbb{R})$ and we get $H^1_c(\Omega_\nu, \mathbb{R}) = H^1(\Omega_\nu, \mathbb{R}) = 0$. The exact sequence of the pair $(\overline{\Omega}_\nu, \partial\Omega_\nu)$ yields

$$\mathbb{R} = H^0(\overline{\Omega}_\nu, \mathbb{R}) \longrightarrow H^0(\partial\Omega_\nu, \mathbb{R}) \longrightarrow H^1(\overline{\Omega}_\nu, \partial\Omega_\nu\,; \mathbb{R}) \simeq H^1_c(\Omega_\nu, \mathbb{R}) = 0,$$

thus $H^0(\partial\Omega_\nu, \mathbb{R}) = \mathbb{R}$. Finally, the Mayer-Vietoris sequence applied to small neighborhoods of the two copies of $\overline{\Omega}_\nu$ in Y_ν gives an exact sequence

$$H^0(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} \longrightarrow H^0(\partial\Omega_\nu, \mathbb{R}) \longrightarrow H^1(Y_\nu, \mathbb{R}) \longrightarrow H^1(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} = 0$$

where the first map is onto. Hence $H^1(Y_\nu, \mathbb{R}) = 0$. □

Proof End of the proof of the uniformization theorem. Extend the almost complex structure of $\overline{\Omega}_\nu$ in an arbitrary way to Y_ν, e.g. by an extension of a Riemannian metric. Then Y_ν becomes a compact Riemann surface of genus 0, thus $Y_\nu \simeq \mathbb{P}^1$ and we obtain in particular a holomorphic embedding $\Phi_\nu : \Omega_\nu \longrightarrow \mathbb{C}$. Fix a point $a \in \Omega_0$ and a non zero linear form $\xi^\star \in T_aX$. We can take the composition of Φ_ν with an affine linear map $\mathbb{C} \to \mathbb{C}$ so that $\Phi_\nu(a) = 0$ and $d\Phi_\nu(a) = \xi^\star$. By the well-known properties of injective holomorphic maps, (Φ_ν) is then uniformly bounded on every small disk centered at a, thus also on every compact subset of X by a connectedness argument. Hence (Φ_ν) has a subsequence converging towards an injective holomorphic map $\Phi : X \longrightarrow \mathbb{C}$. □