Chapter IX
Finiteness Theorems for q-Convex Spaces and Stein Spaces

1. Topological Preliminaries

1.A. Krull Topology of $\mathcal{O}_n$-Modules

We shall use in an essential way different kind of topological results. The first of these concern the topology of modules over a local ring and depend on the Artin-Rees and Krull lemmas. Let R be a noetherian local ring; "local" means that R has a unique maximal ideal $\mathfrak{m}$, or equivalently, that R has an ideal $\mathfrak{m}$ such that every element $\alpha \in R \smallsetminus \mathfrak{m}$ is invertible.

(1.1) Nakayama lemma. *Let E be a finitely generated R-module such that $\mathfrak{m}E = E$. Then $E = \{0\}$.*

Proof. By induction on the number of generators of E: if E is generated by $x_1, \ldots, x_p$, the hypothesis $E = \mathfrak{m}E$ shows that $x_p = \alpha_1 x_1 + \cdots + \alpha_p x_p$ with $\alpha_j \in \mathfrak{m}$; as $1 - \alpha_p \in R \smallsetminus \mathfrak{m}$ is invertible, we see that x_p can be expressed in terms of $x_1, \ldots, x_{p-1}$ if $p > 1$ and that $x_1 = 0$ if $p = 1$. □

(1.2) Artin-Rees lemma. *Let F be a finitely generated R-module and let E be a submodule. There exists an integer s such that*

$$E \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F) \quad \text{for} \quad k \geq s.$$

Proof. Let R_t be the polynomial ring $R[\mathfrak{m}t] = R + \mathfrak{m}t + \cdots + \mathfrak{m}^k t^k + \cdots$ where t is an indeterminate. If $g_1, \ldots, g_p$ is a set of generators of the ideal $\mathfrak{m}$ over R, we see that the ring R_t is generated by $g_1 t, \ldots, g_p t$ over R, hence R_t is also noetherian. Now, we consider the R_t-modules

$$E_t = \bigoplus E\, t^k, \qquad F_t = \bigoplus\, (\mathfrak{m}^k F)\, t^k.$$

Then F_t is generated over R_t by the generators of F over R, hence the submodule $E_t \cap F_t$ is finitely generated. Let s be the highest exponent of t in a set of generators $P_1(t), \ldots, P_N(t)$ of $E_t \cap F_t$. If we identify the components of t^k in the extreme terms of the equality

$$\bigoplus \,(E \cap \mathfrak{m}^k F)\, t^k = E_t \cap F_t = \sum_j \Big(\bigoplus_k \mathfrak{m}^k\, t^k\Big) P_j(t),$$

we get

$$E \cap \mathfrak{m}^k F \subset \sum_{l \le s} \mathfrak{m}^{k-l}(E \cap \mathfrak{m}^l F) \subset \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F).$$

The opposite inclusion is clear. □

(1.3) Krull lemma. *Let F be a finitely generated R-module and let E be a submodule. Then*

a) $\bigcap_{k \ge 0} \mathfrak{m}^k F = \{0\}$.

b) $\bigcap_{k \ge 0}(E + \mathfrak{m}^k F) = E$.

Proof. a) Put $G = \bigcap_{k \ge 0} \mathfrak{m}^k F \subset F$. By the Artin-Rees lemma, there exists $s \in \mathbb{N}$ such that $G \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(G \cap \mathfrak{m}^s F)$. Taking $k = s + 1$, we find $G \subset \mathfrak{m}G$, hence $\mathfrak{m}G = G$ and $G = \{0\}$ by the Nakayama lemma.

b) By applying a) to the quotient module F/E we get $\bigcap \mathfrak{m}^k(F/E) = \{0\}$. Property b) follows. □

Now assume that $R = \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ and $\mathfrak{m} = (z_1, \ldots, z_n)$. Then $\mathcal{O}_n/\mathfrak{m}^k$ is a finite dimensional vector space generated by the monomials z^α, $|\alpha| < k$. It follows that $E/\mathfrak{m}^k E$ is a finite dimensional vector space for any finitely generated $\mathcal{O}_n$-module E. As $\bigcap \mathfrak{m}^k E = \{0\}$ by 1.3 a), there is an injection

$$(1.4) \quad E \hookrightarrow \prod_{k \in \mathbb{N}} E/\mathfrak{m}^k E.$$

We endow E with the Hausdorff topology induced by the product, i.e. with the weakest topology that makes all projections $E \longrightarrow E/\mathfrak{m}^k E$ continuous for the complex vector space topology on $E/\mathfrak{m}^k E$. This topology is called the *Krull topology* (or rather, the *analytic Krull topology*; the "algebraic" Krull topology would be obtained by taking the discrete topology on $E/\mathfrak{m}^k E$). For $E = \mathcal{O}_n$, this is the topology of simple convergence on coefficients, defined by the collection of semi-norms $\sum c_\alpha z^\alpha \longmapsto |c_\alpha|$. Observe that this topology is not complete: the completion of $\mathcal{O}_n$ can be identified with the ring of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$. In general, the completion is the inverse limit $\widehat{E} = \varprojlim E/\mathfrak{m}^k E$. Every $\mathcal{O}_n$-homomorphism $E \longrightarrow F$ is continuous, because the induced finite dimensional linear maps $E/\mathfrak{m}^k E \longrightarrow F/\mathfrak{m}^k F$ are continuous.

(1.5) Theorem. *Let $E \subset F$ be finitely generated $\mathcal{O}_n$-modules. Then:*

a) *The map* $F \longrightarrow G = F/E$ *is open, i.e. the Krull topology of* G *is the quotient of the Krull topology of* F ;

b) E *is closed in* F *and the topology induced by* F *on* E *coincides with the Krull topology of* E.

Proof. a) is an immediate consequence of the fact that the surjective finite dimensional linear maps $F/\mathfrak{m}^k F \longrightarrow G/\mathfrak{m}^k G$ are open.

b) Let $\overline{E}$ be the closure of E in F. The image of $\overline{E}$ in $F/\mathfrak{m}^k F$ is mapped into the closure of the image of E. As every subspace of a finite dimensional space is closed, the images of E and $\overline{E}$ must coincide, i.e. $\overline{E} + \mathfrak{m}^k F = E + \mathfrak{m}^k F$. Therefore

$$E \subset \overline{E} \subset \bigcap (E + \mathfrak{m}^k F) = E$$

thanks to 1.3 b). The topology induced by F on E is the weakest that makes all projections $E \longrightarrow E/E \cap \mathfrak{m}^k F$ continuous (via the injections $E/E \cap \mathfrak{m}^k F \hookrightarrow F/\mathfrak{m}^k F$). However, the Artin-Rees lemma gives

$$\mathfrak{m}^k E \subset E \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F) \subset \mathfrak{m}^{k-s} E \quad \text{for} \quad k \geq s,$$

so the topology induced by F coincides with that induced by $\prod E/\mathfrak{m}^k E$. $\square$

1.B. Compact Pertubations of Linear Operators

We now recall some basic results in the perturbation theory of linear operators. These results will be needed in order to obtain a finiteness criterion for cohomology groups.

(1.6) Definition. *Let* E, F *be Hausdorff locally convex topological vector spaces and* $g : E \longrightarrow F$ *a continuous linear operator.*

a) g *is said to be compact if there exists a neighborhood* U *of* 0 *in* E *such that the image* $\overline{g(U)}$ *is compact in* F.

b) g *is said to be a monomorphism if* g *is a topological isomorphism of* E *onto a closed subspace of* F*, and a quasi-monomorphism if* $\ker g$ *is finite dimensional and* $\widetilde{g} : E/\ker g \longrightarrow F$ *a monomorphism.*

c) g *is said to be an epimorphism if* g *is surjective and open, and a quasi-epimorphism if* g *is an epimorphism of* E *onto a closed finite codimensional subspace* $F' \subset F$.

d) g *is said to be a quasi-isomorphism if* g *is simultaneously a quasi-monomorphism and a quasi-epimorphism.*

(1.7) Lemma. *Assume that* E, F *are Fréchet spaces. Then*

a) *g is a (quasi-) monomorphism if and only if $g(E)$ is closed in F and g is injective (resp. and* $\ker g$ *is finite dimensional).*

b) *g is a (quasi-) epimorphism if and only if g is surjective (resp. $g(E)$ is finite codimensional).*

Proof. a) If $g(E)$ is closed, the map $\widetilde{g} : E/\ker g \longrightarrow g(E)$ is a continuous bijective linear map between Fréchet spaces, so $\widetilde{g}$ is a topological isomorphism by Banach's theorem.

b) If g is surjective, Banach's theorem implies that g is open, thus g is an epimorphism. If $g(E)$ is finite codimensional, let S be a supplementary subspace of $g(E)$ in F, $\dim S < +\infty$. Then the map

$$G \;:\; (E/\ker g) \oplus S \longrightarrow F, \qquad \widetilde{x} \oplus y \longmapsto \widetilde{g}(\widetilde{x}) + y$$

is a bijective linear map between Fréchet spaces, so it is a topological isomorphism. In particular $g(E) = G\big((E/\ker g) \oplus \{0\}\big)$ is closed as an image of a closed subspace. Hence $g(E)$ is also a Fréchet space and $g : E \longrightarrow g(E)$ is an epimorphism. □

(1.8) Theorem. *Let $h : E \longrightarrow F$ be a compact linear operator.*

a) *If $g : E \longrightarrow F$ is a quasi-monomorphism, then $g + h$ is a quasi-monomorphism.*

b) *If E, F are Fréchet spaces and if $g : E \longrightarrow F$ is a quasi-epimorphism, then $g + h$ is a quasi-epimorphism.*

Proof. Set $f = g + h$ and let U be an open convex symmetric neighborhood of 0 in E such that $K = \overline{h(U)}$ is compact.

a) It is sufficient to show that there is a finite dimensional subspace $E' \subset E$ such that $f_{\restriction E'}$ is a monomorphism. If we take E' equal to a supplementary subspace of $\ker g$, we see that we may assume g injective. Then g is a monomorphism, so we may assume in fact that E is a subspace of F and that g is the inclusion. Let V be an open convex symmetric neighborhood of 0 in F such that $U = V \cap E$. There exists a closed finite codimensional subspace $F' \subset F$ such that $K \cap F' \subset 2^{-1}V$ because $\bigcap_{F'} K \cap F' = \{0\}$. If we replace E by $E' = h^{-1}(F')$ and U by $U' = U \cap E'$, we get

$$K' := \overline{h(U')} \subset K \cap F' \subset 2^{-1}V.$$

Hence, we may assume without loss of generality that $K \subset 2^{-1}V$. Then we show that $f = g + h$ is actually a monomorphism. If Ω is an arbitrary open neighborhood of 0 in E, we have to check that there exists a neighborhood W of 0 in F such that $f(x) \in W \Longrightarrow x \in \Omega$. There is an integer N such that $2^{-N}K \cap E \subset \Omega$. We choose W convex and so small that

$$(W + 2^{-N}K) \cap E \subset \Omega \quad \text{and} \quad 2^N W + K \subset 2^{-1}V.$$

Let $x \in E$ be such that $f(x) \in W$. Then $x \in 2^n U$ for n large enough and we infer

$$x = f(x) - h(x) \in W + 2^n K \subset 2^{n-1}V \quad \text{provided that} \quad n \geq -N.$$

Thus $x \in 2^{n-1}V \cap E = 2^{n-1}U$. By induction we finally get $x \in 2^{-N}U$, so

$$x \in (W + 2^{-N}K) \cap E \subset \Omega.$$

b) By Lemma 1.7 b), we only have to show that there is a finite dimensional subspace $S \subset F$ such that the induced map

$$\widetilde{f} : E \longrightarrow F \longrightarrow F/S$$

is surjective. If we take S equal to a supplementary subspace of $g(E)$ and replace g, h by the induced maps $\widetilde{g}, \widetilde{h} : E \longrightarrow F/S$, we may assume that g itself is surjective. Then g is open, so $V = g(U)$ is a convex open neighborhood of 0 in F. As K is compact, there exists a finite set of elements $b_1, \ldots, b_N \in K$ such that $K \subset \bigcup(b_j + 2^{-1}V)$. If we take now $S = \mathrm{Vect}(b_1, \ldots, b_N)$, we obtain $\widetilde{K} \subset 2^{-1}\widetilde{V}$ where $\widetilde{K}$ is the closure of $\widetilde{h}(U)$ and $V = \widetilde{g}(U)$, so we may assume in addition that $K \subset 2^{-1}V$. Then we show that $f = g + h$ is actually surjective. Let $y_0 \in V$. There exists $x_0 \in U$ such that $g(x_0) = y_0$, thus

$$y_1 = y_0 - f(x_0) = -h(x_0) \in K \subset 2^{-1}V.$$

By induction, we construct $x_n \in 2^{-n}U$ such that $g(x_n) = y_n$ and

$$y_{n+1} = y_n - f(x_n) = -h(x_n) \in 2^{-n}K \subset 2^{-n-1}V.$$

Hence $y_{n+1} = y_0 - f(x_0 + \cdots + x_n)$ tends to 0 in F, but we still have to make sure that the series $\sum x_n$ converges in E. Let U_p be a fundamental system of convex neighborhoods of 0 in E such that $U_{p+1} \subset 2^{-1}U_p$. For each p, K is contained in the union of the open sets $g(2^n U_p \cap 2^{-1}U)$ when $n \in \mathbb{N}$, equal to $g(2^{-1}U) = 2^{-1}V$. There exists an increasing sequence $N(p)$ such that $K \subset g(2^{N(p)}U_p \cap 2^{-1}U)$, thus

$$2^{1-n}K \subset g(2^{N(p)+1-n}U_p \cap 2^{-n}U).$$

As $y_n \in 2^{1-n}K$, we see that we can choose $x_n \in 2^{N(p)+1-n}\, U_p \cap 2^{-n}\, U$ for $N(p) < n \leq N(p+1)$; then

$$x_{N(p)+1} + \cdots + x_{N(p+1)} \in (1 + 2^{-1} + \cdots\ )\, U_p \subset 2\, U_p.$$

As E is complete, the series $x = \sum x_n$ converges towards an element x such that $f(x) = y_0$, and f is surjective. □

The following important finiteness theorem due to L. Schwartz can be easily deduced from this.

(1.9) Theorem. *Let $(E^\bullet, d)$ and $(F^\bullet, \delta)$ be complexes of Fréchet spaces with continuous differentials, and $\rho^\bullet : E^\bullet \longrightarrow F^\bullet$ a continuous complex morphism. If ρ^q is compact and $H^q(\rho^\bullet) : H^q(E^\bullet) \longrightarrow H^q(F^\bullet)$ surjective, then $H^q(F^\bullet)$ is a Hausdorff finite dimensional space.*

Proof. Consider the operators

$$g, h \ : \ Z^q(E^\bullet) \oplus F^{q-1} \longrightarrow Z^q(F^\bullet),$$
$$g(x \oplus y) = \rho^q(x) + \delta^{q-1}(y), \qquad h(x \oplus y) = -\rho^q(x).$$

As $Z^q(E^\bullet) \subset E^q$, $Z^q(F^\bullet) \subset F^q$ are closed, all our spaces are Fréchet spaces. Moreover the hypotheses imply that h is compact and g is surjective since $H^q(\rho^\bullet)$ is surjective. Hence g is an epimorphism and $f = g + h = 0 \oplus \delta^{q-1}$ is a quasi-epimorphism by 1.8 b). Therefore $B^q(F^\bullet)$ is closed and finite codimensional in $Z^q(F^\bullet)$, thus $H^q(F^\bullet)$ is Hausdorff and finite dimensional. □

(1.10) Remark. If $\rho^\bullet : E^\bullet \longrightarrow F^\bullet$ is a continuous morphism of Fréchet complexes and if $H^q(\rho^\bullet)$ is surjective, then $H^q(\rho^\bullet)$ is in fact open, because the above map g is open. If $H^q(\rho^\bullet)$ is bijective, it follows that $H^q(\rho^\bullet)$ is necessarily a topological isomorphism (however $H^q(E^\bullet)$ and $H^q(F^\bullet)$ need not be Hausdorff). □

1.C. Abstract Mittag-Leffler Theorem

We will also need the following abstract Mittag-Leffler theorem, which is a very efficient tool in order to deal with cohomology groups of inverse limits.

(1.11) Proposition. *Let $(E^\bullet_\nu, \delta)_{\nu\in\mathbb{N}}$ be a sequence of Fréchet complexes together with morphisms $E^\bullet_{\nu+1} \longrightarrow E^\bullet_\nu$. We assume that the image of $E^\bullet_{\nu+1}$ in $E^\bullet_\nu$ is dense and we let $E^\bullet = \varprojlim E^\bullet_\nu$ be the inverse limit complex.*

a) *If all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$, $\nu \in \mathbb{N}$, are surjective, then the limit $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ is surjective.*
b) *If all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$, $\nu \in \mathbb{N}$, have a dense range, then $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ has a dense range.*
c) *If all maps $H^{q-1}(E^\bullet_{\nu+1}) \longrightarrow H^{q-1}(E^\bullet_\nu)$ have a dense range and all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$ are injective, $\nu \in \mathbb{N}$, then $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ is injective.*
d) *Let $\varphi^\bullet : F^\bullet \longrightarrow E^\bullet$ be a morphism of Fréchet complexes that has a dense range. If every map $H^q(F^\bullet) \longrightarrow H^q(E^\bullet_\nu)$ has a dense range, then $H^q(F^\bullet) \longrightarrow H^q(E^\bullet)$ has a dense range.*

Proof. If x is an element of $E^\bullet$ or of $E^\bullet_\mu$, $\mu \geq \nu$, we denote by x^ν its canonical image in $E^\bullet_\nu$. Let d_ν be a translation invariant distance that defines the topology of $E^\bullet_\nu$. After replacement of $d_\nu(x, y)$ by

$$d'_\nu(x,y) = \max_{\mu\le\nu}\big\{d_\mu(x^\mu,y^\mu)\big\},\qquad x,y\in E^\bullet_\nu,$$

we may assume that all maps $E^\bullet_{\nu+1}\longrightarrow E^\bullet_\nu$ are Lipschitz continuous with coefficient 1.

a) Let $x_0\in Z^q(E^\bullet_0)$ represent a given cohomology class $\overline{x}_0\in H^q(E^\bullet_0)$. We construct by induction a convergent sequence $x_\nu\in Z^q(E^\bullet_\nu)$ such that $\overline{x}_\nu$ is mapped onto $\overline{x}_0$. If x_ν is already chosen, we can find by assumption $x_{\nu+1}\in Z^q(E^\bullet_{\nu+1})$ such that $\overline{x}^\nu_{\nu+1}=\overline{x}_\nu$, i.e. $x^\nu_{\nu+1}=x_\nu+\delta y_\nu$ for some $y_\nu\in E^{q-1}_\nu$. If we replace $x_{\nu+1}$ by $x_{\nu+1}-\delta y_{\nu+1}$ where $y_{\nu+1}\in E^{q-1}_{\nu+1}$ yields an approximation $y^\nu_{\nu+1}$ of y_ν, we may assume that $\max\{d_\nu(y_\nu,0),d_\nu(\delta y_\nu,0)\}\le 2^{-\nu}$. Then (x_ν) converges to a limit $\xi\in Z^q(E^\bullet)$ and we have $\xi^0=x_0+\delta\sum y^0_\nu$.

b) The density assumption for cohomology groups implies that the map

$$Z^q(E^\bullet_{\nu+1})\times E^{q-1}_\nu\longrightarrow Z^q(E^\bullet_\nu),\qquad (x_{\nu+1},y_\nu)\longmapsto x^\nu_{\nu+1}+\delta y_\nu$$

has a dense range. If we approximate y_ν by elements coming from $E^{q-1}_{\nu+1}$, we see that the map $Z^q(E^\bullet_{\nu+1})\longrightarrow Z^q(E^\bullet_\nu)$ has also a dense range. If $x_0\in Z^q(E^\bullet_0)$, we can find inductively a sequence $x_\nu\in Z^q(E^\bullet_\nu)$ such that $d_\nu(x^\nu_{\nu+1},x_\nu)\le\varepsilon 2^{-\nu-1}$ for all ν, thus (x_ν) converges to an element $\xi\in Z^q(E^\bullet)$ such that $d_0(\xi^0,x_0)\le\varepsilon$ and $Z^q(E^\bullet)\longrightarrow Z^q(E^\bullet_0)$ has a dense range.

c) Let $x\in Z^q(E^\bullet)$ be such that $\overline{x}^0\in H^q(E^\bullet_0)$ is zero. By assumption, the image of $\overline{x}$ in $H^q(E^\bullet_\nu)$ must be also zero, so we can write $x^\nu=dy_\nu$, $y_\nu\in E^{q-1}_\nu$. We have $z_\nu=y^\nu_{\nu+1}-y_\nu\in Z^{q-1}(E^\bullet_\nu)$. Let $z_{\nu+1}\in Z^{q-1}(E^\bullet_{\nu+1})$ be such that $z^\nu_{\nu+1}$ approximates z_ν. If we replace $y_{\nu+1}$ by $y_{\nu+1}-z_{\nu+1}$, we still have $x^{\nu+1}=dy_{\nu+1}$ and we may assume in addition that $d_\nu(y^\nu_{\nu+1},y_\nu)\le 2^{-\nu}$. Then (y_ν) converges towards an element $y\in E^{q-1}$ such that $x=dy$, thus $\overline{x}=0$ and $H^q(E^\bullet)\longrightarrow H^q(E^\bullet_0)$ is injective.

d) For every class $\overline{y}\in H^q(E^\bullet)$, the hypothesis implies the existence of a sequence $x_\nu\in Z^q(F^\bullet)$ such that $\varphi^q(\overline{x}_\nu)^\nu$ converges to $\overline{y}^\nu$, that is, $d_\nu(y^\nu,\varphi^q(x_\nu)^\nu+\delta z_\nu)$ tends to 0 for some sequence $z_\nu\in E^{q-1}_\nu$. Approximate z_ν by $\varphi^{q-1}(w_\nu)^\nu$ for some $w_\nu\in F^{q-1}$ and replace x_ν by $x'_\nu=x_\nu+\delta w_\nu$. Then $\varphi^q(x'_\nu)$ converges to y in $Z^q(E^\bullet)$. □

2. q-Convex Spaces

2.A. q-Convex Functions

The concept of q-convexity, first introduced in (Rothstein 1955) and further developed by (Andreotti-Grauert 1962), generalizes the concepts of pseudoconvexity already considered in chapters 1 and 8. Let M be a complex manifold, $\dim_{\mathbb{C}} M=n$. A function $v\in C^2(M,\mathbb{R})$ is said to be strongly (resp. weakly) q-convex at a point $x\in M$ if $id'd''v(x)$ has at least $(n-q+1)$

strictly positive (resp. nonnegative) eigenvalues, or equivalently if there exists a $(n-q+1)$-dimensional subspace $F \subset T_xM$ on which the complex Hessian H_xv is positive definite (resp. semi-positive). Weak 1-convexity is thus equivalent to plurisubharmonicity. Some authors use different conventions for the number of positive eigenvalues in q-convexity. The reason why we introduce the number $n-q+1$ instead of q is mainly due to the following result:

(2.1) Proposition. *If $v \in C^2(M,\mathbb{R})$ is strongly (weakly) q-convex and if Y is a submanifold of M, then $v_{\restriction Y}$ is strongly (weakly) q-convex.*

Proof. Let $d = \dim Y$. For every $x \in Y$, there exists $F \subset T_xM$ with $\dim F = n-q+1$ such that Hv is (semi-) positive on F. Then $G = F \cap T_xY$ has dimension $\geq (n-q+1)-(n-d) = d-q+1$, and $H(v_{\restriction Y})$ is (semi-) positive on $G \subset T_xY$. Hence $v_{\restriction Y}$ is strongly (weakly) q-convex at x. □

(2.2) Proposition. *Let $v_j \in C^2(M,\mathbb{R})$ be a weakly (strongly) q_j-convex function, $1 \leq j \leq s$, and $\chi \in C^2(\mathbb{R}^s,\mathbb{R})$ a convex function that is increasing (strictly increasing) in all variables. Then $v = \chi(v_1,\ldots,v_s)$ is weakly (strongly) q-convex with $q-1 = \sum(q_j-1)$. In particular $v_1+\cdots+v_s$ is weakly (strongly) q-convex.*

Proof. A simple computation gives

$$(2.3)\quad Hv = \sum_j \frac{\partial\chi}{\partial t_j}(v_1,\ldots,v_s)\,Hv_j + \sum_{j,k}\frac{\partial^2\chi}{\partial t_j\partial t_k}(v_1,\ldots,v_s)\,d'v_j\otimes\overline{d'v_k},$$

and the second sum defines a semi-positive hermitian form. In every tangent space T_xM there exists a subspace F_j of codimension q_j-1 on which Hv_j is semi-positive (positive definite). Then $F = \bigcap F_j$ has codimension $\leq q-1$ and Hv is semi-positive (positive definite) on F. □

The above result cannot be improved, as shown by the trivial example

$$v_1(z) = -2|z_1|^2+|z_2|^2+|z_3|^2, \qquad v_2(z) = |z_1|^2-2|z_2|^2+|z_3|^2 \quad \text{on } \mathbb{C}^3,$$

in which case $q_1 = q_2 = 2$ but v_1+v_2 is only 3-convex. However, formula (2.3) implies the following result.

(2.4) Proposition. *Let $v_j \in C^2(M,\mathbb{R})$, $1 \leq j \leq s$, be such that every convex linear combination $\sum \alpha_j v_j$, $\alpha_j \geq 0$, $\sum \alpha_j = 1$, is weakly (strongly) q-convex. If $\chi \in C^2(\mathbb{R}^s,\mathbb{R})$ is a convex function that is increasing (strictly increasing) in all variables, then $\chi(v_1,\ldots,v_s)$ is weakly (strongly) q-convex.* □

The invariance property of Prop. 2.1 immediately suggests the definition of q-convexity on complex spaces or analytic schemes:

(2.5) Definition. *Let $(X, \mathcal{O}_X)$ be an analytic scheme. A function v on X is said to be strongly (resp. weakly) q-convex of class C^k on X if X can be covered by patches $G : U \xrightarrow{\simeq} A$, $A \subset \Omega \subset \mathbb{C}^N$ such that for each patch there exists a function $\widetilde{v}$ on Ω with $\widetilde{v}_{\restriction A} \circ G = v_{\restriction U}$, which is strongly (resp. weakly) q-convex of class C^k.*

The notion of q-convexity on a patch U does not depend on the way U is embedded in $\mathbb{C}^N$, as shown by the following lemma.

(2.6) Lemma. *Let $G : U \longrightarrow A \subset \Omega \subset \mathbb{C}^N$ and $G' : U' \longrightarrow A' \subset \Omega' \subset \mathbb{C}^{N'}$ be two patches of X. Let $\widetilde{v}$ be a strongly (weakly) q-convex function on Ω and $v = \widetilde{v}_{\restriction A} \circ G$. For every $x \in U \cap U'$ there exists a strongly (weakly) q-convex function $\widetilde{v}'$ on a neighborhood $W' \subset \Omega'$ of $G'(x)$ such that $\widetilde{v}'_{\restriction A' \cap W'} \circ G'$ coincides with v on $G'^{-1}(W')$.*

Proof. The isomorphisms

$$G' \circ G^{-1} : A \supset G(U \cap U') \longrightarrow G'(U \cap U') \subset A'$$
$$G \circ G'^{-1} : A' \supset G'(U \cap U') \longrightarrow G(U \cap U') \subset A$$

are restrictions of holomorphic maps $H : W \longrightarrow \Omega'$, $H' : W' \longrightarrow \Omega$ defined on neighborhoods $W \ni G(x)$, $W' \ni G'(x)$; we can shrink W' so that $H'(W') \subset W$. If we compose the automorphism $(z, z') \longmapsto (z, z' - H(z))$ of $W \times \mathbb{C}^{N'}$ with the function $v(z) + |z'|^2$ we see that the function $\varphi(z, z') = \widetilde{v}(z) + |z' - H(z)|^2$ is strongly (weakly) q-convex on $W \times \Omega'$. Now, W' can be embedded in $W \times \Omega'$ via the map $z' \longmapsto \big(H'(z'), z'\big)$, so that the composite function

$$\widetilde{v}'(z') = \varphi\big(H'(z'), z'\big) = \widetilde{v}\big(H'(z')\big) + |z' - H \circ H'(z')|^2$$

is strongly (weakly) q-convex on W' by Prop. 2.1. Since $H \circ G = G'$ and $H' \circ G' = G$ on $G'^{-1}(W')$, we have $\widetilde{v}' \circ G' = \widetilde{v} \circ G = v$ on $G'^{-1}(W')$ and the lemma follows. □

A consequence of this lemma is that Prop. 2.2 is still valid for an analytic scheme X (all the extensions $\widetilde{v}_j$ near a given point $x \in X$ can be obtained with respect to the same local embedding).

(2.7) Definition. *An analytic scheme $(X, \mathcal{O}_X)$ is said to be strongly (resp. weakly) q-convex if X has a C^∞ exhaustion function ψ which is strongly (resp. weakly) q-convex outside an exceptional compact set $K \subset X$. We say that X is strongly q-complete if ψ can be chosen so that $K = \emptyset$. By convention, a compact scheme X is said to be strongly 0-complete, with exceptional compact set $K = X$.*

We consider the sublevel sets

(2.8) $X_c = \{x \in X \,;\, \psi(x) < c\}, \qquad c \in \mathbb{R}.$

If $K \subset X_c$, we may select a convex increasing function χ such that $\chi = 0$ on $]-\infty, c]$ and $\chi' > 0$ on $]c, +\infty[$. Then $\chi \circ \psi = 0$ on X_c, so that $\chi \circ \psi$ is weakly q-convex everywhere in virtue of (2.3). In the weakly q-convex case, we may therefore always assume $K = \emptyset$. The following properties are almost immediate consequences of the definition:

(2.9) Theorem.

a) *A scheme X is strongly (weakly) q-convex if and only if the reduced space X_{red} is strongly (weakly) q-convex.*

b) *If X is strongly (weakly) q-convex, every closed analytic subset Y of X_{red} is strongly (weakly) q-convex.*

c) *If X is strongly (weakly) q-convex, every sublevel set X_c containing the exceptional compact set K is strongly (weakly) q-convex.*

d) *If U_j is a weakly q_j-convex open subset of X, $1 \le j \le s$, the intersection $U = U_1 \cap \ldots \cap U_s$ is weakly q-convex with $q - 1 = \sum (q_j - 1)$; U is strongly q-convex (resp. q-complete) as soon as one of the sets U_j is strongly q_j-convex (resp. q_j-complete).*

Proof. a) is clear, since Def. 2.5 does not involve the structure sheaf $\mathcal{O}_X$. In cases b) and c), let ψ be an exhaustion of the required type on X. Then $\psi_{\restriction Y}$ and $1/(c - \psi)$ are exhaustions on Y and X_c respectively (this is so only if Y is closed). Moreover, these functions are strongly (weakly) q-convex on $Y \smallsetminus (K \cap Y)$ and $X_c \smallsetminus K$, thanks to Prop. 2.1 and 2.2. For property d), note that a sum $\psi = \psi_1 + \cdots + \psi_s$ of exhaustion functions on the sets U_j is an exhaustion on U, choose the ψ_j's weakly q_j-convex everywhere, and apply Prop. 2.2. □

(2.10) Corollary. *Any finite intersection $U = U_1 \cap \ldots \cap U_s$ of weakly 1-convex open subsets is weakly 1-convex. The set U is strongly 1-convex (resp. 1-complete) as soon as one of the sets U_j is strongly 1-convex (resp. 1-complete).*

2.B. Neighborhoods of q-complete subspaces

We prove now a rather useful result asserting the existence of q-complete neighborhoods for q-complete subvarieties. The case $q = 1$ goes back to (Siu 1976), who used a much more complicated method. The first step is an approximation-extension theorem for strongly q-convex functions.

(2.11) Proposition. *Let Y be an analytic set in a complex space X and ψ a strongly q-convex C^∞ function on Y. For every continuous function $\delta > 0$ on Y, there exists a strongly q-convex C^∞ function φ on a neighborhood V of Y such that $\psi \le \varphi_{\restriction Y} < \psi + \delta$.*

Proof. Let Z_k be a stratication of Y as given by Prop. II.5.6, i.e. Z_k is an increasing sequence of analytic subsets of Y such that $Y = \bigcup Z_k$ and $Z_k \smallsetminus Z_{k-1}$ is a smooth k-dimensional manifold (possibly empty for some k's). We shall prove by induction on k the following statement:

There exists a C^∞ function φ_k on X which is strongly q-convex along Y and on a closed neighborhood $\overline{V}_k$ of Z_k in X, such that $\psi \le \varphi_{k\restriction Y} < \psi + \delta$.

We first observe that any smooth extension φ_{-1} of ψ to X satisfies the requirements with $Z_{-1} = V_{-1} = \emptyset$. Assume that V_{k-1} and φ_{k-1} have been constructed. Then $Z_k \smallsetminus V_{k-1} \subset Z_k \smallsetminus Z_{k-1}$ is contained in $Z_{k,\mathrm{reg}}$. The closed set $Z_k \smallsetminus V_{k-1}$ has a locally finite covering (A_λ) in X by open coordinate patches $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda}$ in which Z_k is given by equations $z'_\lambda = (z_{\lambda,k+1}, \ldots, z_{\lambda,N_\lambda}) = 0$. Let θ_λ be C^∞ functions with compact support in A_λ such that $0 \le \theta_\lambda \le 1$ and $\sum \theta_\lambda = 1$ on $Z_k \smallsetminus V_{k-1}$. We set

$$\varphi_k(x) = \varphi_{k-1}(x) + \sum \theta_\lambda(x)\, \varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4} |z'_\lambda|^2) \quad \text{on} \quad X.$$

For $\varepsilon_\lambda > 0$ small enough, we will have $\psi \le \varphi_{k-1\restriction Y} \le \varphi_{k\restriction Y} < \psi + \delta$. Now, we check that φ_k is still strongly q-convex along Y and near any $x_0 \in \overline{V}_{k-1}$, and that φ_k becomes strongly q-convex near any $x_0 \in Z_k \smallsetminus V_{k-1}$. We may assume that $x_0 \in \operatorname{Supp} \theta_\mu$ for some μ, otherwise φ_k coincides with φ_{k-1} in a neighborhood of x_0. Select μ and a small neighborhood $W \subset\subset \Omega_\mu$ of x_0 such that

a) if $x_0 \in Z_k \smallsetminus V_{k-1}$, then $\theta_\mu(x_0) > 0$ and $A_\mu \cap W \subset\subset \{\theta_\mu > 0\}$;

b) if $x_0 \in A_\lambda$ for some λ (there is only a finite set I of such λ's), then $A_\mu \cap W \subset\subset A_\lambda$ and $z_{\lambda \restriction A_\mu \cap W}$ has a holomorphic extension $\widetilde{z}_\lambda$ to $\overline{W}$;

c) if $x_0 \in \overline{V}_{k-1}$, then $\varphi_{k-1\restriction A_\mu \cap W}$ has a strongly q-convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$;

d) if $x_0 \in Y \smallsetminus \overline{V}_{k-1}$, then $\varphi_{k-1 \restriction Y \cap W}$ has a strongly q-convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$.

Otherwise take an arbitrary smooth extension $\widetilde{\varphi}_{k-1}$ of $\varphi_{k-1\restriction A_\mu \cap W}$ to $\overline{W}$ and let $\widetilde{\theta}_\lambda$ be an extension of $\theta_{\lambda \restriction A_\mu \cap W}$ to $\overline{W}$. Then

$$\widetilde{\varphi}_k = \widetilde{\varphi}_{k-1} + \sum \widetilde{\theta}_\lambda\, \varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4} |\widetilde{z}'_\lambda|^2)$$

is an extension of $\varphi_{k\restriction A_\mu \cap W}$ to $\overline{W}$, resp. of $\varphi_{k \restriction Y \cap W}$ to $\overline{W}$ in case d). As the function $\log(1 + \varepsilon_\lambda^{-4}|\widetilde{z}'_\lambda|^2)$ is plurisubharmonic and as its first derivative $\langle \widetilde{z}'_\lambda, d\widetilde{z}'_\lambda \rangle\, (\varepsilon_\lambda^4 + |\widetilde{z}'_\lambda|^2)^{-1}$ is bounded by $O(\varepsilon_\lambda^{-2})$, we see that

$$id'd''\widetilde{\varphi}_k \ge id'd''\widetilde{\varphi}_{k-1} - O(\textstyle\sum \varepsilon_\lambda).$$

Therefore, for ε_λ small enough, $\widetilde{\varphi}_k$ remains q-convex on $\overline{W}$ in cases c) and d). Since all functions $\widetilde{z}'_\lambda$ vanish along $Z_k \cap W$, we have

$$id'd''\widetilde{\varphi}_k \ge id'd''\widetilde{\varphi}_{k-1} + \sum_{\lambda \in I} \theta_\lambda\, \varepsilon_\lambda^{-1}\, id'd''|\widetilde{z}'_\lambda|^2 \ge id'd''\widetilde{\varphi}_{k-1} + \theta_\mu\, \varepsilon_\mu^{-1}\, id'd''|z'_\mu|^2$$

at every point of $Z_k \cap W$. Moreover $id'd''\widetilde{\varphi}_{k-1}$ has at most $(q-1)$-negative eigenvalues on TZ_k since $Z_k \subset Y$, whereas $id'd''|z'_\mu|^2$ is positive definite in the normal directions to Z_k in Ω_μ. In case a), we thus find that $\widetilde{\varphi}_k$ is strongly q-convex on $\overline{W}$ for ε_μ small enough; we also observe that only finitely many conditions are required on each ε_λ if we choose a locally finite covering of $\bigcup \operatorname{Supp}\theta_\lambda$ by neighborhoods W as above. Therefore, for ε_λ small enough, φ_k is strongly q-convex on a neighborhood $\overline{V}'_k$ of $Z_k \smallsetminus V_{k-1}$. The function φ_k and the set $V_k = V'_k \cup V_{k-1}$ satisfy the requirements at order k. It is clear that we can choose the sequence φ_k stationary on every compact subset of X; the limit φ and the open set $V = \bigcup V_k$ fulfill the proposition. □

The second step is the existence of almost plurisubharmonic functions having poles along a prescribed analytic set. By an almost plurisubharmonic function on a manifold, we mean a function that is locally equal to the sum of a plurisubharmonic function and of a smooth function, or equivalently, a function whose complex Hessian has bounded negative part. On a complex space, we require that our function can be locally extended as an almost plurisubharmonic function in the ambient space of an embedding.

(2.12) Lemma. *Let Y be an analytic subvariety in a complex space X. There is an almost plurisubharmonic function v on X such that $v = -\infty$ on Y with logarithmic poles and $v \in C^\infty(X \smallsetminus Y)$.*

Proof. Since $\mathcal{I}_Y \subset \mathcal{O}_X$ is a coherent subsheaf, there is a locally finite covering of X by patches A_λ isomorphic to analytic sets in balls $B(0, r_\lambda) \subset \mathbb{C}^{N_\lambda}$, such that $\mathcal{I}_Y$ admits a system of generators $g_\lambda = (g_{\lambda,j})$ on a neighborhood of each set $\overline{A}_\lambda$. We set

$$v_\lambda(z) = \log|g_\lambda(z)|^2 - \frac{1}{r_\lambda^2 - |z - z_\lambda|^2} \quad \text{on} \quad A_\lambda,$$
$$v(z) = M_{(1,\ldots,1)}\big(\ldots, v_\lambda(z), \ldots\big) \quad \text{for} \;\; \lambda \;\; \text{such that} \;\; A_\lambda \ni z,$$

where M_η is the regularized max function defined in I-3.37. As the generators $(g_{\lambda,j})$ can be expressed in terms of one another on a neighborhood of $\overline{A}_\lambda \cap \overline{A}_\mu$, we see that the quotient $|g_\lambda|/|g_\mu|$ remains bounded on this set. Therefore none of the values $v_\lambda(z)$ for $A_\lambda \ni z$ and z near ∂A_λ contributes to the value of v, since $1/(r_\lambda^2 - |z - z_\lambda|^2)$ tends to $+\infty$ on ∂A_λ. It follows that v is smooth on $X \smallsetminus Y$; as each v_λ is almost plurisubharmonic on A_λ, we also see that v is almost plurisubharmonic on X. □

(2.13) Theorem. *Let X be a complex space and Y a strongly q-complete analytic subset. Then Y has a fundamental family of strongly q-complete neighborhoods V in X.*

Proof. By Prop. 2.11 applied to a strongly q-convex exhaustion of Y and $\delta = 1$, there exists a strongly q-convex function φ on a neighborhood W_0 of Y such that $\varphi_{\restriction Y}$ is an exhaustion. Let W_1 be a neighborhood of Y such that $\overline{W}_1 \subset W_0$ and such that $\varphi_{\restriction \overline{W}_1}$ is an exhaustion. We are going to show that every neighborhood $W \subset W_1$ of Y contains a strongly q-complete neighborhood V. If v is the function given by Lemma 2.12, we set

$$\widetilde{v} = v + \chi \circ \varphi \quad \text{on } \overline{W}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function. If χ grows fast enough, we get $\widetilde{v} > 0$ on ∂W and the $(q-1)$-codimensional subspace on which $id'd''\varphi$ is positive definite (in some ambient space) is also positive definite for $id'd''\widetilde{v}$ provided that χ' be large enough to compensate the bounded negative part of $id'd''v$. Then $\widetilde{v}$ is strongly q-convex. Let θ be a smooth convex increasing function on $]-\infty, 0[$ such that $\theta(t) = 0$ for $t < -3$ and $\theta(t) = -1/t$ on $]-1, 0[$. The open set $V = \{z \in W\,;\, \widetilde{v}(z) < 0\}$ is a neighborhood of Y and $\widetilde{\psi} = \varphi + \theta \circ \widetilde{v}$ is a strongly q-convex exhaustion of V. □

2.C. Runge Open Subsets

In order to extend the classical Runge theorem into an approximation result for sheaf cohomology groups, we need the concept of a q-Runge open subset.

(2.14) Definition. *An open subset U of a complex space X is said to be q-Runge (resp. q-Runge complete) in X if for every compact subset $L \subset U$ there exists a smooth exhaustion function ψ on X and a sublevel set X_b of ψ such that $L \subset X_b \subset\subset U$ and ψ is strongly q-convex on $X \smallsetminus \overline{X}_b$ (resp. on the whole space X).*

(2.15) Example. If X is strongly q-complete and if ψ is a strongly q-convex exhaustion function of X, then every sublevel set X_c of ψ is q-Runge complete in X : every compact set $L \subset X_c$ satisfies $L \subset X_b \subset\subset X_c$ for some $b < c$. More generally, if X is strongly q-convex and if ψ is strongly q-convex on $X \smallsetminus K$, every sublevel set X_c containing K is q-Runge in X.

Later on, we shall need the following technical result.

(2.16) Proposition. *Let Y be an analytic subset of a complex space X. If U is a q-Runge complete open subset of Y and L a compact subset, there exist a neighborhood V of Y in X and a strongly q-convex exhaustion $\widetilde{\psi}$ on V such that $U = Y \cap V$ and $L \subset Y \cap V_b \subset\subset U$ for some sublevel set V_b of $\widetilde{\psi}$.*

Proof. Let ψ be a strongly q-convex exhaustion on Y with $L \subset \{\psi < b\} \subset\subset U$ as in Def. 2.14. Then $L \subset \{\psi < b-\delta\}$ for some number $\delta > 0$ and Lemma 2.11 gives a strongly q-convex function φ on a neighborhood W_0 of Y so that $\psi \le \varphi_{\restriction Y} < \psi + \delta$. The neighborhood V and the function $\widetilde{\psi} = \varphi + \theta \circ \widetilde{v}$ constructed in the proof of Th. 2.13 are the desired ones: we have $\psi \le \widetilde{\psi}_{\restriction Y} = \varphi_{\restriction Y} < \psi + \delta$, thus

$$L \subset Y \cap V_{b-\delta} \subset \{\psi < b\} \subset\subset U. \qquad \square$$

3. q-Convexity Properties in Top Degrees

It is obvious by definition that a n-dimensional complex manifold M is strongly q-complete for $q \ge n+1$ (an arbitrary smooth function is then strongly q-convex !). If M is connected and non compact, (Greene and Wu 1975) have shown that M is strongly n-complete, i.e. there is a smooth exhaustion function ψ on M such that $id'd''\psi$ has at least one positive eigenvalue everywhere. We need the following lemmas.

(3.1) Lemma. *Let ψ be a strongly q-convex function on M and $\varepsilon > 0$ a given number. There exists a hermitian metric ω on M such that the eigenvalues $\gamma_1 \le \ldots \le \gamma_n$ of the Hessian form $id'd''\psi$ with respect to ω satisfy $\gamma_1 \ge -\varepsilon$ and $\gamma_q = \ldots = \gamma_n = 1$.*

Proof. Let ω_0 be a fixed hermitian metric, $A_0 \in C^\infty(\mathrm{End}\, TM)$ the hermitian endomorphism associated to the hermitian form $id'd''\psi$ with respect to ω_0, and $\gamma_1^0 \le \ldots \le \gamma_n^0$ the eigenvalues of A_0 (or $id'd''\psi$). We can choose a function $\eta \in C^\infty(M, \mathbb{R})$ such that $0 < \eta(x) \le \gamma_q^0(x)$ at each point $x \in M$. Select a positive function $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\theta(t) \ge |t|/\varepsilon \quad \text{for} \quad t \le 0, \qquad \theta(t) \ge t \quad \text{for} \quad t \ge 0, \qquad \theta(t) = t \quad \text{for} \quad t \ge 1.$$

We let ω be the hermitian metric defined by the hermitian endomorphism

$$A(x) = \eta(x)\, \theta[(\eta(x))^{-1} A_0(x)]$$

where $\theta[\eta^{-1} A_0] \in C^\infty(\mathrm{End}\, TM)$ is defined as in Lemma VII-6.2. By construction, the eigenvalues of $A(x)$ are $\alpha_j(x) = \eta(x)\theta\big(\gamma_j^0(x)/\eta(x)\big) > 0$ and we have

$$\begin{aligned}
&\alpha_j(x) \ge |\gamma_j^0(x)|/\varepsilon && \text{for} \quad \gamma_j^0(x) \le 0,\\
&\alpha_j(x) \ge \gamma_j^0(x) && \text{for} \quad \gamma_j^0(x) \ge 0,\\
&\alpha_j(x) = \gamma_j^0(x) && \text{for} \quad j \ge q \quad \big(\text{then } \gamma_j^0(x) \ge \eta(x)\big).
\end{aligned}$$

The eigenvalues of $id'd''\psi$ with respect to ω are $\gamma_j(x) = \gamma_j^0(x)/\alpha_j(x)$ and they have the required properties. $\square$

On a hermitian manifold (M, ω), we consider the Laplace operator Δ_ω defined by

$$(3.2)\quad \Delta_\omega v = \mathrm{Trace}_\omega(id'd''v) = \sum_{1\le j,k\le n} \omega^{jk}(z)\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}$$

where (ω^{jk}) is the conjugate of the inverse matrix of (ω_{jk}). Note that Δ_ω may differ from the usual Laplace-Beltrami operator if ω is not Kähler. We say that v is strongly ω-subharmonic if $\Delta_\omega v > 0$. This property implies clearly that v is strongly n-convex; however, as

$$\Delta_\omega \chi(v_1,\ldots,v_s) = \sum_j \frac{\partial \chi}{\partial t_j}(v_1,\ldots,v_s)\,\Delta_\omega v_j + \sum_{j,k}\frac{\partial^2\chi}{\partial t_j \partial t_k}(v_1,\ldots,v_s)\,\langle d'v_j, d'v_k\rangle_\omega,$$

subharmonicity has the advantage of being preserved by all convex increasing transformations. Conversely, if ψ is strongly n-convex and ω chosen as in Lemma 3.1 with ε small enough, we get $\Delta_\omega\psi \ge 1-(n-1)\varepsilon > 0$, thus ψ is strongly subharmonic for a suitable metric ω.

(3.3) Lemma. *Let $U, W \subset M$ be open sets such that for every connected component U_s of U there is a connected component $W_{t(s)}$ of W such that $W_{t(s)} \cap U_s \ne \emptyset$ and $W_{t(s)} \setminus \overline{U}_s \ne \emptyset$. Then there exists a function $v \in C^\infty(M,\mathbb{R})$, $v \ge 0$, with support contained in $\overline{U}\cup\overline{W}$, such that v is strongly ω-subharmonic and > 0 on U.*

Proof. We first prove that the result is true when U, W are small cylinders with the same radius and axis. Let $a_0 \in M$ be a given point and $z_1,\ldots,z_n$ holomorphic coordinates centered at a_0. We set $\mathrm{Re}\,z_j = x_{2j-1}$, $\mathrm{Im}\,z_j = x_{2j}$, $x' = (x_2,\ldots,x_{2n})$ and $\omega = \sum \widetilde{\omega}_{jk}(x)dx_j\otimes dx_k$. Let U be the cylinder $|x_1|<r$, $|x'|<r$, and W the cylinder $r-\varepsilon < x_1 < r+\varepsilon$, $|x'|<r$. There are constants $c, C > 0$ such that

$$\sum \widetilde{\omega}^{jk}(x)\xi_j\xi_k \ge c|\xi|^2 \quad\text{and}\quad \sum|\widetilde{\omega}^{jk}(x)| \le C \quad \text{on } \overline{U}.$$

Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be a nonnegative function equal to 0 on $]-\infty,-r]\cup[r+\varepsilon,+\infty[$ and strictly convex on $]-r,r]$. We take explicitly $\chi(x_1) = (x_1+r)\exp(-1/(x_1+r)^2)$ on $]-r,r]$ and

$$v(x) = \chi(x_1)\exp\big(1/(|x'|^2-r^2)\big) \quad \text{on } U\cup W, \quad v=0 \quad \text{on } M\setminus(U\cup W).$$

We have $v\in C^\infty(M,\mathbb{R})$, $v>0$ on U, and a simple computation gives

$$\begin{aligned}\frac{\Delta_\omega v(x)}{v(x)} &= \widetilde{\omega}^{11}(x)\big(4(x_1+r)^{-5}-2(x_1+r)^{-3}\big)\\ &+\sum_{j>1}\widetilde{\omega}^{1j}(x)\big(1+2(x_1+r)^{-2}\big)(-2x_j)(r^2-|x'|^2)^{-2}\\ &+\sum_{j,k>1}\widetilde{\omega}^{jk}(x)\Big(x_jx_k\big(4-8(r^2-|x'|^2)\big)-2(r^2-|x'|^2)^2\delta_{jk}\Big)(r^2-|x'|^2)^{-4}.\end{aligned}$$

For r small, we get

$$\frac{\Delta_\omega v(x)}{v(x)} \geq 2c(x_1+r)^{-5} - C_1(x_1+r)^{-2}|x'|(r^2-|x'|^2)^{-2} + (2c|x'|^2 - C_2 r^4)(r^2-|x'|^2)^{-4}$$

with constants C_1, C_2 independent of r. The negative term is bounded by $C_3(x_1+r)^{-4} + c|x'|^2(r^2-|x'|^2)^{-4}$, hence

$$\Delta_\omega v/v(x) \geq c(x_1+r)^{-5} + (c|x'|^2 - C_2 r^4)(r^2-|x'|^2)^{-4}.$$

The last term is negative only when $|x'| < C_4 r^2$, in which case it is bounded by $C_5 r^{-4} < c(x_1+r)^{-5}$. Hence v is strongly ω-subharmonic on U.

Next, assume that U and W are connected. Then $U \cup W$ is connected. Fix a point $a \in W \setminus \overline{U}$. If $z_0 \in U$ is given, we choose a path $\Gamma \subset U \cup W$ from z_0 to a which is piecewise linear with respect to holomorphic coordinate patches. Then we can find a finite sequence of cylinders (U_j, W_j) of the type described above, $1 \leq j \leq N$, whose axes are segments contained in Γ, such that

$$U_j \cup W_j \subset U \cup W, \quad \overline{W}_j \subset U_{j+1} \quad \text{and} \quad z_0 \in U_0, \quad a \in W_N \subset W \setminus \overline{U}.$$

For each such pair, we have a function $v_j \in C^\infty(M)$ with support in $\overline{U}_j \cup \overline{W}_j$, $v_j \geq 0$, strongly ω-subharmonic and > 0 on U_j. By induction, we can find constants $C_j > 0$ such that $v_0 + C_1 v_1 + \cdots + C_j v_j$ is strongly ω-subharmonic on $U_0 \cup \ldots \cup U_j$ and ω-subharmonic on $M \setminus \overline{W}_j$. Then

$$w_{z_0} = v_0 + C_1 v_1 + \ldots + C_N v_N \geq 0$$

is ω-subharmonic on U and strongly ω-subharmonic > 0 on a neighborhood Ω_0 of the given point z_0. Select a denumerable covering of U by such neighborhoods Ω_p and set $v(z) = \sum \varepsilon_p w_{z_p}(z)$ where ε_p is a sequence converging sufficiently fast to 0 so that $v \in C^\infty(M, \mathbb{R})$. Then v has the required properties.

In the general case, we find for each pair $(U_s, W_{t(s)})$ a function v_s with support in $\overline{U}_s \cup \overline{W}_{t(s)}$, strongly ω-subharmonic and > 0 on U_s. Any convergent series $v = \sum \varepsilon_s v_s$ yields a function with the desired properties. □

(3.4) Lemma. *Let X be a connected, locally connected and locally compact topological space. If U is a relatively compact open subset of X, we let $\widetilde{U}$ be the union of U with all compact connected components of $X \setminus U$. Then $\widetilde{U}$ is open and relatively compact in X, and $X \setminus \widetilde{U}$ has only finitely many connected components, all non compact.*

Proof. A rather easy exercise of general topology. Intuitively, $\widetilde{U}$ is obtained by "filling the holes" of U in X. □

(3.5) Theorem (Greene-Wu 1975). *Every n-dimensional connected non compact complex manifold M has a strongly subharmonic exhaustion function with respect to any hermitian metric ω. In particular, M is strongly n-complete.*

Proof. Let $\varphi \in C^\infty(M, \mathbb{R})$ be an arbitrary exhaustion function. There exists a sequence of connected smoothly bounded open sets $\Omega'_\nu \subset\subset M$ such that $\overline{\Omega'_\nu} \subset \Omega'_{\nu+1}$ and $M = \bigcup \Omega'_\nu$. Let $\Omega_\nu = \widetilde{\Omega}'_\nu$ be the relatively compact open set given by Lemma 3.4. Then $\overline{\Omega}_\nu \subset \Omega_{\nu+1}$, $M = \bigcup \Omega_\nu$ and $M \smallsetminus \Omega_\nu$ has no compact connected component. We set

$$U_1 = \Omega_2, \qquad U_\nu = \Omega_{\nu+1} \smallsetminus \overline{\Omega_{\nu-2}} \quad \text{for} \quad \nu \geq 2.$$

Then $\partial U_\nu = \partial\Omega_{\nu+1} \cup \partial\Omega_{\nu-2}$; any connected component $U_{\nu,s}$ of U_ν has its boundary $\partial U_{\nu,s} \not\subset \partial\Omega_{\nu-2}$, otherwise $\overline{U}_{\nu,s}$ would be open and closed in $M \smallsetminus \Omega_{\nu-2}$, hence $\overline{U}_{\nu,s}$ would be a compact component of $M \smallsetminus \Omega_{\nu-2}$. Therefore $\partial U_{\nu,s}$ intersects $\partial\Omega_{\nu+1} \subset U_{\nu+1}$. If $\partial U_{\nu+1,t(s)}$ is a connected component of $U_{\nu+1}$ containing a point of $\partial U_{\nu,s}$, then $U_{\nu+1,t(s)} \cap U_{\nu,s} \neq \emptyset$ and $U_{\nu+1,t(s)} \smallsetminus \overline{U}_{\nu,s} \neq \emptyset$. Lemma 7 implies that there is a nonnegative function $v_\nu \in C^\infty(M, \mathbb{R})$ with support in $U_\nu \cup U_{\nu+1}$, which is strongly ω-subharmonic on U_ν. An induction yields constants C_ν such that

$$\psi_\nu = \varphi + C_1 v_1 + \cdots + C_\nu v_\nu$$

is strongly ω-subharmonic on $\overline{\Omega_\nu} \subset U_0 \cup \ldots \cup U_\nu$, thus $\psi = \varphi + \sum C_\nu v_\nu$ is a strongly ω-subharmonic exhaustion function on M. $\square$

By an induction on the dimension, the above result can be generalized to an arbitrary complex space (or analytic scheme), as was first shown by T. Ohsawa.

(3.6) Theorem (Ohsawa 1984). *Let X be a complex space of maximal dimension n.*

a) *X is always strongly $(n+1)$-complete.*

b) *If X has no compact irreducible component of dimension n, then X is strongly n-complete.*

c) *If X has only finitely many irreducible components of dimension n, then X is strongly n-convex.*

Proof. We prove a) and b) by induction on $n = \dim X$. For $n = 0$, property b) is void and a) is obvious (any function can then be considered as strongly 1-convex). Assume that a) has been proved in dimension $\leq n-1$. Let X' be the union of X_{sing} and of the irreducible components of X of dimension at most $n-1$, and $M = X \smallsetminus X'$ the n-dimensional part of X_{reg}. As $\dim X' \leq n-1$, the induction hypothesis shows that X' is strongly n-complete. By Th. 2.13, there exists a strongly n-convex exhaustion function φ' on a neighborhood V' of X'. Take a closed neighborhood $\overline{V} \subset V'$ and an arbitrary exhaustion φ on X that extends $\varphi'_{\restriction \overline{V}}$. Since every function on a n-dimensional manifold is strongly $(n+1)$-convex, we conclude that X is at worst $(n+1)$-complete, as stated in a).

In case b), the hypothesis means that the connected components M_j of $M = X \smallsetminus X'$ have non compact closure $\overline{M}_j$ in X. On the other hand, Lemma 3.1 shows that there exists a hermitian metric ω on M such that $\varphi_{\restriction M \cap V}$ is strongly ω-subharmonic. Consider the open sets $U_{j,\nu} \subset M_j$ provided by Lemma 3.7 below. By the arguments already used in Th. 3.5, we can find a strongly ω-subharmonic exhaustion $\psi = \varphi + \sum_{j,\nu} C_{j,\nu} v_{j,\nu}$ on X, with $v_{j,\nu}$ strongly ω-subharmonic on $U_{j,\nu}$, $\operatorname{Supp} v_{j,\nu} \subset U_{j,\nu} \cup U_{j,\nu+1}$ and $C_{j,\nu}$ large. Then ψ is strongly n-convex on X.

(3.7) Lemma. *For each j, there exists a sequence of open sets $U_{j,\nu} \subset\subset M_j$, $\nu \in \mathbb{N}$, such that*

a) *$M_j \smallsetminus V' \subset \bigcup_\nu U_{j,\nu}$ and $(U_{j,\nu})$ is locally finite in $\overline{M}_j$;*

b) *for every connected component $U_{j,\nu,s}$ of $U_{j,\nu}$ there is a connected component $U_{j,\nu+1,t(s)}$ of $U_{j,\nu+1}$ such that $U_{j,\nu+1,t(s)} \cap U_{j,\nu,s} \neq \emptyset$ and $U_{j,\nu+1,t(s)} \smallsetminus \overline{U}_{j,\nu,s} \neq \emptyset$.*

Proof. By Lemma 3.4 applied to the space $\overline{M}_j$, there exists a sequence of relatively compact connected open sets $\Omega_{j,\nu}$ in $\overline{M}_j$ such that $\overline{M}_j \smallsetminus \Omega_{j,\nu}$ has no compact connected component, $\overline{\Omega}_{j,\nu} \subset \Omega_{j,\nu+1}$ and $\overline{M}_j = \bigcup \Omega_{j,\nu}$. We define a compact set $K_{j,\nu} \subset M_j$ and an open set $W_{j,\nu} \subset \overline{M}_j$ containing $K_{j,\nu}$ by

$$K_{j,\nu} = (\overline{\Omega}_{j,\nu} \smallsetminus \Omega_{j,\nu-1}) \smallsetminus V', \qquad W_{j,\nu} = \Omega_{j,\nu+1} \smallsetminus \overline{\Omega}_{j,\nu-2}.$$

By induction on ν, we construct an open set $U_{j,\nu} \subset\subset W_{j,\nu} \smallsetminus X' \subset M_j$ and a finite set $F_{j,\nu} \subset \partial U_{j,\nu} \smallsetminus \overline{\Omega}_{j,\nu}$. We let $F_{j,-1} = \emptyset$. If these sets are already constructed for $\nu-1$, the compact set $K_{j,\nu} \cup F_{j,\nu-1}$ is contained in the open set $W_{j,\nu}$, thus contained in a finite union of connected components $W_{j,\nu,s}$. We can write $K_{j,\nu} \cup F_{j,\nu-1} = \bigcup L_{j,\nu,s}$ where $L_{j,\nu,s}$ is contained in $W_{j,\nu,s} \smallsetminus X' \subset M_j$. The open set $W_{j,\nu,s} \smallsetminus X'$ is connected and non contained in $\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s}$, otherwise its closure $\overline{W}_{j,\nu,s}$ would have no boundary point $\in \partial \Omega_{j,\nu+1}$, thus would be open and compact in $\overline{M}_j \smallsetminus \Omega_{j,\nu-2}$, contradiction. We select a point $a_s \in (W_{j,\nu,s} \smallsetminus X') \smallsetminus (\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s})$ and a smoothly bounded connected open set $U_{j,\nu,s} \subset\subset W_{j,\nu,s} \smallsetminus X'$ containing $L_{j,\nu,s}$ with $a_s \in \partial U_{j,\nu,s}$. Finally, we set $U_{j,\nu} = \bigcup_s U_{j,\nu,s}$ and let $F_{j,\nu}$ be the set of all points a_s. By construction, we have $U_{j,\nu} \supset K_{j,\nu} \cup F_{j,\nu-1}$, thus $\bigcup U_{j,\nu} \supset \bigcup K_{j,\nu} = M_j \smallsetminus V'$, and $\partial U_{j,\nu,s} \ni a_s$ with $a_s \in F_{j,\nu} \subset U_{j,\nu+1}$. Property b) follows. □

Proof of Theorem 3.6 c) (end). Let $Y \subset X$ be the union of X_{sing} with all irreducible components of X that are non compact or of dimension $< n$.

Then $\dim Y \le n-1$, so Y is n-convex and Th. 2.13 implies that there is an exhaustion function $\psi \in C^\infty(X, \mathbb{R})$ such that ψ is strongly n-convex on a neighborhood V of Y. Then the complement $K = X \smallsetminus V$ is compact and ψ is strongly n-convex on $X \smallsetminus K$. □

(3.8) Proposition. *Let M be a connected non compact n-dimensional complex manifold and U an open subset of M. Then U is n-Runge complete in M if and only if $M \smallsetminus U$ has no compact connected component.* □

Proof. First observe that a strongly n-convex function cannot have any local maximum, so it satisfies the maximum principle. If $M \smallsetminus U$ has a compact connected component T, then T has a compact neighborhood L in M such that $\partial L \subset U$. We have $\max_L \psi = \max_{\partial L} \psi$ for every strongly n-convex function, thus $\partial L \subset M_b$ implies $L \subset M_b$; thus we cannot find a sublevel set M_b such that $\partial L \subset M_b \subset\subset U$, and U is not n-Runge in M.

On the other hand, assume that $M \smallsetminus U$ has no compact connected component and let L be a compact subset of U. Let ω be any hermitian metric on M and φ a strongly ω-subharmonic exhaustion function on M. Set $b = 1 + \sup_L \varphi$ and

$$P = \{x \in M \smallsetminus U \,;\, \varphi(x) \le b\}.$$

As $M \smallsetminus U$ has no compact connected component, all its components T_α contain a point y_α in

$$W = \{x \in X \,;\, \varphi(x) > b + 1\}.$$

For every point $x \in P$ with $x \in T_\alpha$, there exists a connected open set $V_x \subset\subset M \smallsetminus L$ containing x such that $\partial V_x \ni y_\alpha$ ($M \smallsetminus L$ is a neighborhood of $M \smallsetminus U$ and we can consider a tubular neighborhood of a path from x to y_α in $M \smallsetminus L$). The compact set P can be covered by a finite number of open sets V_{x_j}. Then Lemma 3.3 yields functions v_j with support in $\overline{V}_{x_j} \cup \overline{W}$ which are strongly ω-subharmonic on V_{x_j}. Let χ be a convex increasing function such that $\chi(t) = 0$ on $]-\infty, b]$ and $\chi'(t) > 0$ on $]b, +\infty[$. Consider the function

$$\psi = \varphi + \sum C_j v_j + \chi \circ \varphi.$$

First, choose C_j large enough so that $\psi \ge b$ on P. Then choose χ increasing fast enough so that ψ is strongly ω-subharmonic on $\overline{W}$. Then ψ is a strongly n-convex exhaustion function on M, and as $\psi \ge \varphi$ on M and $\psi = \varphi$ on L, we see that

$$L \subset \{x \in M \,;\, \psi(x) < b\} \subset U.$$

This proves that U is n-Runge complete in M. □

4. Andreotti-Grauert Finiteness Theorems

4.A. Case of Vector Bundles over Manifolds

The crucial point in the proof of the Andreotti-Grauert theorems is the following special case, which is easily obtained by the methods of chapter 8.

(4.1) Proposition. *Let M be a strongly q-complete manifold with $q \geq 1$, and E a holomorphic vector bundle over M. Then:*

a) $H^k\big(M, \mathcal{O}(E)\big) = 0$ *for $k \geq q$.*

b) *Let U be a q-Runge complete open subset of M. Every d''-closed form $h \in C^\infty_{0,q-1}(U, E)$ can be approximated uniformly with all derivatives on every compact subset of U by a sequence of global d''-closed forms $\widetilde{h}_\nu \in C^\infty_{0,q-1}(M, E)$.*

Proof. We replace E by $\widetilde{E} = \Lambda^n TM \otimes E$; then we can work with forms of bidegree (n, k) instead of $(0, k)$. Let ψ be a strongly q-convex exhaustion function on M and ω the metric given by Lemma 3.1. Select a function $\rho \in C^\infty(M, \mathbb{R})$ which increases rapidly at infinity so that the hermitian metric $\widetilde{\omega} = e^\rho \omega$ is complete on M. Denote by E_χ the bundle E endowed with the hermitian metric obtained by multiplication of a fixed metric of E by the weight $\exp(-\rho \circ \psi)$ where $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is a convex increasing function. We apply Th. VIII-4.5 for the bundle E_χ over the complete hermitian manifold $(M, \widetilde{\omega})$. Then

$$ic(E_\chi) = ic(E) + id'd''(\chi \circ \psi) \otimes \mathrm{Id}_E \geq_{\mathrm{Nak}} ic(E) + \chi' \circ \psi \; id'd''\psi \otimes \mathrm{Id}_E \,.$$

The eigenvalues of $id'd''\psi$ with respect to $\widetilde{\omega}$ are $e^{-\rho}\gamma_j$, so Lemma VII-7.2 and Prop. VI-8.3 yield

$$\begin{aligned}
[ic(E_\chi), \Lambda] + T_{\widetilde{\omega}} &\geq [ic(E), \Lambda] + T_{\widetilde{\omega}} + \chi' \circ \psi \; [id'd''\psi, \Lambda] \otimes \mathrm{Id}_E \\
&\geq [ic(E), \Lambda] + T_{\widetilde{\omega}} + \chi' \circ \psi \; e^{-\rho}(\gamma_1 + \cdots + \gamma_k) \otimes \mathrm{Id}_E
\end{aligned}$$

when this curvature tensor acts on (n, k)-forms. For $k \geq q$, we have

$$\gamma_1 + \cdots + \gamma_k \geq 1 - (q-1)\varepsilon > 0 \quad \text{ if } \varepsilon \leq 1/q.$$

We choose χ_0 increasing fast enough so that all the eigenvalues of the above curvature tensor are ≥ 1 when $\chi = \chi_0$. Then for every $g \in C^\infty_{n,k}(M, E)$ with $D''g = 0$ the equation $D''f = g$ can be solved with an estimate

$$\int_M |f|^2 e^{-\chi \circ \psi} dV \leq \int_M |g|^2 e^{-\chi \circ \psi} dV,$$

where $\chi = \chi_0 + \chi_1$ and where χ_1 is a convex increasing function chosen so that the integral of g converges. This gives a). In order to prove b), let

$h \in C^\infty_{n,q-1}(U,E)$ be such that $D''h = 0$ and let L be an arbitrary compact subset of U. Thanks to Def. 2.14, we can choose ψ such that there is a sublevel set M_b with $L \subset M_b \subset\subset U$. Select $b_0 < b$ so that $L \subset M_{b_0}$, and let $\theta \in C^\infty(\mathbb{R},\mathbb{R})$ be a convex increasing function such that $\theta = 0$ on $]-\infty, b_0[$ and $\theta \geq 1$ on $]b,+\infty[$. Let $\eta \in \mathcal{D}(U)$ be a cut-off function such that $\eta = 1$ on M_b. We solve the equation $D''f = g$ for $g = D''(\eta h)$ with the weight $\chi = \chi_0 + \nu\theta\circ\psi$ and let ν tend to infinity. As g has compact support in $U \smallsetminus M_b$ and $\chi\circ\psi \geq \chi_0\circ\psi + \nu$ on this set, we find a solution f_ν such that

$$\int_{M_{b_0}} |f_\nu|^2 e^{-\chi_0\circ\psi} dV \leq \int_M |f_\nu|^2 e^{-\chi\circ\psi} dV \leq \int_{U\smallsetminus M_b} |g|^2 e^{-\chi\circ\psi} dV \leq Ce^{-\nu},$$

thus f_ν converges to 0 in $L^2(M_{b_0})$ and $h_\nu = \eta h - f_\nu \in C^\infty_{n,q-1}(M,E)$ is a D''-closed form converging to h in $L^2(M_{b_0})$. However, if we choose the minimal solution such that $\delta''_\chi f_\nu = 0$ as in Rem. VIII-4.6, we get $\Delta''_\chi f_\nu = \delta''_\chi g$ on M and in particular $\Delta''_{\chi_0} f_\nu = 0$ on M_{b_0}. Gårding's inequality VI-3.3 applied to the elliptic operator Δ''_{χ_0} shows that f_ν converges to 0 with all derivatives on L, hence h_ν converges to h on L. Now, replace L by an exhaustion L_ν of U by compact sets; some diagonal subsequence h_ν converges to h in $C^\infty_{n,q-1}(U,E)$. □

4.B. A Local Vanishing Result for Sheaves

Let $(X,\mathcal{O}_X)$ be an analytic scheme and $\mathcal{S}$ a coherent sheaf of $\mathcal{O}_X$-modules. We wish to extend Prop. 4.1 to the cohomology groups $H^k(X,\mathcal{S})$. The first step is to show that the result holds on small open sets, and this is done by means of local resolutions of $\mathcal{S}$.

For a given point $x \in X$, we choose a patch $(A, \mathcal{O}_\Omega/\mathcal{J})$ of X containing x, where A is an analytic subset of $\Omega \subset \mathbb{C}^N$ and $\mathcal{J}$ a sheaf of ideals with zero set A. Let $i_A : A \longrightarrow \Omega$ be the inclusion. Then $(i_A)_\star\mathcal{S}$ is a coherent $\mathcal{O}_\Omega$-module supported on A. In particular there is a neighborhood $W_0 \subset \Omega$ of x and a surjective sheaf morphism

$$\mathcal{O}^{p_0} \longrightarrow (i_A)_\star\mathcal{S} \quad \text{on} \quad W_0, \qquad (u_1,\ldots,u_{p_0}) \longmapsto \sum_{1\leq j\leq p_0} u_j G_j$$

where $G_1,\ldots,G_{p_0} \in \mathcal{S}(A\cap W_0)$ are generators of $(i_A)_\star\mathcal{S}$ on W_0. If we repeat the procedure inductively for the kernel of the above surjective morphism, we get a *homological free resolution* of $(i_A)_\star\mathcal{S}$:

$$(4.3) \quad \mathcal{O}^{p_l} \longrightarrow \cdots \longrightarrow \mathcal{O}^{p_1} \longrightarrow \mathcal{O}^{p_0} \longrightarrow (i_A)_\star\mathcal{S} \longrightarrow 0 \quad \text{on} \quad W_l$$

of arbitrary large length l, on neighborhoods $W_l \subset W_{l-1} \subset \ldots \subset W_0$. In particular, after replacing Ω by W_{2N} and A by $A\cap W_{2N}$, we may assume that $(i_A)_\star\mathcal{S}$ has a resolution of length $2N$ on Ω. In this case, we shall say that $A \subset \Omega$ is a $\mathcal{S}$-*distinguished patch* of X.

(4.4) Lemma. *Let $A \subset \Omega$ be a $\mathcal{S}$-distinguished patch of X and U a strongly q-convex open subset of A. Then*

$$H^k(U, \mathcal{S}) = 0 \quad \textit{for} \;\; k \geq q.$$

Proof. Theorem 2.13 shows that there exists a strongly q-convex open set $V \subset \Omega$ such that $U = A \cap V$. Let us denote by $\mathcal{Z}^l$ the kernel of $\mathcal{O}^{p_l} \longrightarrow \mathcal{O}^{p_{l-1}}$ for $l \geq 1$ and $\mathcal{Z}^0 = \ker\big(\mathcal{O}^{p_0} \longrightarrow (i_A)_\star \mathcal{S}\big)$. There are exact sequences

$$0 \longrightarrow \mathcal{Z}^0 \longrightarrow \mathcal{O}^{p_0} \longrightarrow (i_A)_\star \mathcal{S} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{Z}^l \longrightarrow \mathcal{O}^{p_l} \longrightarrow \mathcal{Z}^{l-1} \longrightarrow 0, \qquad 1 \leq l \leq 2N.$$

For $k \geq q$, Prop. 4.1 a) gives $H^k(V, \mathcal{O}^{p_l}) = 0$, therefore we get

$$H^k(U, \mathcal{S}) \simeq H^k\big(V, (i_A)_\star \mathcal{S}\big) \simeq H^{k+1}(V, \mathcal{Z}^0) \simeq \ldots \simeq H^{k+2N+1}(V, \mathcal{Z}^{2N}),$$

and the last group vanishes because $\operatorname{topdim} V \leq \dim_{\mathbb{R}} V = 2N$. $\square$

4.C. Topological Structure on Spaces of Sections and on Cohomology Groups

Let $V \subset \Omega$ be a strongly 1-complete open set relatively to a $\mathcal{S}$-distinguished patch $A \subset \Omega$ and let $U = A \cap V$. By the proof of Lemma 4.4, we have

$$H^1(V, \mathcal{Z}^0) \simeq H^{2N+1}(V, \mathcal{Z}^{2N}) = 0,$$

hence we get an exact sequence

$$(4.5) \quad 0 \longrightarrow \mathcal{Z}^0(V) \longrightarrow \mathcal{O}^{p_0}(V) \longrightarrow \mathcal{S}(U) \longrightarrow 0.$$

We are going to show that the Fréchet space structure on $\mathcal{O}^{p_0}(V)$ induces a natural Fréchet space structure on the groups of sections of $\mathcal{S}$ over any open subset. We first note that $\mathcal{Z}^0(V)$ is closed in $\mathcal{O}^{p_0}(V)$. Indeed, let $f_\nu \in \mathcal{Z}^0(V)$ be a sequence converging to a limit $f \in \mathcal{O}^{p_0}(V)$ uniformly on compact subsets of V. For every $x \in V$, the germs $(f_\nu)_x$ converge to f_x with respect to the topology defined by (1.4) on $\mathcal{O}^{p_0}$. As $\mathcal{Z}^0_x$ is closed in $\mathcal{O}^{p_0}_x$ in view of Th. 1.5 b), we get $f_x \in \mathcal{Z}^0_x$ for all $x \in V$, thus $f \in \mathcal{Z}^0(V)$.

(4.6) Proposition. *The quotient topology on $\mathcal{S}(U)$ is independent of the choices made above.*

Proof. For a smaller set $U' = A \cap V'$ where V' is a strongly 1-convex open subset of V, the restriction map $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V')$ is continuous, thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is continuous. If (V_α) is a countable covering of V by such sets and $U_\alpha = A \cap V_\alpha$, we get an injection of $\mathcal{S}(U)$ onto the closed subspace of the product $\prod \mathcal{S}(U_\alpha)$ consisting of families which are compatible in the intersections. Therefore, the Fréchet topology induced by the product coincides with the original topology of $\mathcal{S}(U)$. If we choose other generators $H_1, \ldots, H_{q_0}$ for $(i_A)_\star\mathcal{S}$, the germs $H_{j,x}$ can be expressed in terms of the $G_{j,x}$'s, thus we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{p_0}(V) & \xrightarrow{G} & \mathcal{S}(U) & \longrightarrow & 0 \\ \downarrow & & \| & & \\ \mathcal{O}^{q_0}(V) & \xrightarrow{H} & \mathcal{S}(U) & \longrightarrow & 0 \end{array}$$

provided that U and V are small enough. If we express the generators G_j in terms of the H_j's, we find a similar diagram with opposite vertical arrows and we conclude easily that the topology obtained in both cases is the same. Finally, it remains to show that the topology of $\mathcal{S}(U)$ is independent of the embedding $A \subset \Omega$ near a given point $x \in X$. We compare the given embedding with the Zariski embedding $(A, x) \subset \Omega'$ of minimal dimension d. After shrinking A and changing coordinates, we may assume $\Omega = \Omega' \times \mathbb{C}^{N-d}$ and that the embedding $i_A : A \longrightarrow \Omega$ is the composite of $i'_A : A \longrightarrow \Omega'$ and of the inclusion $j : \Omega' \longrightarrow \Omega' \times \{0\} \subset \Omega$. For $V' \subset \Omega'$ sufficient small and $U' = A \cap V'$, we have a surjective map $G' : \mathcal{O}^{p_0}(V') \longrightarrow \mathcal{S}(U')$ obtained by choosing generators G'_j of $(i'_A)^\star\mathcal{S}$ on a neighborhood of x in Ω'. Then we consider the open set $V = V' \times \mathbb{C}^{N-d} \subset \Omega$ and the surjective map onto $\mathcal{S}(U')$ equal to the composite

$$\mathcal{O}^{p_0}(V) \xrightarrow{j^\star} \mathcal{O}^{p_0}(V') \xrightarrow{G'} \mathcal{S}(U).$$

This map corresponds to a choice of generators $G_j \in (i_A)^\star\mathcal{S}(V)$ equal to the functions G'_j, considered as functions independent of the last variables $z_{d+1}, \ldots, z_N$. Since $j^\star$ is open, it is obvious that the quotient topology on $\mathcal{S}(U')$ is the same for both embeddings. □

Now, there is a natural topology on the cohomology groups $H^k(X, \mathcal{S})$. In fact, let (U_α) be a countable covering of X by strongly 1-complete open sets, such that each U_α is contained in a $\mathcal{S}$-distinguished patch. Since the intersections $U_{\alpha_0\ldots\alpha_k}$ are again strongly 1-complete, the covering $\mathcal{U}$ is acyclic by Lemma 4.4 and Leray's theorem shows that $H^k(X, \mathcal{S})$ is isomorphic to $\check{H}^q(\mathcal{U}, \mathcal{S})$. We consider the product topology on the spaces of Čech cochains $C^k(\mathcal{U}, \mathcal{S}) = \prod \mathcal{S}(U_{\alpha_0\ldots\alpha_k})$ and the quotient topology on $\check{H}^k(\mathcal{U}, \mathcal{S})$. It is clear that $\check{H}^0(\mathcal{U}, \mathcal{S})$ is a Fréchet space; however the higher cohomology groups $\check{H}^k(\mathcal{U}, \mathcal{S})$ need not be Hausdorff because the coboundary groups may be non closed in the cocycle groups. The resulting topology on $H^k(X, \mathcal{S})$ is independent of the choice of the covering: in fact we only have to check that the bijective continuous map $\check{H}^k(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^k(\mathcal{U}', \mathcal{S})$ is a topological isomorphism if $\mathcal{U}'$ is a refinement of $\mathcal{U}$, and this follows from Rem. 1.10 applied to the morphism of Čech complexes $C^\bullet(\mathcal{U}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{U}', \mathcal{S})$.

Finally, observe that when $\mathcal{S}$ is the locally free sheaf associated to a holomorphic vector bundle E on a smooth manifold X, the topology on $H^k\big(X, \mathcal{O}(E)\big)$ is the same as the topology associated to the Fréchet space structure on the Dolbeault complex $\big(C^\infty_{0,\bullet}(X,E), d''\big)$: by the analogue of formula (IV-6.11) we have a bijective continuous map

$$\begin{aligned} \check{H}^k\big(\mathcal{U}, \mathcal{O}(E)\big) &\longrightarrow H^k\big(C^\infty_{0,\bullet}(X,E)\big) \\ \{(c_{\alpha_0\ldots\alpha_k})\} &\longmapsto f(z) = \sum_{\alpha_0,\ldots,\alpha_q} c_{\alpha_0\ldots\alpha_q}(z)\,\theta_{\alpha_q}\, d''\theta_{\alpha_0}\wedge\ldots\wedge d''\theta_{\alpha_{q-1}} \end{aligned}$$

where (θ_α) is a partition of unity subordinate to $\mathcal{U}$. As in Rem. 1.10, the continuity of the inverse follows by the open mapping theorem applied to the surjective map

$$Z^k\big(C^\bullet(\mathcal{U}, \mathcal{O}(E))\big) \oplus C^\infty_{0,k-1}(X,E) \longrightarrow Z^k\big(C^\infty_{0,\bullet}(X,E)\big).$$

We shall need a few simple additional results.

(4.7) Proposition. *The following properties hold:*

a) *For every $x \in X$, the map $\mathcal{S}(X) \longrightarrow \mathcal{S}_x$ is continuous with respect to the topology of $\mathcal{S}_x$ defined by* (1.4).

b) *If $\mathcal{S}'$ is a coherent analytic subsheaf of $\mathcal{S}$, the space of global sections $\mathcal{S}'(X)$ is closed in $\mathcal{S}(X)$.*

c) *If $U' \subset U$ are open in X, the restriction maps $H^k(U,\mathcal{S}) \longrightarrow H^k(U',\mathcal{S})$ are continuous.*

d) *If U' is relatively compact in U, the restriction operator $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is compact.*

e) *Let $\mathcal{S} \longrightarrow \mathcal{S}'$ be a morphism of coherent sheaves over X. Then the induced maps $H^k(X,\mathcal{S}) \longrightarrow H^k(X,\mathcal{S}')$ are continuous.*

Proof. a) Let $V \subset \Omega$ be a strongly 1-convex open neighborhood of x relatively to a $\mathcal{S}$-distinguished patch $A \subset \Omega$. The map $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}_x$ is continuous, and the same is true for $\mathcal{O}^{p_0}_x \longrightarrow \mathcal{S}_x$ by §1. Therefore the composite $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{S}_x$ and its factorization $\mathcal{S}(U) \longrightarrow \mathcal{S}_x$ are continuous.

b) is a consequence of the above property a) and of the fact that each stalk $\mathcal{S}'_x$ is closed in $\mathcal{S}_x$ (cf. 1.5 b)).

c) The restriction map $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is continuous, and the case of higher cohomology groups follows immediately.

d) Assume first that $U = A \cap V$ and $U' = A \cap V'$, where $A \subset \Omega$ is a $\mathcal{S}$-distinguished patch and $V' \subset\subset V$ are strongly 1-convex open subsets of Ω. The operator $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V')$ is compact by Montel's theorem, thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is also compact. In the general case, select a finite family of strongly 1-convex sets $U'_\alpha \subset\subset U_\alpha \subset U$ such that (U'_α) covers $\overline{U}'$ and U_α is contained in some distinguished patch. There is a commutative diagram

$$\begin{array}{ccccc}
\mathcal{S}(U) & \longrightarrow & & \longrightarrow & \mathcal{S}(U') \\
\downarrow & & & & \downarrow \\
\prod \mathcal{S}(U_\alpha) & \longrightarrow & \prod \mathcal{S}(U'_\alpha) & \longrightarrow & \prod \mathcal{S}(U' \cap U'_\alpha)
\end{array}$$

where the right vertical arrow is a monomorphism and where the first arrow in the bottom line is compact. Thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is compact.

e) It is enough to check that $\mathcal{S}(U) \longrightarrow \mathcal{S}'(U)$ is continuous, and for this we may assume that $U = A \cap V$ where V is a small neighborhood of a given point x. Let $G_1, \ldots, G_{p_0}$ be generators of $\mathcal{S}_x$, $G'_1, \ldots, G'_{p_0}$ their images in $\mathcal{S}'_x$. Complete these elements in order to obtain a system of generators $(G'_1, \ldots, G'_{q_0})$ of $\mathcal{S}'_x$. For V small enough, the map $\mathcal{S}(U) \longrightarrow \mathcal{S}'(U)$ is induced by the inclusion $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V) \times \{0\} \subset \mathcal{O}^{q_0}(V)$, hence continuous. □

4.D. Cartan-Serre Finiteness Theorem

The above results enable us to prove a finiteness theorem for cohomology groups over compact analytic schemes.

(4.8) Theorem (Cartan-Serre)**.** *Let $\mathcal{S}$ be a coherent analytic sheaf over an analytic scheme $(X, \mathcal{O}_X)$. If X is compact, all cohomology groups $H^k(X, \mathcal{S})$ are finite dimensional* (*and Hausdorff*)*.*

Proof. There exist finitely many strongly 1-complete open sets $U'_\alpha \subset\subset U_\alpha$ such that each U_α is contained in some $\mathcal{S}$-distinguished patch and such that $\bigcup U'_\alpha = X$. By Prop. 4.7 d), the restriction map on Čech cochains

$$C^\bullet(\mathcal{U}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{U}', \mathcal{S})$$

defines a compact morphism of complexes of Fréchet spaces. As the coverings $\mathcal{U} = (U_\alpha)$ and $\mathcal{U}' = (U'_\alpha)$ are acyclic by 4.4, the induced map

$$\check{H}^k(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^k(\mathcal{U}', \mathcal{S})$$

is an isomorphism, both spaces being isomorphic to $H^k(X, \mathcal{S})$. We conclude by Schwartz' theorem 1.9. □

4.E. Local Approximation Theorem

We show that a local analogue of the approximation result 4.1 b) holds for a sheaf $\mathcal{S}$ over an analytic scheme $(X, \mathcal{O}_X)$.

(4.9) Lemma. *Let $A \subset \Omega$ be a $\mathcal{S}$-distinguished patch of X, and $U' \subset U \subset A$ open subsets such that U' is q-Runge complete in U. Then the restriction map*

$$H^{q-1}(U, \mathcal{S}) \longrightarrow H^{q-1}(U', \mathcal{S})$$

has a dense range.

Proof. Let L be an arbitrary compact subset of U'. Proposition 2.16 applied with $Y = U$ embedded in some neighborhood in Ω shows that there is a neighborhood V of U in Ω such that $A \cap V = U$ and a strongly q-convex function ψ on V such that $L \subset U_b \subset\subset U'$ for some $U_b = A \cap V_b$. The proof of Lemma 4.4 gives $H^q(V, \mathcal{Z}^0) = H^q(V_b, \mathcal{Z}^0) = 0$ and the cohomology exact sequences of $0 \to \mathcal{Z}^0 \to \mathcal{O}^{p_0} \to i_A^\star \mathcal{S} \to 0$ over V and V_b yield a commutative diagram of continuous maps

$$\begin{array}{ccccc} H^{q-1}\big(V, \mathcal{O}^{p_0}\big) & \longrightarrow & H^{q-1}\big(V, i_A^\star \mathcal{S}\big) & = & H^{q-1}\big(U, \mathcal{S}\big) \\ \downarrow & & \downarrow & & \downarrow \\ H^{q-1}\big(V_b, \mathcal{O}^{p_0}\big) & \longrightarrow & H^{q-1}\big(V_b, i_A^\star \mathcal{S}\big) & = & H^{q-1}\big(U_b, \mathcal{S}\big) \end{array}$$

where the horizontal arrows are surjective. Since V_b is q-Runge complete in V, the left vertical arrow has a dense range by Prop. 4.1 b). As U' is the union of an increasing sequence of sets U_{b_ν}, we only have to show that the range remains dense in the inverse limit $H^{q-1}(U', \mathcal{S})$. For that, we apply Property 1.11 d) on a suitable covering of U. Let $\mathcal{W}$ be a countable basis of the topology of U, consisting of strongly 1-convex open subsets contained in $\mathcal{S}$-distinguished patches. We let $\mathcal{W}'$ (resp. $\mathcal{W}_\nu$) be the subfamily of $W \in \mathcal{W}$ such that $W \subset\subset U'$ (resp. $W \subset\subset U_{b_\nu}$). Then $\mathcal{W}$, $\mathcal{W}'$, $\mathcal{W}_\nu$ are acyclic coverings of U, U', U_{b_ν} and each restriction map $C^\bullet(\mathcal{W}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{W}_\nu, \mathcal{S})$ is surjective. Property 1.11 d) can thus be applied and the lemma follows. □

4.F. Statement and Proof of the Andreotti-Grauert Theorem

(4.10) Theorem (Andreotti-Grauert 1962). *Let $\mathcal{S}$ be a coherent analytic sheaf over a strongly q-convex analytic scheme $(X, \mathcal{O}_X)$. Then*

a) *$H^k(X, \mathcal{S})$ is Hausdorff and finite dimensional for $k \ge q$.*

Moreover, let U be a q-Runge open subset of X, $q \ge 1$. Then

b) *the restriction map $H^k(X, \mathcal{S}) \to H^k(U, \mathcal{S})$ is an isomorphism for $k \ge q$;*

c) *the restriction map $H^{q-1}(X, \mathcal{S}) \to H^{q-1}(U, \mathcal{S})$ has a dense range.*

The compact case $q = 0$ of 4.10 a) is precisely the Cartan-Serre finiteness theorem. For $q \ge 1$, the special case when X is strongly q-complete and $U = \emptyset$ yields the following very important consequence.

(4.11) Corollary. *If X is strongly q-complete, then*

$$H^k(X, \mathcal{S}) = 0 \quad \textit{for} \;\; k \ge q.$$

Assume that $q \geq 1$ and let ψ be a smooth exhaustion on X that is strongly q-convex on $X \smallsetminus K$. We first consider sublevel sets $X_d \supset X_c \supset K$, $d > c$, and verify assertions 4.10 b), c) for all restriction maps

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S}), \qquad k \geq q-1.$$

The basic idea, already contained in (Andreotti-Grauert 1962), is to deform X_c into X_d through a sequence of strongly q-convex open sets (G_j) such that G_{j+1} is obtained from G_j by making a small bump.

(4.12) Lemma. *There exist a sequence of strongly q-convex open sets $G_0 \subset \ldots \subset G_s$ and a sequence of strongly q-complete open sets $U_0, \ldots, U_{s-1}$ in X such that*

a) $G_0 = X_c$, $G_s = X_d$, $G_{j+1} = G_j \cup U_j$ *for* $0 \leq j \leq s-1$;

b) $G_j = \{x \in X\,;\, \psi_j(x) < c_j\}$ *where ψ_j is an exhaustion function on X that is strongly q-convex on $X \smallsetminus K$*;

c) *U_j is contained in a $\mathcal{S}$-distinguished patch $A_j \subset \Omega_j$ of X*;

d) *$G_j \cap U_j$ is strongly q-complete and q-Runge complete in U_j.*

Proof. There exists a finite covering of the compact set $\overline{X}_d \smallsetminus X_c$ by $\mathcal{S}$-distinguished patches $A_j \subset \Omega_j$, $0 \leq j < s$, where $\Omega_j \subset \mathbb{C}^{N_j}$ is a euclidean ball and $K \cap A_j = \emptyset$. Let $\theta_j \in \mathcal{D}(X)$ be a family of functions such that $\operatorname{Supp}\theta_j \subset A_j$, $\theta_j \geq 0$, $\sum \theta_j \leq 1$ and $\sum \theta_j = 1$ on a neighborhood of $\overline{X}_d \smallsetminus X_c$. We can find $\varepsilon_0 > 0$ so small that

$$\psi_j = \psi - \varepsilon \sum_{0 \leq k < j} \theta_k$$

is still strongly q-convex on $X \smallsetminus K$ for $0 \leq j \leq s$ and $\varepsilon \leq \varepsilon_0$. We have $\psi_0 = \psi$ and $\psi_s = \psi - \varepsilon$ on $\overline{X}_d \smallsetminus X_c$, thus

$$G_j = \{x \in X\,;\, \psi_j(x) < c\}, \qquad 0 \leq j \leq s$$

is an increasing sequence of strongly q-convex open sets such that $G_0 = X_c$, $G_s = X_{c+\varepsilon}$. Moreover, as $\psi_{j+1} - \psi_j = -\varepsilon\theta_j$ has support in A_j, we have

$$G_{j+1} = G_j \cup U_j \quad \text{where} \quad U_j = G_{j+1} \cap A_j.$$

It follows that conditions a), b), c) are satisfied with $c+\varepsilon$ instead of d. Finally, the functions

$$\varphi_j = 1/(c - \psi_{j+1}) + 1/(r_j^2 - |z - z_j|^2), \qquad \widetilde{\varphi}_j = 1/(c - \psi_j) + 1/(r_j^2 - |z - z_j|^2)$$

are strongly q-convex exhaustions on U_j and $G_j \cap U_j = G_j \cap A_j$. Let L be an arbitrary compact subset of $G_j \cap U_j$ and $a = \sup_L \psi_j < c$. Select $b \in]a, c[$ and set

$$\psi_{j,\eta} = \psi_j + \eta\varphi_j \quad \text{on} \ \ U_j, \quad \eta > 0.$$

Then $\psi_{j,\eta}$ is an exhaustion of U_j. As φ_j is bounded below, we have

$$L \subset \{\psi_{j,\eta} < b\} \subset\subset \{\psi_j < c\} \cap U_j = G_j \cap U_j$$

for η small enough. Moreover

$$(1-\alpha)\psi_j + \alpha\psi_{j+1} = \psi - \varepsilon \sum_{0 \le k < j} \theta_k - \alpha\varepsilon\,\theta_j$$

is strongly q-convex for all $\alpha \in [0,1]$ and $\varepsilon \le \varepsilon_0$ small enough, so Prop. 2.4 implies that $\psi_{j,\eta}$ is strongly q-convex. By definition, $G_j \cap U_j$ is thus q-Runge complete in U_j, and Lemma 4.12 is proved with $X_{c+\varepsilon}$ instead of X_d. In order to achieve the proof, we consider an increasing sequence $c = c_0 < c_1 < \ldots < c_N = d$ with $c_{k+1} - c_k \le \varepsilon_0$ and perform the same construction for each pair $X_{c_k} \subset X_{c_{k+1}}$, with c replaced by c_k and $\varepsilon = c_{k+1} - c_k$. □

(4.13) Proposition. *For every sublevel set $X_c \supset K$, the group $H^k(X_c, \mathcal{S})$ is Hausdorff and finite dimensional when $k \ge q$. Moreover, for $d > c$, the restriction map*

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S})$$

is an isomorphism when $k \ge q$ and has a dense range when $k = q - 1$.

Proof. Thanks to Lemma 4.12, we are led to consider the restriction maps

$$(4.14) \quad H^k(G_{j+1}, \mathcal{S}) \longrightarrow H^k(G_j, \mathcal{S}).$$

Let us apply the Mayer-Vietoris exact sequence IV-3.11 to $G_{j+1} = G_j \cup U_j$. For $k \ge q$ we have $H^k(U_j, \mathcal{S}) = H^k(G_j \cap U_j, \mathcal{S}) = 0$ by Lemma 4.4. Hence we get an exact sequence

$$\begin{array}{l} H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S}) \longrightarrow H^{q-1}(G_j \cap U_j, \mathcal{S}) \longrightarrow \\ \quad H^k(G_{j+1}, \mathcal{S}) \longrightarrow \qquad H^k(G_j, \mathcal{S}) \qquad \longrightarrow \quad 0 \quad \longrightarrow \cdots, \qquad k \ge q. \end{array}$$

In this sequence, all the arrows are induced by restriction maps, so they define continuous linear operators. We already infer that the map (4.14) is bijective for $k > q$ and surjective for $k = q$. There exist a $\mathcal{S}$-acyclic covering $\mathcal{V} = (V_\alpha)$ of X_d and a finite family $\mathcal{V}' = (V'_{\alpha_1}, \ldots, V'_{\alpha_p})$ of open sets such that $V'_{\alpha_j} \subset\subset V_{\alpha_j}$ and $\bigcup V'_{\alpha_j} \supset \overline{X}_c$. Let $\mathcal{W}$ be a locally finite $\mathcal{S}$-acyclic covering of X_c which refines $\mathcal{V}' \cap X_c = (V'_{\alpha_j} \cap X_c)$. The refinement map

$$C^\bullet(\mathcal{V}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{V}' \cap X_c, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{W}, \mathcal{S})$$

is compact because the first arrow is, and it induces a surjective map

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S}) \quad \text{for} \ \ k \ge q.$$

By Schwartz' theorem 1.9, we conclude that $H^k(X_c, \mathcal{S})$ is Hausdorff and finite dimensional for $k \geq q$. This is equally true for $H^q(G_j, \mathcal{S})$ because G_j is also a global sublevel set $\{x \in X\,;\, \psi_j(x) < c_j\}$ containing K. Now, the Mayer-Vietoris exact sequence implies that the composite

$$H^{q-1}(U_j, \mathcal{S}) \longrightarrow H^{q-1}(G_j \cap U_j, \mathcal{S}) \xrightarrow{\partial} H^q(G_{j+1}, \mathcal{S})$$

is equal to zero. However, the first arrow has a dense range by Lemma 4.9. As the target space is Hausdorff, the second arrow must be zero; we obtain therefore the injectivity of $H^q(G_{j+1}, \mathcal{S}) \longrightarrow H^q(G_j, \mathcal{S})$ and an exact sequence

$$\begin{array}{ccccc} H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow & H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S}) & \longrightarrow & H^{q-1}(G_j \cap U_j, \mathcal{S}) & \longrightarrow 0 \\ & g \oplus u & \longmapsto & u_{\restriction G_j \cap U_j} - g_{\restriction G_j \cap U_j}. & \end{array}$$

The argument used in Rem. 1.10 shows that the surjective arrow is open. Let $g \in H^{q-1}(G_j, \mathcal{S})$ be given. By Lemma 4.9, we can approximate $g_{\restriction G_j \cap U_j}$ by a sequence $u_{\nu \restriction G_j \cap U_j}$, $u_\nu \in H^{q-1}(U_j, \mathcal{S})$. Then $w_\nu = u_{\nu \restriction G_j \cap U_j} - g_{\restriction G_j \cap U_j}$ tends to zero. As the second map in the exact sequence is open, we can find a sequence

$$g'_\nu \oplus u'_\nu \in H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S})$$

converging to zero which is mapped on w_ν. Then $(g - g'_\nu) \oplus (u_\nu - u'_\nu)$ is mapped on zero, and there exists a sequence $f_\nu \in H^{q-1}(G_{j+1}, \mathcal{S})$ which coincides with $g - g'_\nu$ on G_j and with $u_\nu - u'_\nu$ on U_j. In particular $f_{\nu \restriction G_j}$ converges to g and we have shown that

$$H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow H^{q-1}(G_j, \mathcal{S})$$

has a dense range. □

Proof of Andreotti-Grauert's Theorem 4.10. Let $\mathcal{W}$ be a countable basis of the topology of X consisting of strongly 1-convex open sets W_α contained in $\mathcal{S}$-distinguished patches of X. Let $L \subset U$ be an arbitrary compact subset. Select a smooth exhaustion function ψ on X such that ψ is strongly q-convex on $X \smallsetminus \overline{X}_b$ and $L \subset X_b \subset\subset U$ for some sublevel set X_b of ψ; choose $c > b$ such that $X_c \subset\subset U$. For every $d \in \mathbb{R}$, we denote by $\mathcal{W}_d \subset \mathcal{W}$ the collection of sets $W_\alpha \in \mathcal{W}$ such that $W_\alpha \subset X_d$. Then $\mathcal{W}_d$ is a $\mathcal{S}$-acyclic covering of X_d. We consider the sequence of Čech complexes

$$E^\bullet_\nu = C^\bullet(\mathcal{W}_{c+\nu}, \mathcal{S}), \qquad \nu \in \mathbb{N}$$

together with the surjective projection maps $E^\bullet_{\nu+1} \longrightarrow E^\bullet_\nu$, and their inverse limit $E^\bullet = C^\bullet(\mathcal{W}, \mathcal{S})$. Then we have $H^k(E^\bullet) = H^k(X, \mathcal{S})$ and $H^k(E^\bullet_\nu) = H^k(X_{c+\nu}, \mathcal{S})$. Propositions 1.11 (a,b,c) and 4.13 imply that $H^k(X, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S})$ is bijective for $k \geq q$ and has a dense range for $k = q-1$. It already follows that $H^k(X, \mathcal{S})$ is Hausdorff for $k \geq q$. Now, take an increasing sequence of open sets X_{c_ν} equal to sublevel sets of a sequence of exhaustions

ψ_ν, such that $U = \bigcup X_{c_\nu}$. Then all groups $H^k(X_{c_\nu}, \mathcal{S})$ are in bijection with $H^k(X, \mathcal{S})$ for $k \geq q$, and the image of $H^{q-1}(X_{c_{\nu+1}}, \mathcal{S})$ in $H^{q-1}(X_{c_\nu}, \mathcal{S})$ is dense because it contains the image of $H^{q-1}(X, \mathcal{S})$. Proposition 1.11 (a,b,c) again shows that $H^k(U, \mathcal{S}) \longrightarrow H^k(X_{c_0}, \mathcal{S})$ is bijective for $k \geq q$, and d) shows that $H^{q-1}(X, \mathcal{S}) \longrightarrow H^{q-1}(U, \mathcal{S})$ has a dense range. The theorem follows. □

A combination of Andreotti-Grauert's theorem with Th. 3.6 yields the following important consequence.

(4.15) Corollary. *Let $\mathcal{S}$ be a coherent sheaf over an analytic scheme $(X, \mathcal{O}_X)$ with* $\dim X \leq n$.

a) *We have* $H^k(X, \mathcal{S}) = 0$ *for all* $k \geq n+1$;

b) *If X has no compact irreducible component of dimension n, then we have* $H^n(X, \mathcal{S}) = 0$.

c) *If X has only finitely many n-dimensional compact irreducible components, then $H^n(X, \mathcal{S})$ is finite dimensional.* □

The special case of 4.15 b) when X is smooth and $\mathcal{S}$ locally free has been first proved by (Malgrange 1955), and the general case is due to (Siu 1969). Another consequence is the following approximation theorem for coherent sheaves over manifolds, which results from Prop. 3.8.

(4.16) Proposition. *Let $\mathcal{S}$ be a coherent sheaf over a non compact connected complex manifold M with* $\dim M = n$. *Let $U \subset M$ be an open subset such that the complement $M \smallsetminus U$ has no compact connected component. Then the restriction map $H^{n-1}(M, \mathcal{S}) \longrightarrow H^{n-1}(U, \mathcal{S})$ has a dense range.* □

5. Grauert's Direct Image Theorem

The goal of this section is to prove the following fundamental result on direct images of coherent analytic sheaves, due to (Grauert 1960).

(5.1) Direct image theorem. *Let X, Y be complex analytic schemes and let $F : X \to Y$ be a proper analytic morphism. If $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module, the direct images $R^qF_\star\mathcal{S}$ are coherent $\mathcal{O}_Y$-modules.*

We give below a beautiful proof due to (Kiehl-Verdier 1971), which is much simpler than Grauert's original proof; this proof rests on rather deep properties of nuclear modules over nuclear Fréchet algebras. We first introduce the basic concept of topological tensor product. Our presentation owes much to the seminar lectures by (Douady-Verdier 1973).

5.A. Topological Tensor Products and Nuclear Spaces

The algebra of holomorphic functions on a product space $X \times Y$ is a completion $\mathcal{O}(X) \mathbin{\widehat{\otimes}} \mathcal{O}(Y)$ of the algebraic tensor product $\mathcal{O}(X) \otimes \mathcal{O}(Y)$. We are going to describe the construction and the basic properties of the required topological tensor products $\widehat{\otimes}$.

Let E, F be (real or complex) vector spaces equipped with semi-norms p and q, respectively. Then $E \otimes F$ can be equipped with any one of the two natural semi-norms $p \otimes_\pi q$, $p \otimes_\varepsilon q$ defined by

$$p \otimes_\pi q(t) = \inf\Big\{ \sum_{1\le j\le N} p(x_j)\,q(y_j)\,;\ t = \sum_{1\le j\le N} x_j \otimes y_j\,,\ x_j \in E\,,\ y_j \in F \Big\},$$
$$p \otimes_\varepsilon q(t) = \sup_{||\xi||_p\le 1,\,||\eta||_q\le 1} \big|\xi\otimes\eta(t)\big|, \qquad \xi \in E',\quad \eta\in F'\,;$$

the inequalities in the last line mean that ξ, η satisfy $|\xi(x)| \le p(x)$ and $|\eta(y)| \le q(y)$ for all $x \in E$, $y \in F$. Then clearly $p \otimes_\varepsilon q \le p \otimes_\pi q$, for

$$p \otimes_\varepsilon q\Big(\sum x_j \otimes y_j\Big) \le \sum p \otimes_\varepsilon q(x_j \otimes y_j) \le \sum p(x_j)\,q(y_j).$$

Given $x \in E$, $y \in F$, the Hahn-Banach theorem implies that there exist ξ, η such that $||\xi||_p = ||\eta||_q = 1$ with $\xi(x) = p(x)$ and $\eta(y) = q(y)$, hence $p \otimes_\varepsilon q(x \otimes y) \ge p(x)\,q(y)$. On the other hand $p \otimes_\pi q(x \otimes y) \le p(x)\,q(y)$, thus

$$p \otimes_\varepsilon q(x \otimes y) = p \otimes_\pi q(x\otimes y) = p(x)\,q(y).$$

(5.2) Definition. *Let E, F be locally convex topological vector spaces. The topological tensor product $E \mathbin{\widehat{\otimes}}_\pi F$ (resp. $E \mathbin{\widehat{\otimes}}_\varepsilon F$) is the Hausdorff completion of $E \otimes F$, equipped with the family of semi-norms $p \otimes_\pi q$ (resp. $p \otimes_\varepsilon q$) associated to fundamental families of semi-norms on E and F.*

Since we may also write

$$p \otimes_\pi q(t) = \inf\Big\{ \sum |\lambda_j|\,;\ t = \sum \lambda_j\, x_j \otimes y_j\,,\ p(x_j) \le 1\,,\ q(y_j) \le 1 \Big\}$$

where the λ_j's are scalars, we see that the closed unit ball $B(p \mathbin{\widehat{\otimes}}_\pi q)$ in $E \mathbin{\widehat{\otimes}}_\pi F$ is the closed convex hull of $B(p) \otimes B(q)$. From this, we easily infer that the topological dual space $(E \mathbin{\widehat{\otimes}}_\pi F)'$ is isomorphic to the space of continuous bilinear forms on $E \times F$. Another simple consequence of this interpretation of $B(p \mathbin{\widehat{\otimes}}_\pi q)$ is example a) below.

(5.3) Examples.

a) For all discrete spaces I, J, there is an isometry

$$\ell^1(I) \mathbin{\widehat{\otimes}}_\pi \ell^1(J) \simeq \ell^1(I \times J).$$

b) For Banach spaces (E,p), (F,q), the closed unit ball in $E \mathbin{\widehat{\otimes}_\varepsilon} F$ is dual to the unit ball $B(p' \mathbin{\widehat{\otimes}_\pi} q')$ of $E' \mathbin{\widehat{\otimes}_\pi} F'$ through the natural pairing extending the algebraic pairing of $E \otimes F$ and $E' \otimes F'$. If $c_0(I)$ denotes the space of bounded sequences on I converging to zero at infinity, we have $c_0(I)' = \ell^1(I)$, hence by duality $c_0(I) \mathbin{\widehat{\otimes}_\varepsilon} c_0(J)$ is isometric to $c_0(I \times J)$.

c) If X, Y are compact topological spaces and if $C(X)$, $C(Y)$ are their algebras of continuous functions with the sup norm, then

$$C(X) \mathbin{\widehat{\otimes}_\varepsilon} C(Y) \simeq C(X \times Y).$$

Indeed, $C(X)'$ is the space of finite Borel measures equipped with the mass norm. Thus for $f \in C(X) \otimes C(Y)$, the $\otimes_\varepsilon$-seminorm is given by

$$||f||_\varepsilon = \sup_{||\mu|| \le 1,\, ||\nu|| \le 1} \mu \otimes \nu(f) = \sup_{X \times Y} |f|;$$

the last equality is obtained by taking Dirac measures δ_x, δ_y for μ, ν (the inequality $\le$ is obvious). Now $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$ by the Stone-Weierstrass theorem, hence its completion is $C(X \times Y)$, as desired. □

Let $f : E_1 \to E_2$ and $g : F_1 \to F_2$ be continuous morphisms. For all semi-norms p_2, q_2 on E_2, F_2, there exist semi-norms p_1, q_1 on E_1, F_1 and constants $||f|| = ||f||_{p_1,p_2}$, $||g|| = ||g||_{q_1,q_2}$ such that $p_2 \circ f \le ||f||\, p_1$ and $q_2 \circ g \le ||g||\, q_1$. Then we find

$$(p_2 \otimes_\pi q_2) \circ (f \otimes g) \le ||f||\, ||g||\, p_1 \otimes_\pi q_1$$

and a similar formula with $p_j \otimes_\varepsilon q_j$. It follows that there are well defined continuous maps

$$(5.4') \qquad f \mathbin{\widehat{\otimes}_\pi} g : E_1 \mathbin{\widehat{\otimes}_\pi} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\pi} F_2,$$
$$(5.4'') \qquad f \mathbin{\widehat{\otimes}_\varepsilon} g : E_1 \mathbin{\widehat{\otimes}_\varepsilon} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\varepsilon} F_2.$$

Another simple fact is that $\widehat{\otimes}_\pi$ preserves open morphisms:

(5.5) Proposition. *If $f : E_1 \to E_2$ and $g : F_1 \to F_2$ are epimorphisms, then $f \mathbin{\widehat{\otimes}_\pi} g : E_1 \mathbin{\widehat{\otimes}_\pi} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\pi} F_2$ is an epimorphism.*

Proof. Recall that when E is locally convex complete and F Hausdorff, a morphism $u : E \to F$ is open if and only if $\overline{u(V)}$ is a neighborhood of 0 for every neighborhood of 0 (this can be checked essentially by the same proof as 1.8 b)). Here, for any semi-norms p, q on E_1, F_1 the closure of $f \mathbin{\widehat{\otimes}_\pi} g\big(B(p \mathbin{\widehat{\otimes}_\pi} q)\big)$ contains the closed convex hull of $f\big(B(p)\big) \otimes g\big(B(q)\big)$ in which $f\big(B(p)\big)$ and $g\big(B(q)\big)$ are neighborhoods of 0, so it is a neighborhood of 0 in $E \mathbin{\widehat{\otimes}_\pi} F$. □

If $E_1 \subset E_2$ is a closed subspace, every continuous semi-norm p_1 on E_1 is the restriction of a continuous semi-norm on E_2, and every linear form

$\xi_1 \in E_1'$ such that $||\xi_1||_{p_1} \le 1$ can be extended to a linear form $\xi_2 \in E_2$ such that $||\xi_2||_{p_2} = ||\xi_1||_{p_1}$ (Hahn-Banach theorem); similar properties hold for a closed subspace $F_1 \subset F_2$. We infer that

$$(p_2 \otimes_\varepsilon q_2)_{\restriction E_1 \otimes F_1} = p_1 \otimes_\varepsilon q_1\,,$$

thus $E_1 \mathbin{\widehat{\otimes}_\varepsilon} F_1$ is a closed subspace of $E_2 \mathbin{\widehat{\otimes}_\varepsilon} F_2$. In other words:

(5.6) Proposition. *If $f : E_1 \to E_2$ and $g : F_1 \to F_2$ are monomorphisms, then $f \mathbin{\widehat{\otimes}_\varepsilon} g : E_1 \mathbin{\widehat{\otimes}_\varepsilon} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\varepsilon} F_2$ is a monomorphism.* □

Unfortunately, 5.5 fails for $\widehat{\otimes}_\varepsilon$ and 5.6 fails for $\widehat{\otimes}_\pi$, even with Fréchet or Banach spaces. It follows that neither $\widehat{\otimes}_\pi$ nor $\widehat{\otimes}_\varepsilon$ are exact functors in the category of Fréchet spaces. In order to circumvent this difficulty, it is necessary to work in a suitable subcategory.

(5.7) Definition. *A morphism $f : E \to F$ of complete locally convex spaces is said to be nuclear if f can be written as*

$$f(x) = \sum \lambda_j\, \xi_j(x)\, y_j$$

where (λ_j) is a sequence of scalars with $\sum |\lambda_j| < +\infty$, $\xi_j \in E'$ an equicontinuous sequence of linear forms and $y_j \in F$ a bounded sequence.

When E and F are Banach spaces, the space of nuclear morphisms is isomorphic to $E' \mathbin{\widehat{\otimes}_\pi} F$ and the nuclear norm $||f||_\nu$ is defined to be the norm in this space, namely

$$(5.8)\qquad ||f||_\nu = \inf\Big\{ \sum |\lambda_j|\,;\ f = \sum \lambda_j\, \xi_j \otimes y_j,\ ||\xi_j|| \le 1,\ ||y_j|| \le 1 \Big\}.$$

For general spaces E, F, the equicontinuity of (ξ_j) means that there is a seminorm p on E and a constant C such that $|\xi_j(x)| \le C\,p(x)$ for all j. Then the definition shows that $f : E \to F$ is nuclear if and only if f can be factorized as $E \to E_1 \to F_1 \to F$ where $E_1 \to F_1$ is a nuclear morphism of Banach spaces: indeed we need only take E_1 be equal to the Hausdorff completion $\widehat{E}_p$ of (E, p) and let F_1 be the subspace of F generated by the closed balanced convex hull of $\{y_j\}$ (= unit ball in F_1) ; moreover, if $u : S \to E$ and $v : F \to T$ are continuous, the nuclearity of f implies the nuclearity of $v \circ f \circ u$; its nuclear decomposition is then $v \circ f \circ u = \sum \lambda_j\, (\xi_j \circ u) \otimes v(y_j)$.

(5.9) Remark. Every nuclear morphism is compact: indeed, we may assume in Def. 5.7 that (y_j) converges to 0 and $\sum |\lambda_j| \le 1$, otherwise we replace y_j by $\varepsilon_j y_j$ with ε_j converging to zero such that $\sum |\lambda_j/\varepsilon_j| \le 1$; then, if $U \subset F$ is a neighborhood of 0 such that $|\xi_j(U)| \le 1$ for all j, the image $f(U)$ is contained in the closed convex hull of the compact set $\{y_j\} \cup \{0\}$, which is compact.

(5.10) Proposition. *If E, F, G are Banach spaces and if $f : E \to F$ is nuclear, there is a continuous morphism*

$$f \mathbin{\widehat{\otimes}} \mathrm{Id}_G : E \mathbin{\widehat{\otimes}}_\varepsilon G \longrightarrow F \mathbin{\widehat{\otimes}}_\pi G$$

extending $f \otimes \mathrm{Id}_G$, such that $||f \mathbin{\widehat{\otimes}} \mathrm{Id}_G|| \le ||f||_\nu$.

Proof. If $f = \sum \lambda_j \, \xi_j \otimes y_j$ as in (5.8), then for any $t \in E \otimes G$ we have

$$(f \otimes \mathrm{Id}_G)(t) = \sum \lambda_j \big(\xi_j \otimes \mathrm{Id}_G(t)\big) \otimes y_j$$

where $(\xi_j \otimes \mathrm{Id}_G)(t) \in G$ has norm

$$||(\xi_j \otimes \mathrm{Id}_G)(t)|| = \sup_{\eta \in G',\, ||\eta|| \le 1} \big|\eta\big(\xi_j \otimes \mathrm{Id}_G(t)\big)\big| = \sup_\eta \big|\xi_j \otimes \eta(t)\big| \le ||t||_\varepsilon.$$

Therefore $||f \otimes \mathrm{Id}_G(t)||_\pi \le \sum |\lambda_j| \, ||t||_\varepsilon$, and the infimum over all decompositions of f yields

$$||f \otimes \mathrm{Id}_G(t)||_\pi \le ||f||_\nu ||t||_\varepsilon.$$

Proposition 5.10 follows. □

If E is a Fréchet space and (p_j) an increasing sequence of semi-norms on E defining the topology of E, we have

$$E = \varprojlim \widehat{E}_{p_j},$$

where $\widehat{E}_{p_j}$ is the Hausdorff completion of (E, p_j) and $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ the canonical morphism. Here $\widehat{E}_{p_j}$ is a Banach space for the induced norm $\widehat{p}_j$.

(5.11) Definition. *A Fréchet space E is said to be nuclear if the topology of E can be defined by an increasing sequence of semi-norms p_j such that each canonical morphism*

$$\widehat{E}_{p_{j+1}} \longrightarrow \widehat{E}_{p_j}$$

of Banach spaces is nuclear.

If E, F are arbitrary locally convex spaces, we always have a continuous morphism $E \mathbin{\widehat{\otimes}}_\pi F \to E \mathbin{\widehat{\otimes}}_\varepsilon F$, because $p \otimes_\varepsilon q \le p \otimes_\pi q$. If E, say, is nuclear, this morphism yields in fact an isomorphism $E \mathbin{\widehat{\otimes}}_\varepsilon F \simeq E \mathbin{\widehat{\otimes}}_\pi F$: indeed, by Prop. 5.10, we have $p_j \mathbin{\widehat{\otimes}}_\pi q \le C_j \, p_{j+1} \mathbin{\widehat{\otimes}}_\varepsilon q$ where C_j is the nuclear norm of $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$. Hence, when E or F is nuclear, we will identify $E \mathbin{\widehat{\otimes}}_\pi F$ and $E \mathbin{\widehat{\otimes}}_\varepsilon F$ and omit ε or π in the notation $E \mathbin{\widehat{\otimes}} F$.

(5.12) Example. Let $D = \prod D(0, R_j)$ be a polydisk in $\mathbb{C}^n$. For any $t \in \,]0,1[$, we equip $\mathcal{O}(D)$ with the semi-norm

$$p_t(f) = \sup_{tD} |f|.$$

The completion of $\big(\mathcal{O}(D), p_t\big)$ is the Banach space E_t of holomorphic functions on tD which are continuous up to the boundary. We claim that for $t' < t < 1$ the restriction map

$$\rho_{t,t'} : E_{t'} \longrightarrow E_t$$

is nuclear. In fact, for $f \in \mathcal{O}(D)$, we have $f(z) = \sum a_\alpha z^\alpha$ where $a_\alpha = a_\alpha(f)$ satisfies the Cauchy inequalities $|a_\alpha(f)| \le p_{t'}(f)/(t'R)^\alpha$ for all $\alpha \in \mathbb{N}^n$. The formula $f = \sum a_\alpha(f)\, e_\alpha$ with $e_\alpha(z) = z^\alpha$ shows that

$$||\rho_{t,t'}||_\nu \le \sum ||a_\alpha||_{p_{t'}} ||e_\alpha||_{p_t} \le \sum (t'R)^{-\alpha}(tR)^\alpha = (1 - t/t')^{-n} < +\infty.$$

We infer that $\mathcal{O}(D)$ is a nuclear Fréchet space. It is also in a natural way a fully nuclear Fréchet algebra (see Def. 5.39 below). $\square$

(5.13) Proposition. *Let E be a nuclear space. A morphism $f : E \to F$ is nuclear if and only if f admits a factorization $E \to M \to F$ through a Banach space M.*

Proof. By definition, a nuclear map $f : E \to F$ always has a factorization through a Banach space (even if E is not nuclear). Conversely, if E is nuclear, any continuous linear map $E \to M$ into a Banach space M is continuous for some semi-norm p_j on E, so this map has a factorization

$$E \to \widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j} \to M$$

in which the second arrow is nuclear. Hence any map $E \to M \to F$ is nuclear. $\square$

(5.14) Proposition.

a) *If E, F are nuclear spaces, then $E \mathbin{\widehat{\otimes}} F$ is nuclear.*

b) *Any closed subspace or quotient space of a nuclear space is nuclear.*

c) *Any countable product of nuclear spaces is nuclear.*

d) *Any countable inverse limit of nuclear spaces is nuclear.*

Proof. a) If $f : E_1 \to F_1$ and $g : E_2 \to F_2$ are nuclear morphisms of Banach spaces, it is easy to check that $f \mathbin{\widehat{\otimes}_\pi} g$ and $f \mathbin{\widehat{\otimes}_\varepsilon} g$ are nuclear with $||f \mathbin{\widehat{\otimes}_?} g||_\nu \le ||f||_\nu ||g||_\nu$ in both cases. Property a) follows by applying this to the canonical morphisms $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ and $\widehat{F}_{q_{j+1}} \to \widehat{F}_{q_j}$.

c) Let E_k, $k \in \mathbb{N}$, be nuclear spaces and $F = \prod E_k$. If (p^k_j) is an increasing family of semi-norms on E_k as in Def. 5.11, then the topology of F is defined by the family of semi-norms

$$q_j(x) = \max_{0 \le k \le j} p_j^k(x_k), \qquad x = (x_k) \in F.$$

Then $\widehat{F}_{q_j} = \bigoplus_{0 \le k \le j} \widehat{E}_{k,p_j^k}$ and

$$\big(\widehat{F}_{q_{j+1}} \to \widehat{F}_{q_j}\big) = \bigoplus_{0 \le k \le j} \big(\widehat{E}_{k,p_{j+1}^k} \to \widehat{E}_{k,p_j^k}\big) \oplus \big(\widehat{E}_{j+1,p_{j+1}^{j+1}} \to \{0\}\big)$$

is easily seen to be nuclear.

b) If $F \subset E$ is closed, then $\widehat{F}_{p_j}$ can be identified to a closed subspace of $\widehat{E}_{p_j}$, the map $\widehat{F}_{p_{j+1}} \to \widehat{F}_{p_j}$ is the restriction of $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ and we have $\widehat{E/F}_{p_j} \simeq \widehat{E}_{p_j}/\widehat{F}_{p_j}$. It is not true in general that the restriction or quotient of a nuclear morphism is nuclear, but this is true for a binuclear $=$ (nuclear $\circ$ nuclear) morphism, as shown by Lemma 5.15 b) below. Hence $\widehat{F}_{p_{2j+2}} \to \widehat{E}_{p_{2j}}$ and $\widehat{E/F}_{p_{2j+2}} \to \widehat{E/F}_{p_{2j}}$ are nuclear, so (p_{2j}) is a fundamental family of semi-norms on F or E/F, as required in Def. 5.11.

d) follows immediately from b) and c), since $\varprojlim E_k$ is a closed subspace of $\prod E_k$. □

(5.15) Lemma. *Let E, F, G be Banach spaces.*

a) *If $f : E \to F$ is nuclear, then f can be factorized through a Hilbert space H as a morphism $E \to H \to F$.*

b) *Let $g : F \to G$ be another nuclear morphism. If $\mathrm{Im}(g \circ f)$ is contained in a closed subspace T of G, then $g \circ f : E \to T$ is nuclear. If $\ker(g \circ f)$ contains a closed subspace S of E, the induced map $(g \circ f)^\sim : E/S \to G$ is nuclear.*

Proof. a) Write $f = \sum_{j \in I} \xi_j \otimes y_j \in E' \mathbin{\widehat{\otimes}_\pi} F$ with $\sum ||\xi_j||\,||y_j|| < +\infty$. Without loss of generality, we may suppose $||\xi_j|| = ||y_j||$. Then f is the composition

$$E \longrightarrow \ell^2(I) \longrightarrow F, \qquad x \longmapsto \big(\xi_j(x)\big), \quad (\lambda_j) \longmapsto \sum \lambda_j y_j.$$

b) Decompose g into $g = v \circ u$ as in a) and write $g \circ f$ as the composition

$$E \xrightarrow{\ f\ } F \xrightarrow{\ u\ } H \xrightarrow{\ v\ } G$$

where H is a Hilbert space. If $\mathrm{Im}(g \circ f) \subset T$ and if $T \subset G$ is closed, then $H_1 = v^{-1}(T)$ is a closed subspace of H containing $\mathrm{Im}(u \circ f)$. Therefore $g \circ f : E \to T$ is the composition

$$E \xrightarrow{\ f\ } F \xrightarrow{\ u\ } H \xrightarrow{\ \mathrm{pr}^\perp\ } H_1 \xrightarrow{\ v\restriction H_1\ } T$$

where f is nuclear and $g \circ f : E \longrightarrow T$ is nuclear. Similar proof for $(g \circ f)^\sim : E/S \to G$ by using decompositions $f = v \circ u : E \to H \to F$ and

$$(g \circ f)^{\sim} : E/S \xrightarrow{\widetilde{u}} H/H_1 \simeq H_1^{\perp} \xrightarrow{v_{\restriction H_1^{\perp}}} F \xrightarrow{g} G$$

where $H_1 = \overline{u(S)}$ satisfies $H_1 \subset \ker(g \circ v) \subset H$. □

(5.16) Corollary. *Let E be a nuclear space and let $E \to F$ be a nuclear morphism.*

a) *If $f(E)$ is contained in a closed subspace T of F, then the morphism $f_1 : E \to T$ induced by f is nuclear.*

b) *If $\ker f$ contains a closed subspace S of E, then $\widetilde{f} : E/S \to F$ is nuclear.*

Proof. Let $E \xrightarrow{u} M \xrightarrow{v} F$ be a factorization of f through a Banach space M. In case a), resp. b), $M_1 = v^{-1}(T)$ is a closed subspace of M, resp. $M/\overline{u(S)}$ is a Banach space, and we have factorizations

$$f_1 : E \xrightarrow{u_1} M_1 \xrightarrow{v_1} T, \qquad \widetilde{f} : E/S \xrightarrow{\widetilde{u}} M/\overline{u(S)} \xrightarrow{\widetilde{v}} F$$

where u_1, $\widetilde{u}$ are induced by u and v_1, $\widetilde{v}$ by v. Hence f_1 and $\widetilde{f}$ are nuclear. □

(5.17) Proposition. *Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of Fréchet spaces and let F be a Fréchet space. If E_2 or F is nuclear, there is an exact sequence*

$$0 \longrightarrow E_1 \mathbin{\widehat{\otimes}} F \longrightarrow E_2 \mathbin{\widehat{\otimes}} F \longrightarrow E_3 \mathbin{\widehat{\otimes}} F \longrightarrow 0.$$

Proof. If E_2 is nuclear, then so are E_1 and E_3 by Prop. 5.14 b). Hence $E_1 \mathbin{\widehat{\otimes}} F \to E_2 \mathbin{\widehat{\otimes}} F$ is a monomorphism and $E_2 \mathbin{\widehat{\otimes}} F \to E_3 \mathbin{\widehat{\otimes}} F$ an epimorphism by Prop. 5.6 and 5.5. It only remains to show that

$$\mathrm{Im}\,\big(E_1 \mathbin{\widehat{\otimes}} F \longrightarrow E_2 \mathbin{\widehat{\otimes}} F\big) = \ker\,\big(E_2 \mathbin{\widehat{\otimes}} F \longrightarrow E_3 \mathbin{\widehat{\otimes}} F\big)$$

and for this, we need only show that the left hand side is dense in the right hand side (we already know it is closed). Let $\varphi \in (E_2 \mathbin{\widehat{\otimes}} F)'$ be a linear form, viewed as a continuous bilinear form on $E_2 \times F$. If φ vanishes on the image of $E_1 \mathbin{\widehat{\otimes}} F$, then φ induces a continuous bilinear form on $E_3 \times F$ by passing to the quotient. Hence φ must vanish on the kernel of $E_2 \mathbin{\widehat{\otimes}} F \to E_3 \mathbin{\widehat{\otimes}} F$, and our density statement follows by the Hahn-Banach theorem. □

5.B. Künneth Formula for Coherent Sheaves

As an application of the above general concepts, we now show how topological tensor products can be used to compute holomorphic functions and cohomology of coherent sheaves on product spaces.

(5.18) Proposition. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex analytic scheme $(X, \mathcal{O}_X)$. Then $\mathcal{F}(X)$ is a nuclear space.*

Proof. Let $A \subset \Omega \subset \mathbb{C}^N$ be an open patch of X such that the image sheaf $(i_A)_\star \mathcal{F}_{\restriction A}$ on Ω has a resolution

$$\mathcal{O}_\Omega^{p_1} \longrightarrow \mathcal{O}_\Omega^{p_0} \longrightarrow (i_A)_\star \mathcal{F}_{\restriction A} \longrightarrow 0$$

and let $D \subset\subset \Omega$ be a polydisk. As D is Stein, we get an exact sequence

$$(5.19) \quad \mathcal{O}^{p_1}(D) \longrightarrow \mathcal{O}^{p_0}(D) \longrightarrow \mathcal{F}(A \cap D) \longrightarrow 0.$$

Hence $\mathcal{F}(A \cap D)$ is a quotient of the nuclear space $\mathcal{O}^{p_0}(D)$ and so $\mathcal{F}(A \cap D)$ is nuclear by (5.14 b). Let (U_α) be a countable covering of X by open sets of the form $A \cap D$. Then $\mathcal{F}(X)$ is a closed subspace of $\prod \mathcal{F}(U_\alpha)$, thus $\mathcal{F}(X)$ is nuclear by (5.14 b,c). □

(5.20) Proposition. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on complex analytic schemes X, Y respectively. Then there is a canonical isomorphism*

$$\mathcal{F} \boxtimes \mathcal{G}(X \times Y) \simeq \mathcal{F}(X) \mathbin{\widehat{\otimes}} \mathcal{G}(Y).$$

Proof. We show the proposition in several steps of increasing generality.

a) $X = D \subset \mathbb{C}^n$, $Y = D' \subset \mathbb{C}^p$ *are polydisks,* $\mathcal{F} = \mathcal{O}_X$, $\mathcal{G} = \mathcal{O}_Y$.

Let $p_t(f) = \sup_{tD} |f|$, $p'_t(f) = \sup_{tD'} |f|$ and $q_t(f) = \sup_{t(D \times D')} |f|$ be the semi-norms defining the topology of $\mathcal{O}(D)$, $\mathcal{O}(D')$ and $\mathcal{O}(D \times D')$, respectively. Then $\widehat{E}_{p_t}$ is a closed subspace of the space $C(t\overline{D})$ of continuous functions on $t\overline{D}$ with the sup norm, and we have $p_t \otimes_\varepsilon p'_t = q_t$ by example (5.3 c). Now, $\mathcal{O}(D) \otimes \mathcal{O}(D')$ is dense in $\mathcal{O}(D \times D')$, hence its completion with respect to the family (q_t) is $\mathcal{O}(D) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{O}(D') = \mathcal{O}(D \times D')$.

b) X *is embedded in a polydisk* $D \subset \mathbb{C}^n$, $X = A \cap D \overset{i}{\hookrightarrow} D$,
$i_\star \mathcal{F}$ *is the cokernel of a morphism* $\mathcal{O}_D^{p_1} \longrightarrow \mathcal{O}_D^{p_0}$,
$Y = D' \subset \mathbb{C}^p$ *is a polydisk and* $\mathcal{G} = \mathcal{O}_Y$.

By taking the external tensor product with $\mathcal{O}_Y$, we get an exact sequence

$$(5.21) \quad \mathcal{O}_{D \times Y}^{p_1} \longrightarrow \mathcal{O}_{D \times Y}^{p_0} \longrightarrow i_\star \mathcal{F} \boxtimes \mathcal{O}_Y \longrightarrow 0.$$

Then we find a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}^{p_1}(D) \widehat{\otimes}\, \mathcal{O}(Y) & \longrightarrow & \mathcal{O}^{p_0}(D) \widehat{\otimes}\, \mathcal{O}(Y) & \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{O}(Y) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
\mathcal{O}^{p_1}(D \times Y) & \longrightarrow & \mathcal{O}^{p_0}(D \times Y) & \longrightarrow & \mathcal{F} \boxtimes \mathcal{O}_Y(X \times Y) & \longrightarrow & 0
\end{array}$$

in which the first line is exact as the image of (5.19) by the exact functor $\bullet \mathbin{\widehat{\otimes}} \mathcal{O}(Y)$, and the second line is exact because the exact sequence of sheaves (5.21) gives an exact sequence of spaces of sections on the Stein space $D \times Y$; note that $i_\star \mathcal{F} \boxtimes \mathcal{O}_Y(D \times Y) = \mathcal{F} \boxtimes \mathcal{O}_Y(X \times Y)$. As the first two vertical arrows are isomorphisms by a), the third one is also an isomorphism.

c) *X, $\mathcal{F}$ are as in* b),
Y is embedded in a polydisk $D' \subset \mathbb{C}^p$, $Y = A' \cap D' \overset{j}{\hookrightarrow} D'$
and $j_\star \mathcal{G}$ is the cokernel of $\mathcal{O}_{D'}^{q_1} \longrightarrow \mathcal{O}_{D'}^{q_0}$.

Taking the external tensor product with $\mathcal{F}$, we get an exact sequence

$$\mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_1} \longrightarrow \mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_0} \longrightarrow \mathcal{F} \boxtimes j_\star \mathcal{G} \longrightarrow 0$$

and with the same arguments as above we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}(X) \widehat{\otimes}\, \mathcal{O}^{q_1}(D') & \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{O}^{q_0}(D') & \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(Y) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
\mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_1}(X \times D') & \longrightarrow & \mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_0}(X \times D') & \longrightarrow & \mathcal{F} \boxtimes \mathcal{G}(X \times Y) & \longrightarrow & 0.
\end{array}$$

d) *X, $\mathcal{F}$ are as in* b),c) *and Y, $\mathcal{G}$ are arbitrary.*

Then Y can be covered by open sets $U_\alpha = A_\alpha \cap D_\alpha$ embedded in polydisks D_α, on which the image of $\mathcal{G}$ admits a two-step resolution. We have $\mathcal{F} \boxtimes \mathcal{G}(X \times U_\alpha) \simeq \mathcal{F}(X) \,\widehat{\otimes}\, \mathcal{G}(U_\alpha)$ by c), and the same is true over the intersections $X \times U_{\alpha\beta}$ because $U_{\alpha\beta} = U_\alpha \cap U_\beta$ can be embedded by the cross product embedding $j_\alpha \times j_\beta : U_{\alpha\beta} \to D_\alpha \times D_\beta$. We have an exact sequence

$$0 \longrightarrow \mathcal{G}(Y) \longrightarrow \prod_\alpha \mathcal{G}(U_\alpha) \longrightarrow \prod_{\alpha,\beta} \mathcal{G}(U_{\alpha\beta})$$

where the last arrow is $(c_\alpha) \mapsto (c_\beta - c_\alpha)$, and a commutative diagram with exact lines

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(Y) & \longrightarrow & \prod \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(U_\alpha) & \longrightarrow & \prod \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(U_{\alpha\beta}) \\
& \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
0 \longrightarrow & \mathcal{F} \boxtimes \mathcal{G}(X \times Y) & \longrightarrow & \prod \mathcal{F} \boxtimes \mathcal{G}(X \times U_\alpha) & \longrightarrow & \prod \mathcal{F} \boxtimes \mathcal{G}(X \times U_{\alpha\beta}).
\end{array}$$

Therefore the first vertical arrow is an isomorphism.

e) *X, $\mathcal{F}$, Y, $\mathcal{G}$ are arbitrary.*

This case is treated exactly in the same way as d) by reversing the roles of $\mathcal{F}$, $\mathcal{G}$ and by using d) to get the isomorphism in the last two vertical arrows. □

(5.22) Corollary. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves over complex analytic schemes X, Y and let $\pi : X \times Y \to X$ be the projection. Suppose that $H^\bullet(Y, \mathcal{G})$ is Hausdorff.*

a) *If X is Stein, then $H^q(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \mathcal{F}(X) \,\widehat{\otimes}\, H^q(Y, \mathcal{G})$.*

b) *In general, for every open set $U \subset X$,*

$$\big(R^q \pi_\star(\mathcal{F} \boxtimes \mathcal{G})\big)(U) = \mathcal{F}(U) \,\widehat{\otimes}\, H^q(Y, \mathcal{G}).$$

c) *If $H^q(Y, \mathcal{G})$ is finite dimensional, then*

$$R^q \pi_\star(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes H^q(Y, \mathcal{G}).$$

Proof. a) Let $\mathcal{V} = (V_\alpha)$ be a countable Stein covering of Y. By the Leray theorem, $H^\bullet(Y, \mathcal{G})$ is equal to the cohomology of the Čech complex $C^\bullet(\mathcal{V}, \mathcal{G})$. Similarly $X \times \mathcal{V} = (X \times V_\alpha)$ is a Stein covering of $X \times Y$ and we have

$$H^q(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^q\big(C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G})\big).$$

However, Prop. 5.20 shows that $C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F}(X) \mathbin{\widehat{\otimes}} C^\bullet(\mathcal{V}, \mathcal{G})$. Our assumption that $C^\bullet(\mathcal{V}, \mathcal{G})$ has Hausdorff cohomology implies that the cocycle and coboundary groups are (nuclear) Fréchet spaces, and that each cohomology group can be computed by means of short exact sequences in this category. By Prop. 5.17, we thus get the desired equality

$$H^q\big(C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G})\big) = \mathcal{F}(X) \mathbin{\widehat{\otimes}} H^q\big(C^\bullet(\mathcal{V}, \mathcal{G})\big).$$

b) The presheaf $U \mapsto \mathcal{F}(U) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G})$ is in fact a sheaf, because the tensor product with the nuclear space $H^q(Y, \mathcal{G})$ preserves the exactness of all sequences

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_\alpha) \longrightarrow \prod \mathcal{F}(U_{\alpha\beta})$$

associated to arbitrary coverings (U_α) of U. Property b) thus follows from a) and from the fact that $R^q\pi_\star(\mathcal{F} \boxtimes \mathcal{G})$ is the sheaf associated to the presheaf $U \mapsto H^q(U \times Y, \mathcal{F} \boxtimes \mathcal{G})$.

c) is an immediate consequence of b), since the finite dimensionality of $H^q(Y, \mathcal{G})$ implies that this space is Hausdorff. □

(5.23) Künneth formula. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves over complex analytic schemes X, Y and suppose that the cohomology spaces $H^\bullet(X, \mathcal{F})$ and $H^\bullet(Y, \mathcal{G})$ are Hausdorff. Then there is an isomorphism*

$$\bigoplus_{p+q=k} H^p(X, \mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G}) \xrightarrow{\ \simeq\ } H^k(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

$$\bigoplus \alpha_p \otimes \beta_q \longmapsto \sum \alpha_p \smile \beta_q.$$

Proof. Consider the Leray spectral sequence associated to the coherent sheaf $\mathcal{S} = \mathcal{F} \boxtimes \mathcal{G}$ and to the projection $\pi : X \times Y \to X$. By Cor. 5.22 b) and a use of Čech cohomology, we find

$$E_2^{p,q} = H^p(X, R^q\pi_\star \mathcal{F} \boxtimes \mathcal{G}) = H^p(X, \mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G}).$$

It remains to show that the Leray spectral sequence degenerates in E_2. For this, we argue as in the proof of Th. IV-15.9. In that proof, we defined a morphism of the double complex $C^{p,q} = \mathcal{F}^{[p]}(X) \otimes \mathcal{G}^{[q]}(Y)$ into the double complex that defines the Leray spectral sequence (in IV-15.9, we only considered the sheaf theoretic external tensor product $\mathcal{F} \boxtimes \mathcal{G}$, but there is an obvious morphism of that one into the analytic tensor product). We get a morphism of spectral sequences which induces at the E_2-level the obvious morphism

$$H^p(X,\mathcal{F}) \otimes H^q(Y,\mathcal{G}) \longrightarrow H^p(X,\mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y,\mathcal{G}).$$

It follows that the Leray spectral sequence $E_r^{p,q}$ is obtained for $r \geq 2$ by taking the completion of the spectral sequence of $C^{\bullet,\bullet}$. Since this spectral sequence degenerates in E_2 by the algebraic Künneth theorem, the Leray spectral sequence also satisfies $d_r = 0$ for $r \geq 2$. □

(5.24) Remark. If X or Y is compact, the Künneth formula holds with $\otimes$ instead of $\widehat{\otimes}$, and the assumption that both cohomology spaces are Hausdorff is unnecessary. The proof is exactly the same, except that we use (5.22 c) instead of (5.22 b).

5.C. Modules over Nuclear Fréchet Algebras

Throughout this subsection, we work in the category of nuclear Fréchet spaces. Recall that a topological algebra (commutative, with unit element 1) is an algebra A together with a topological vector space structure such that the multiplication $A \times A \to A$ is continuous. A is said to be a Fréchet (resp. nuclear) algebra if it is Fréchet (resp. nuclear) as a topological vector space.

(5.25) Definition. *A (Fréchet, resp. nuclear) A-module E is a (Fréchet, resp. nuclear) space E with a A-module structure such that the multiplication $A \times E \to E$ is continuous. The module E is said to be nuclearly free if E is of the form $A \mathbin{\widehat{\otimes}} V$ where V is a nuclear Fréchet space.*

Assume that A is nuclear and let E be a nuclear A-module. A *nuclearly free resolution* $L_\bullet$ of E is an exact sequence of A-modules and continuous A-linear morphisms

$$(5.26) \qquad \cdots \longrightarrow L_q \xrightarrow{d_q} L_{q-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow E \longrightarrow 0$$

in which each L_q is a nuclearly free A-module. Such a resolution is said to be *direct* if each map d_q is direct, i.e. if $\operatorname{Im} d_q$ has a topological supplementary space in L_{q-1} (as a vector space over $\mathbb{R}$ or $\mathbb{C}$, not necessarily as a A-module).

(5.27) Proposition. *Every nuclear A-module E admits a direct nuclearly free resolution.*

Proof. We define the "standard resolution" of E to be

$$L_q = A \mathbin{\widehat{\otimes}} \ldots \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} E$$

where A is repeated $(q+1)$ times; the A-module structure of L_q is chosen to be the one given by the first factor and we set $d_0(a_0 \otimes x) = a_0 x$,

$$d_q(a_0\otimes\ldots\otimes a_q\otimes x)=\sum_{0\le i<q}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_q\otimes x$$
$$+(-1)^q a_0\otimes\ldots\otimes a_{q-1}\otimes a_qx.$$

Then there is a homotopy operator $h_q : L_q \to L_{q+1}$ given by $h_q(t) = 1\otimes t$ for all q (h_q, however, is not A-linear). This implies easily that $L_\bullet$ is a direct nuclearly free resolution. □

If E and F are two nuclear A-modules, we define $E\mathbin{\widehat{\otimes}}_A F$ to be

(5.28) $$E\mathbin{\widehat{\otimes}}_A F=\operatorname{coker}\big(E\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}F\xrightarrow{\ d\ }E\mathbin{\widehat{\otimes}}F\big)\quad\text{where}$$
$$d(x\otimes a\otimes y)=ax\otimes y-x\otimes ay.$$

Then $E\mathbin{\widehat{\otimes}}_A F$ is a A-module which it is not necessarily Hausdorff. If $E\mathbin{\widehat{\otimes}}_A F$ is Hausdorff, it is in fact a nuclear A-module by Prop. 5.14. If E is nuclearly free, say $E=A\mathbin{\widehat{\otimes}}V\simeq V\mathbin{\widehat{\otimes}}A$, we have $E\mathbin{\widehat{\otimes}}_A F=V\mathbin{\widehat{\otimes}}F$ (which is thus Hausdorff): indeed, there is an exact sequence

$$V\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}F\longrightarrow V\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}F\longrightarrow V\mathbin{\widehat{\otimes}}F\longrightarrow 0,$$
$$v\otimes a_0\otimes a_1\otimes x\longmapsto v\otimes a_0a_1\otimes x-v\otimes a_0\otimes a_1x,\qquad v\otimes a\otimes x\longmapsto v\otimes ax,$$

obtained by tensoring the standard resolution of F with $V\mathbin{\widehat{\otimes}}$; observe that the tensor product $\widehat{\otimes}$ with a nuclear space preserves exact sequences thanks to Prop. 5.17. We further define $\widehat{\operatorname{Tor}}{}_q^A(E,F)$ to be

(5.29) $$\widehat{\operatorname{Tor}}{}_q^A(E,F)=H_q(E\mathbin{\widehat{\otimes}}_A L_\bullet),$$

where $L_\bullet$ is the standard resolution of F. There is in fact an isomorphism

$$E\mathbin{\widehat{\otimes}}_A L_\bullet\xrightarrow{\ \simeq\ }E\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}\cdots\mathbin{\widehat{\otimes}}A\mathbin{\widehat{\otimes}}F$$
$$x\otimes_A(a_0\otimes a_1\otimes\ldots\otimes a_q\otimes y)\longmapsto a_0x\otimes a_1\otimes\ldots\otimes a_q\otimes y$$

where A is repeated q times in the target space. In this isomorphism, the differential becomes

$$d_q(x\otimes a_1\otimes\ldots\otimes a_q\otimes y)=a_1x\otimes a_2\otimes\ldots\otimes a_q\otimes y$$
$$+\sum_{1\le i<q}(-1)^i x\otimes a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_q\otimes y$$
$$+(-1)^q x\otimes a_1\otimes\ldots\otimes a_{q-1}\otimes a_qy.$$

In particular, we get $\widehat{\operatorname{Tor}}{}_0^A(E,F)=E\mathbin{\widehat{\otimes}}_A F$. Moreover, if we exchange the roles of E and F, we obtain a complex which is isomorphic to the above one up to the sign of d_q, hence $\widehat{\operatorname{Tor}}{}_q^A(E,F)\simeq\widehat{\operatorname{Tor}}{}_q^A(F,E)$. If $E=A\mathbin{\widehat{\otimes}}V$ is nuclearly free, the complex $E\mathbin{\widehat{\otimes}}_A L_\bullet=V\mathbin{\widehat{\otimes}}L_\bullet$ is exact, thus

$$E\text{ or }F\text{ nuclearly free}\Longrightarrow\widehat{\operatorname{Tor}}{}_q^A(E,F)=0\quad\text{for }q\ge 1.$$

(5.30) Proposition. *For any exact sequence* $0 \to E_1 \to E_2 \to E_3 \to 0$ *of nuclear* A*-modules and any nuclear* A*-module* F*, there is an* (*algebraic*) *exact sequence*

$$\begin{aligned}&\cdots \widehat{\mathrm{Tor}}_q^A(E_1,F) \longrightarrow \widehat{\mathrm{Tor}}_q^A(E_2,F) \longrightarrow \widehat{\mathrm{Tor}}_q^A(E_3,F) \longrightarrow \widehat{\mathrm{Tor}}_{q-1}^A(E_1,F)\cdots\\ &\longrightarrow \quad E_1 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow \quad E_2 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow \quad E_3 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow 0.\end{aligned}$$

Proof. As the standard resolution $L_\bullet \to F$ is nuclearly free, $L_q = A \mathbin{\widehat{\otimes}} V_q$ say, then $E_j \mathbin{\widehat{\otimes}_A} L_\bullet = E_j \mathbin{\widehat{\otimes}} V_\bullet$ for $j = 1, 2, 3$, so we have a short exact sequence of complexes

$$0 \longrightarrow E_1 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow E_2 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow E_3 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow 0. \qquad \square$$

(5.31) Corollary. *For any nuclearly free* (*possibly non direct*) *resolution* $L_\bullet$ *of* F*, there is a canonical isomorphism*

$$\widehat{\mathrm{Tor}}_q^A(E,F) \simeq H_q(E \mathbin{\widehat{\otimes}_A} L_\bullet).$$

Proof. Set $B_q = \mathrm{Im}(L_{q+1} \to L_q)$ for all $q \geq 0$ and $B_{-1} = F$. Then apply (5.30) to the short exact sequences $0 \to B_q \to L_q \to B_{q-1} \to 0$ and the fact that L_q is nuclearly free to get

$$\widehat{\mathrm{Tor}}_k^A(E, B_{q-1}) \simeq \begin{cases} \widehat{\mathrm{Tor}}_{k-1}^A(E, B_q) & \text{for } k > 1,\\ \ker(E \mathbin{\widehat{\otimes}_A} B_q \to E \mathbin{\widehat{\otimes}_A} L_q) & \text{for } k = 1.\end{cases}$$

Hence we obtain inductively

$$\begin{aligned}\widehat{\mathrm{Tor}}_q^A(E,F) &= \widehat{\mathrm{Tor}}_q^A(E,B_{-1}) \simeq \ldots \simeq \widehat{\mathrm{Tor}}_1^A(E,B_{q-2})\\ &\simeq \ker(E \mathbin{\widehat{\otimes}_A} B_{q-1} \to E \mathbin{\widehat{\otimes}_A} L_{q-1})\end{aligned}$$

and a commutative diagram

$$\begin{tikzcd}
E \mathbin{\widehat{\otimes}_A} L_{q+1} \arrow[rr] \arrow[dr] & & E \mathbin{\widehat{\otimes}_A} L_q \arrow[r] & E \mathbin{\widehat{\otimes}_A} B_{q-1} \arrow[r] & 0\\
& E \mathbin{\widehat{\otimes}_A} B_q \arrow[ur] & & &
\end{tikzcd}$$

in which the horizontal line is exact (thanks to the surjectivity of the left oblique arrow and the exactness of the sequence with $E \mathbin{\widehat{\otimes}_A} B_q$ as first term). Therefore $\ker(E \mathbin{\widehat{\otimes}_A} B_{q-1} \to E \mathbin{\widehat{\otimes}_A} L_{q-1})$ can be interpreted as the kernel of $E \mathbin{\widehat{\otimes}_A} L_q \to E \mathbin{\widehat{\otimes}_A} L_{q-1}$ modulo the image of $E \mathbin{\widehat{\otimes}_A} L_{q+1} \to E \mathbin{\widehat{\otimes}_A} L_q$, and this is is precisely the definition of $H_q(E \mathbin{\widehat{\otimes}_A} L_\bullet)$. $\square$

Now, we are ready to introduce the crucial concept of transversality.

(5.32) Definition. *We say that two nuclear A-modules E, F are transverse if $E \mathbin{\widehat{\otimes}}_A F$ is Hausdorff and if* $\widehat{\mathrm{Tor}}_q^A(E,F) = 0$ *for $q \ge 1$.*

For example, a nuclearly free A-module $E = A \mathbin{\widehat{\otimes}} V$ is transverse to any nuclear A-module F. Before proving further general properties, we give a fundamental example.

(5.33) Proposition. *Let X, Y be Stein spaces and let $U' \subset U \subset\subset X$, $V \subset\subset Y$ be Stein open subsets. If $\mathcal{F}$ is a coherent sheaf over $X \times Y$, then $\mathcal{O}(U')$ and $\mathcal{F}(U \times V)$ are transverse over $\mathcal{O}(U)$. Moreover*

$$\mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{F}(U \times V) = \mathcal{F}(U' \times V).$$

Proof. Let $\mathcal{L}_\bullet \to \mathcal{F}$ be a free resolution of $\mathcal{F}$ over $U \times V$; such a resolution exists by Cartan's theorem A. Then $\mathcal{L}_\bullet(U \times V)$ is a resolution of $\mathcal{F}(U \times V)$ which is nuclearly free over $\mathcal{O}(U)$, for $\mathcal{O}(U \times V) = \mathcal{O}(U) \mathbin{\widehat{\otimes}} \mathcal{O}(V)$; in particular, we get

$$\begin{aligned} &\mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{O}(U \times V) = \mathcal{O}(U') \mathbin{\widehat{\otimes}} \mathcal{O}(V) = \mathcal{O}(U' \times V), \\ &\mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{L}_\bullet(U \times V) = \mathcal{L}_\bullet(U' \times V). \end{aligned}$$

But $\mathcal{L}_\bullet(U' \times V)$ is a resolution of $\mathcal{F}(U' \times V)$, so

$$\widehat{\mathrm{Tor}}_q^{\mathcal{O}(U)}\big(\mathcal{O}(U'), \mathcal{F}(U \times V)\big) = \begin{cases} \mathcal{F}(U' \times V) & \text{for } q = 0, \\ 0 & \text{for } q \ge 1. \end{cases} \qquad \square$$

(5.34) Properties.

a) *If $0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of nuclear A-modules and if E_2, E_3 are transverse to F, then E_1 is transverse to F.*

b) *Let $A \to A_1 \to A_2$ be homomorphisms of nuclear algebras and let E be a nuclear A-module. if A_1 and A_2 are transverse to E over A, then A_2 is tranverse to $A_1 \mathbin{\widehat{\otimes}}_A E$ over A_1.*

c) *Let $E^\bullet$ be a complex of nuclear A-modules, bounded on the right side, and let M be a nuclear A-module which is transverse to all E^n. If $E^\bullet$ is acyclic in degrees $\ge k$, then $M \mathbin{\widehat{\otimes}}_A E^\bullet$ is also acyclic in degrees $\ge k$.*

d) *Let $E^\bullet$, $F^\bullet$ be complexes of nuclear A-modules, bounded on the right side. Let $f^\bullet : E^\bullet \to F^\bullet$ be a A-linear morphism and let M be a nuclear A-module which is transverse to all E^q and F^q. If $f^\bullet$ induces an isomorphism $H^q(f^\bullet) : H^q(E^\bullet) \to H^q(F^\bullet)$ in degrees $q \ge k$ and an epimorphism in degree $q = k - 1$, then*

$$\mathrm{Id}_M \mathbin{\widehat{\otimes}}_A f^\bullet : M \mathbin{\widehat{\otimes}}_A E^\bullet \to M \mathbin{\widehat{\otimes}}_A F^\bullet$$

has the same property.

Proof. a) is an immediate consequence of the Tôr exact sequence.

To prove b), we need only check that if A_1 is transverse to E over A, then

$$\mathrm{T\hat{o}r}_q^{A_1}(A_2, A_1 \widehat{\otimes}_A E) = \mathrm{T\hat{o}r}_q^{A}(A_2, E), \qquad \forall n \geq 0.$$

Indeed, if $L_\bullet = A \widehat{\otimes} V_\bullet$ is a nuclearly free resolution of E over A, then $A_1 \widehat{\otimes}_A L_\bullet = A_1 \widehat{\otimes} V_\bullet$ is a nuclearly free resolution of $A_1 \widehat{\otimes}_A E$ over A_1, since $H_q(A_1 \widehat{\otimes}_A L_\bullet) = \mathrm{T\hat{o}r}_q^{A}(A_1, E) = 0$ for $q \geq 1$. Hence

$$\begin{aligned}\mathrm{T\hat{o}r}_q^{A_1}(A_2, A_1 \widehat{\otimes}_A E) &= H_q\big(A_2 \widehat{\otimes}_{A_1} (A_1 \widehat{\otimes}_A L_\bullet)\big) = H_q\big(A_2 \widehat{\otimes}_{A_1} (A_1 \widehat{\otimes} V_\bullet)\big)\\ &= H_q(A_2 \widehat{\otimes} V_\bullet) = H_q(A_2 \widehat{\otimes}_A L_\bullet) = \mathrm{T\hat{o}r}_q^{A}(A_2, E).\end{aligned}$$

c) The short exact sequences $0 \to Z^q(E^\bullet) \hookrightarrow E^q \xrightarrow{d^q} Z^{q+1}(E^\bullet) \to 0$ show by backward induction on q that M is transverse to $Z^q(E^\bullet)$ for $q \geq k-1$. Hence for $q \geq k-1$ we obtain an exact sequence

$$0 \longrightarrow M \widehat{\otimes}_A Z^q(E^\bullet) \hookrightarrow M \widehat{\otimes}_A E^q \xrightarrow{d^q} M \widehat{\otimes}_A Z^{q+1}(E^\bullet) \longrightarrow 0,$$

which gives in particular $Z^q(M \widehat{\otimes}_A E^\bullet) = B^q(M \widehat{\otimes}_A E^\bullet) = M \widehat{\otimes}_A Z^q(E^\bullet)$ for $q \geq k$, as desired.

d) is obtained by applying c) to the *mapping cylinder* $C(f^\bullet)$, as defined in the following lemma (the proof is straightforward and left to the reader). □

(5.35) Lemma. *If $f^\bullet : E^\bullet \to F^\bullet$ is a morphism of complexes, the mapping cylinder $C^\bullet = C(f^\bullet)$ is the complex defined by $C^q = E^q \oplus F^{q-1}$ with differential*

$$\begin{pmatrix} d_E^q & 0 \\ -f^q & d_F^{q-1} \end{pmatrix} : E^q \oplus F^{q-1} \longrightarrow E^{q+1} \oplus F^q.$$

Then there is a short exact sequence $0 \to F^{\bullet-1} \to C^\bullet \to E^\bullet \to 0$ and the associated connecting homomorphism $\partial^q : H^q(E^\bullet) \to H^q(F^\bullet)$ is equal to $H^q(f^\bullet)$; in particular, $C^\bullet$ is acyclic in degree q if and only if $H^q(f^\bullet)$ is injective and $H^{q-1}(f^\bullet)$ is surjective. □

5.D. A-Subnuclear Morphisms and Perturbations

We now introduce a notion of nuclearity relatively to an algebra A. This notion is needed for example to describe the properties of the $\mathcal{O}(S)$-linear restriction map $\mathcal{O}(S \times U) \to \mathcal{O}(S \times U')$ when $U' \subset\subset U$.

(5.36) Definition. *Let E and F be Fréchet A-modules over a Fréchet algebra A and let $f : E \to F$ be a A-linear map. We say that*

a) *f is A-nuclear if there exist a scalar sequence (λ_j) with $\sum|\lambda_j| < +\infty$, an equicontinuous family of A-linears maps $\xi_j : E \to A$ and a bounded sequence y_j in F such that for all $x \in E$*

$$f(x) = \sum \lambda_j \, \xi_j(x) y_j .$$

b) *f is A-subnuclear if there exists a Fréchet A-module M and an epimorphism $p : M \to E$ such that $f \circ p$ is A-nuclear; if E is nuclear, we also require M to be nuclear.*

If $f : E \to F$ is A-nuclear and if $u : S \to E$ and $v : F \to T$ are continuous A-linear maps then $v \circ f \circ u$ is A-nuclear; the same is true for A-subnuclear maps. If V and W are nuclear spaces and if $u : V \to W$ is $\mathbb{C}$-nuclear, then $\mathrm{Id}_A \widehat{\otimes} u : A \widehat{\otimes} V \to A \widehat{\otimes} W$ is A-nuclear. From this we infer:

(5.37) Proposition. *Let S, Z be Stein spaces and let $U' \subset\subset U \subset\subset Z$ be Stein open subsets. Then the restriction $\rho : \mathcal{O}(S \times U) \to \mathcal{O}(S \times U')$ is $\mathcal{O}(S)$-nuclear. If $\mathcal{F}$ is a coherent sheaf over $Y \times Z$ with Y Stein and $S \subset\subset Y$, then the restriction map $\rho : \mathcal{F}(S \times U) \to \mathcal{F}(S \times U')$ is $\mathcal{O}(S)$-subnuclear.*

Proof. As $\mathcal{O}(S \times U) = \mathcal{O}(S) \widehat{\otimes} \mathcal{O}(U)$ and $\mathcal{O}(U) \to \mathcal{O}(U')$ is $\mathbb{C}$-nuclear, only the second statement needs a proof. By Cartan's theorem A, there exists a free resolution $\mathcal{L}_\bullet \to \mathcal{F}$ over $S \times U$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_0(S{\times}U) & \longrightarrow & \mathcal{F}(S{\times}U) \\ \rho\downarrow & & \downarrow\rho \\ \mathcal{L}_0(S{\times}U') & \longrightarrow & \mathcal{F}(S{\times}U') \end{array}$$

in which the top horizontal arrow is an $\mathcal{O}(S)$-epimorphism and the left vertical arrow is an $\mathcal{O}(S)$-nuclear map; its composition with the bottom horizontal arrow is thus also $\mathcal{O}(S)$-nuclear. □

Let $f : E \to F$ be a A-linear morphism of Fréchet A-modules. Suppose that $f(E) \subset F_1$ where F_1 is a *closed* A-submodule of F and let $f_1 : E \to F_1$ be the map induced by f. If f is A-nuclear, it is not true in general that f_1 is A-nuclear or A-subnuclear, even if A, E, F are nuclear. However:

(5.38) Proposition. *With the above notations, suppose A, E, F nuclear. Let B be a nuclear Fréchet algebra and let $\rho : A \to B$ be a $\mathbb{C}$-nuclear homomorphism. Suppose that B is transverse to E, F and F/F_1 over A. If $f : E \to F$ is A-subnuclear, then $\mathrm{Id}_B \widehat{\otimes}_A f_1 : B \widehat{\otimes}_A E \to B \widehat{\otimes}_A F_1$ is B-subnuclear.*

Proof. We first show that $\rho \widehat{\otimes}_A f_1 : E = A \widehat{\otimes}_A E \to B \widehat{\otimes}_A F_1$ is $\mathbb{C}$-nuclear. Since a quotient of a $\mathbb{C}$-nuclear map is $\mathbb{C}$-nuclear by Cor. 5.16 b), we may suppose for this that f is A-nuclear. Write

$$f(x) = \sum \lambda_j \, \xi_j(x) y_j, \qquad \xi_j : E \to A, \quad \sum |\lambda_j| < +\infty, \quad y_j \in F,$$
$$\rho(t) = \sum \mu_k \, \eta_k(t) b_k, \qquad \eta_k : A \to \mathbb{C}, \quad \sum |\mu_k| < +\infty, \quad b_k \in B$$

as in the definition of (A-)nuclearity. Then $\rho \mathbin{\widehat{\otimes}_A} f : E \longrightarrow B \mathbin{\widehat{\otimes}_A} F$ is $\mathbb{C}$-nuclear: for any $x \in E$, we have $\rho(\xi_j(x)) = \xi_j(x)\rho(1)$ in the A-module structure of B, hence

$$\begin{aligned} \rho \mathbin{\widehat{\otimes}_A} f(x) = \rho \otimes f(1 \otimes x) &= \sum \lambda_j \, \rho(\xi_j(x)) \mathbin{\widehat{\otimes}_A} y_j \\ &= \sum \lambda_j \mu_k \, (\eta_k \circ \xi_j)(x) \, b_k \mathbin{\widehat{\otimes}_A} y_j. \end{aligned}$$

By our transversality assumptions, $B \mathbin{\widehat{\otimes}_A} F_1$ is a closed subspace of $B \mathbin{\widehat{\otimes}_A} F$. As $\mathrm{Im}(\rho \mathbin{\widehat{\otimes}_A} f) \subset B \mathbin{\widehat{\otimes}_A} F_1$, the induced map $\rho \mathbin{\widehat{\otimes}_A} f_1 : E \to B \mathbin{\widehat{\otimes}_A} F_1$ is $\mathbb{C}$-nuclear by Cor. 5.16 a). Finally, there is a commutative diagram

$$\begin{array}{ccc} B \mathbin{\widehat{\otimes}} E & \xrightarrow{\mathrm{Id}_B \widehat{\otimes} (\rho \widehat{\otimes}_A f_1)} & B \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}_A} F_1) \\ \downarrow & & \downarrow \\ B \mathbin{\widehat{\otimes}_A} E & \xrightarrow{\mathrm{Id}_B \widehat{\otimes}_A f_1} & B \mathbin{\widehat{\otimes}_A} F_1 \end{array}$$

in which the vertical arrows are B-linear epimorphisms. The top horizontal arrow is B-nuclear by the $\mathbb{C}$-nuclearity of $\rho \mathbin{\widehat{\otimes}_A} f_1$, hence $\mathrm{Id}_B \mathbin{\widehat{\otimes}_A} f_1$ is B-subnuclear. □

Example 5.12 suggests the following definition (which is somewhat less general than some other in current use, but sufficient for our purposes).

(5.39) Definition. *We say that a Fréchet algebra A is fully nuclear if the topology of A is defined by an increasing family $(p_t)_{t \in]0,1[}$ of multiplicative semi-norms (that is, $p_t(xy) \le p_t(x)\, p_t(y)$), such that the Banach algebra homomorphism $\widehat{A}_{p_{t'}} \to \widehat{A}_{p_t}$ is nuclear for all $t < t' < 1$.*

If A is fully nuclear and $t \in]0,1]$, we define A_t to be the completion of A equipped with the family of semi-norms $p_{\lambda t}$, $\lambda \in]0,1[$. Then A_t is again a fully nuclear algebra, and for all $t < t' < 1$ the canonical map $A_{t'} \to A_t$ is nuclear: indeed, for $t \le u < u' < t'$, there is a factorization

$$A_{t'} \longrightarrow \widehat{A}_{p_{u'}} \longrightarrow \widehat{A}_{p_u} \longrightarrow A_t.$$

If E is a nuclear A-module, we say that E is fully A-transverse if E is transverse to all A_t over A. Then by 5.34 b), each nuclear space

$$(5.40) \quad E_t = A_t \mathbin{\widehat{\otimes}_A} E$$

is a fully A_t-transverse A_t-module. If $f : E \to F$ is a morphism of fully A-transverse nuclear modules, there is an induced map

(5.40′) $f_t = \mathrm{Id}_{A_t} \widehat{\otimes}_A f : E_t \longrightarrow F_t, \qquad \forall t \in \,]0,1].$

(5.41) Example. Let X be a closed analytic subscheme of an open set $\Omega \subset \mathbb{C}^N$, $D = D(a,R) \subset\subset \Omega$ a polydisk and $U = D \cap X$. We have an epimorphism $\mathcal{O}(D) \to \mathcal{O}(U)$. Denote by $\widetilde{p}_t$ the quotient semi-norm of $p_t(f) = \sup_{D(a,tR)} |f|$ on $\mathcal{O}(U)$. Then $\mathcal{O}(U)$ equipped with $(\widetilde{p}_t)_{t\in]0,1[}$ is a fully nuclear algebra, and $\mathcal{O}(U)_t = \mathcal{O}\big(D(a,tR) \cap X\big)$.

Now, let Y be a Stein space, $V \subset\subset Y$ a Stein open subset and $\mathcal{F}$ a coherent sheaf over $X \times Y$. Then Prop. 5.33 shows that $\mathcal{F}(U \times V)$ is a fully transverse nuclear $\mathcal{O}(U)$-module.

(5.42) Subnuclear perturbation theorem. *Let A be a fully nuclear algebra, let E and F be two fully A-transverse nuclear A-modules and let $f, u : E \to F$ be A-linear maps. Suppose that u is A-subnuclear and that f is an epimorphism. Then for every $t < 1$, the cokernel of*

$$f_t - u_t : E_t \longrightarrow F_t$$

is a finitely generated A_t-module (as an algebraic module; we do not assert that the cokernel is Hausdorff).

Proof. We argue in several steps. The first step is the following special case.

(5.43) Lemma. *Let B be a Banach algebra, S a Fréchet B-module and $v : S \to S$ a B-nuclear morphism. Then* $\mathrm{Coker}(\mathrm{Id}_S - v)$ *is a finitely generated B-module.*

Proof. Let $v(x) = \sum \lambda_j \, \xi_j(x) y_j$ be a B-nuclear decomposition of v. We have a factorization

$$v = \beta \circ \alpha : S \xrightarrow{\ \alpha\ } \ell^1(B) \xrightarrow{\ \beta\ } S$$

where $\alpha(x) = \big(\lambda_j \xi_j(x)\big)$ and $\beta(t_j) = \sum t_j y_j$. Set $w = \alpha \circ \beta : \ell^1(B) \to \ell^1(B)$. As α is B-nuclear, so is w, and α, β induce isomorphisms

$$\mathrm{Coker}(\mathrm{Id}_S - v) \ \underset{\widetilde{\beta}}{\overset{\widetilde{\alpha}}{\rightleftarrows}} \ \mathrm{Coker}\big(\mathrm{Id}_{\ell^1(B)} - w\big).$$

We are thus reduced to the case when S is a Banach module. Then we write $v = v' + v''$ with

$$v'(x) = \sum_{1 \le j \le N} \lambda_j \, \xi_j(x) y_j, \qquad v''(x) = \sum_{j > N} \lambda_j \, \xi_j(x) y_j.$$

For N large enough, we have $||v''|| < 1$, hence $\mathrm{Id}_S - v''$ is an automorphism and $\mathrm{Coker}(\mathrm{Id}_S - v' - v'')$ is generated by the classes of $y_1, \ldots, y_N$. $\square$

Proof of Theorem 5.42. a) We may suppose that E is nuclearly free and that u is A-nuclear, otherwise we replace f, u by their composition with $A \mathbin{\widehat{\otimes}} M \longrightarrow M \xrightarrow{p} E$, where M is nuclear and $p : M \to E$ is an epimorphism such that $u \circ p$ is A-nuclear.

b) As in (5.9), there is a A-nuclear decomposition $u(x) = \sum \lambda_j\, \xi_j(x) y_j$ where (y_j) converges to 0 in F. Since f is an epimorphism, we can find a sequence (x_j) converging to 0 in E such that $f(x_j) = y_j$. Hence we have $u = f \circ v$ where $v(x) = \sum \lambda_j\, \xi_j(x) x_j$ is a A-nuclear endomorphism of E, and the cokernel of $f - u$ is the image by f of the cokernel of $\mathrm{Id}_E - v$.

c) By a), b) we may suppose that $F = E = A \mathbin{\widehat{\otimes}} M$, $f = \mathrm{Id}_E$ and that u is A-nuclear. Let B be the Banach algebra $B = \widehat{A}_{p_t}$. Then $B \mathbin{\widehat{\otimes}}_A E = B \mathbin{\widehat{\otimes}} M$ is a Fréchet B-module and $\mathrm{Id}_B \mathbin{\widehat{\otimes}}_A u$ is B-nuclear. By Lemma 5.42, $\mathrm{Id}_B \mathbin{\widehat{\otimes}}_A \mathrm{Id}_E - \mathrm{Id}_B \mathbin{\widehat{\otimes}}_A u$ has a finitely generated cokernel over B. Now, there is an obvious morphism $B \to A_t$, hence by taking the tensor product with $A_t \mathbin{\widehat{\otimes}}_B \bullet$ we get

$$A_t \mathbin{\widehat{\otimes}}_B (B \mathbin{\widehat{\otimes}}_A E) = A_t \mathbin{\widehat{\otimes}}_B (B \mathbin{\widehat{\otimes}} M) = A_t \mathbin{\widehat{\otimes}} M = A_t \mathbin{\widehat{\otimes}}_A E = E_t$$

and we see that

$$\mathrm{Id}_{E_t} - u_t = \mathrm{Id}_{A_t} \mathbin{\widehat{\otimes}}_A \mathrm{Id}_E - \mathrm{Id}_{A_t} \mathbin{\widehat{\otimes}}_A u$$

has a finitely generated cokernel over A_t. □

5.E. Proof of the Direct Image Theorem

We first prove a functional analytic version of the result, which appears as a relative version of Schwartz' theorem 1.9.

(5.44) Theorem. *Let A be a fully nuclear algebra, $E^\bullet$ and $F^\bullet$ complexes of fully A-transverse nuclear A-modules. Let $f^\bullet : E^\bullet \to F^\bullet$ be a morphism of complexes such that each f^q is A-subnuclear. Suppose that $E^\bullet$ and $F^\bullet$ are bounded on the right and that $H^q(f^\bullet)$ is an isomorphism for each q. Then for every $t < 1$, there is a complex $L^\bullet$ of finitely generated free A_t-modules and a complex morphism $h^\bullet : L^\bullet \to E_t^\bullet$ which induces an isomorphism on cohomology.*

Proof. a) We first show the following statement:

> *Suppose that $E_t^\bullet$ and $F_t^\bullet$ are acyclic in degrees $> q$. Then for every $t' < t$, the cohomology space $H^q(E_{t'}^\bullet) \simeq H^q(F_{t'}^\bullet)$ is a finitely generated $A_{t'}$-module.*

Indeed, the exact sequences $0 \to Z^k(E_t^\bullet) \to E_t^k \to Z^{k+1}(E_t^\bullet) \to 0$ show by backward induction on k that $Z^k(E_t^\bullet)$ is fully A_t-transverse for $k \geq q$. The same is true for $Z^k(F_t^\bullet)$. Then f_t^q is a A_t-subnuclear map from

$Z^q(E_t^\bullet)$ into F_t^q, and its image is contained in the closed subspace $Z^q(F_t^\bullet)$. By Prop. 5.38, for all $t'' < t$, the map $f_{t''}^q = \mathrm{Id}_{A_{t''}} \widehat{\otimes}_{A_t} f_t^q$ is a $A_{t''}$-subnuclear map $Z^q(E_{t''}^\bullet) \to Z^q(F_{t''}^\bullet)$. By Prop. 5.34 d), $H^\bullet(f_{t''}^\bullet)$ is an isomorphism in all degrees, hence

$$d_{t''}^q \oplus f_{t''}^q : F_{t''}^{q-1} \oplus Z^q(E_{t''}^\bullet) \longrightarrow Z^q(F_{t''}^\bullet)$$

is surjective. By the subnuclear perturbation theorem, the map

$$d_{t'}^q \oplus 0 = \mathrm{Id}_{A_{t'}} \widehat{\otimes}_{A_{t''}} \big((d_{t''}^q \oplus f_{t''}^q) - (0 \oplus f_{t''}^q)\big)$$

has a finitely generated $A_{t'}$-cokernel for $t' < t'' < t$, as desired.

b) Let N be an index such that $E^k = F^k = 0$ for $k > N$. Fix a sequence $t < \ldots < t_q < t_{q+1} < \ldots < t_N < 1$. To prove the theorem, we construct by backward induction on q a finitely generated free module L^q over A_{t_q} and morphisms $d^q : L^q \to L_{t_q}^{q+1}$, $h^q : L^q \to E_{t_q}^q$ such that

i) *$L_{\geq q,\, t_q}^\bullet : 0 \to L^q \to L_{t_q}^{q+1} \to \cdots \to L_{t_q}^N \to 0$ is a complex and $h_{\geq q,\, t_q}^\bullet : L_{\geq q,\, t_q}^\bullet \to E_{t_q}^\bullet$ is a complex morphism.*

ii) *The mapping cylinder $M_q^\bullet = C(h_{\geq q,\, t_q}^\bullet)$ defined by $M_q^k = \bigoplus_{k\in\mathbb{Z}} \big(L_{\geq q,\, t_q}^k \oplus E_{t_q}^{k-1}\big)$ is acyclic in degrees $k > q$.*

Suppose that L^k has been constructed for $k \geq q$. Consider the mapping cylinder $N_q^\bullet = C(f_{t_q}^\bullet \circ h_{\geq q,\, t_q}^\bullet)$ and the complex morphism

$$M_q^\bullet \longrightarrow N_q^\bullet, \qquad L_{\geq q,\, t_q}^k \oplus E_{t_q}^{k-1} \longrightarrow L_{\geq q,\, t_q}^k \oplus F_{t_q}^{k-1}$$

given by $\mathrm{Id} \oplus f_{t_q}^{k-1}$. This morphism is A_{t_q}-subnuclear in each degree and induces an isomorphism in cohomology (compare the cohomology of the short exact sequences associated to each mapping cylinder, with the obvious morphism between them). Moreover, $M_q^\bullet$ and $N_q^\bullet$ are acyclic in degrees $k > q$. By step a), the cohomology space $H^q(M_{q,\, t_{q-1}}^\bullet)$ is a finitely generated $A_{t_{q-1}}$-module. Therefore, we can find a finitely generated free $A_{t_{q-1}}$-module L^{q-1} and a morphism

$$d^{q-1} \oplus h^{q-1} : L^{q-1} \to M_{q,\, t_{q-1}}^q = L_{t_{q-1}}^q \oplus E_{t_{q-1}}^{q-1}$$

such that the image is contained in $Z^q(M_{q,\, t_{q-1}}^\bullet)$ and generates the cohomology space $H^q(M_{q,\, t_{q-1}}^\bullet)$. As $M_{q,\, t_{q-1}}^{q-1} = E_{t_{q-1}}^{q-2}$, this means that $M_{q-1}^\bullet$ is also acyclic in degree q. Thus L^{q-1}, together with the maps (d^{q-1}, h^{q-1}) satisfies the induction hypotheses for $q-1$, and $L_t^\bullet$ together with the induced map $h_t^\bullet : L_t^\bullet \to E_t^\bullet$ is the required morphism of complexes. □

Proof of theorem 5.1. Let X, Y be complex analytic schemes, let $F : X \to Y$ be a proper analytic morphism and let $\mathcal{S}$ be a coherent sheaf over X. Fix a point $y_0 \in Y$, a neighborhood of y_0 which is isomorphic to a closed analytic subscheme of a Stein open set $W \subset \mathbb{C}^n$ and a polydisk $D^0 = D(y_0, R_0) \subset\subset W$.

The compact set $K = F^{-1}(\overline{D}^0 \cap Y)$ can be covered by finitely many open subsets $U^0_\alpha \subset\subset X$ which possess embeddings as closed analytic subschemes of Stein open sets $\Omega^0_\alpha \subset \mathbb{C}^{N_\alpha}$. Let $\Omega'_\alpha \subset\subset \Omega_\alpha \subset\subset \Omega^0_\alpha$ be Stein open subsets such that $U_\alpha = U^0_\alpha \cap \Omega_\alpha$ and $U'_\alpha = U^0_\alpha \cap \Omega'_\alpha$ still cover K. Let $i_\alpha : U^0_\alpha \to \Omega^0_\alpha$ and $j : Y \cap D^0 \to D^0$ be the embeddings and $\mathcal{S}_\alpha = \big(i_\alpha \times (j \circ F)\big)_\star \mathcal{S}$ the image sheaf of $\mathcal{S}$ on $\Omega^0_\alpha \times D^0$. Let $D \subset\subset D^0$ be a concentric polydisk. Then $\mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) = \mathcal{S}_\alpha(\Omega_\alpha \times D)$ is a fully transverse $\mathcal{O}(D)$-module by Ex. 5.41, and so is $\mathcal{S}\big(U'_\alpha \cap F^{-1}(D)\big) = \mathcal{S}_\alpha(\Omega'_\alpha \times D)$. Moreover, the restriction map

$$\mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) \longrightarrow \mathcal{S}\big(U'_\alpha \cap F^{-1}(D)\big)$$

is $\mathcal{O}(D)$-subnuclear by Prop. 5.37 applied to $\mathcal{F} = \mathcal{S}_\alpha$. For every Stein open set $V \subset D$, Prop. 5.33 shows that

$$\mathcal{O}(V)\, \widehat{\otimes}_{\mathcal{O}(D)}\, \mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) = \mathcal{S}\big(U_\alpha \cap F^{-1}(V)\big).$$

Denote by $\mathcal{U} \cap F^{-1}(D)$ the collection $\big(U_\alpha \cap F^{-1}(D)\big)$ and use a similar notation with $\mathcal{U}' = (U'_\alpha)$. As $\mathcal{U} \cap F^{-1}(D)$, $\mathcal{U}' \cap F^{-1}(D)$ are Stein coverings of $F^{-1}(D)$, the Leray theorem applied to the alternate Čech complex of $\mathcal{S}$ over $\mathcal{U} \cap F^{-1}(D)$ and $\mathcal{U}' \cap F^{-1}(D)$ gives an isomorphism

$$H^\bullet\big(AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S})\big) = H^\bullet\big(AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})\big) = H^\bullet\big(F^{-1}(D), \mathcal{S}\big).$$

By the above discussion, $AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S})$ and $AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})$ are finite complexes of fully transverse nuclear $\mathcal{O}(D)$-modules, the restriction map

$$AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S}) \longrightarrow AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})$$

is $\mathcal{O}(D)$-subnuclear and induces an isomorphism on cohomology groups. Set $D = D(y_0, R)$ and $D_t = D(y_0, tR)$. Theorem 5.44 shows that for every $t < 1$ there is a complex of finitely generated free $\mathcal{O}$-modules $\mathcal{L}^\bullet$ and a $\mathcal{O}(D_t)$-linear morphism of complexes

$$\mathcal{L}^\bullet(D_t) \to AC^\bullet(\mathcal{U} \cap F^{-1}(D_t), \mathcal{S})$$

which induces an isomorphism on cohomology. Let $V \subset D_t$ be an arbitrary Stein open set. By Prop. 5.34 d) applied with $M = \mathcal{O}(V)$, we conclude that $\mathcal{L}^\bullet(V) \to AC^\bullet(\mathcal{U} \cap F^{-1}(V), \mathcal{S})$ induces an isomorphism on cohomology. If we take the direct limit as V runs over all Stein neighborhoods of a point $y \in Y \cap D_t$, we see that $\mathcal{H}^q(\mathcal{L}^\bullet) \simeq R^q F_\star \mathcal{S}$ over $Y \cap D_t$, hence $R^q F_\star \mathcal{S}$ is $\mathcal{O}_Y$-coherent near y_0. □