CHAPTER 1

Algebras of operators in a finite dimensional Hilbert space

In this chapter, we consider linear operators in a complex finite dimensional vector space, which will be usually denoted by V. We will also denote the identity operator in this space by I.

Although the main goal of this book lies in applications of algebras of operators in infinite-dimensional Hilbert spaces, it is good to understand first everything which is possible for the finite-dimensional case, even for the case $\dim V = 2$. I quote the preface of Ju.I. Ljubič to an excellent book by I.M. Glazman and Ju.I. Ljubič [**GL**]:

> *When somebody talked to him* [I.M. Glazman] *about complicated infinite-dimensional constructions, he usually asked: "And how does that look in the two-dimensional case?" Often enough that shocking question helped to a better understanding of the mathematical situation.*

Unfortunately a very small part of the von Neumann algebras technique exists in the finite-dimensional case. Nevertheless even this small part should not be neglected and will be discussed in this chapter.

1.1. Trace

Let V be a finite-dimensional vector space over $\mathbb{C}$. The set of all linear operators $A : V \to V$ is an algebra denoted by $\mathcal{L}(V)$. The operations in this algebra are defined as follows:

$$(A_1 + A_2)x := A_1x + A_2x, \qquad (A_1A_2)x = A_1(A_2x), \qquad (\lambda A)x = \lambda(Ax),$$
$$\text{where } A, A_1, A_2 \in \mathcal{L}(V),\ x \in V,\ \lambda \in \mathbb{C}.$$

A basis $\{e_1, \dots, e_n\}$ in V gives an isomorphism of $\mathcal{L}(V)$ and the space of all $(n \times n)$ matrices over $\mathbb{C}$,

$$M_n(\mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix},\ a_{ij} \in \mathbb{C} \right\}.$$

Explicitly, this isomorphism is given by the formulas

$$V \cong \mathbb{C}^n\,, \text{where } \ V \ni x \leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ x_j \in \mathbb{C}, \text{if } x = x_1 e_1 + \cdots + x_n e_n,$$

$$\text{If } Ae_i = \sum_{k=1}^{n} a_{ki} e_k, \qquad \text{then } A \leftrightarrow \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Composition of operators in $\mathcal{L}(V)$ corresponds to matrix multiplication in $M_n(\mathbb{C})$

$$A = A_1 A_2 \Rightarrow a_{ij} = \sum_{k=1}^{n} a_{ik}^{(1)} a_{kj}^{(2)} \text{ if } A_1 \leftrightarrow \left(a_{ij}^{(1)}\right), \ A_2 \leftrightarrow \left(a_{ij}^{(2)}\right) \text{ and } A \leftrightarrow (a_{ij}).$$

Definition. The *trace* $\mathrm{Tr} : \mathcal{L}(V) \to \mathbb{C}$ is given by the formula

$$\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}.$$

Exercise 1. (a) $\mathrm{Tr}(A_1 A_2) = \mathrm{Tr}(A_2 A_1), \ A_1, A_2 \in \mathcal{L}(V)$.

(b) Tr does not depend on the choice of the basis in V.

(c) Let $f : \mathcal{L}(V) \to \mathbb{C}$ be a linear functional which has the property

$$f(A_1 A_2) = f(A_2 A_1), \quad A_1, A_2 \in \mathcal{L}(V).$$

Then $f = c\,\mathrm{Tr}$ with a constant $c \in \mathbb{C}$.

Hint to (c). Consider $[\mathcal{L}(V), \mathcal{L}(V)] = \{$linear combinations of $[A, B]$ with $A, B \in \mathcal{L}(V), [A, B] = AB - BA\} \subset \mathcal{L}(V)$. Check that $\dim_{\mathbb{C}} \mathcal{L}(V)/[\mathcal{L}(V), \mathcal{L}(V)] = 1$ (or $\mathrm{codim}_{\mathbb{C}}[\mathcal{L}(V), \mathcal{L}(V)] = 1$) by considering $E_{ij} \in \mathcal{L}(V)$ such that $E_{ij} e_j = e_i$, $E_{ij} e_k = 0$ if $k \neq j$ for a given basis $\{e_1, \dots, e_n\}$.

Hence the linear functional in (c) is unique up to a constant factor.

Exercise 2. Let $\mathrm{GL}(V)$ denote the group of all invertible linear generators in V. Prove that if a linear functional $f : \mathcal{L}(V) \to \mathbb{C}$ satisfies

$$f(C^{-1} A C) = f(A), \ A \in \mathcal{L}(V), C \in \mathrm{GL}(V),$$

then $f = c\,\mathrm{Tr}$.

Hint. Check that the condition in Exercise 1 (c) is satisfied for all invertible A_1, hence for all A_1.

1.2. Projections

Definition 1. $P \in \mathcal{L}(V)$ is called a *projection* if $P^2 = P$.

Definition 2. For $A \in \mathcal{L}(V)$ denote

$$\begin{aligned} \mathrm{Ker}\, A &= \{x | x \in V, Ax = 0\} && \text{(kernel of } A\text{)} \\ \mathrm{Im}\, A &= \{Ax | x \in V\} && \text{(range or image of } A\text{)} \\ \mathrm{Coker}\, A &= V/\,\mathrm{Im}\, A && \text{(cokernel of A)} \end{aligned}$$

Exercise. (a) $P \in \mathcal{L}(V)$ is a projection iff $I - P$ is a projection. (Here I is the identity operator in V).

(b) If $P \in \mathcal{L}(V)$ is a projection then V is decomposed into a direct set of subspaces $V = \operatorname{Im} P \dot{+} \operatorname{Ker} P$. (Here $V = V_1 \dot{+} V_2$ means that V_1, V_2 are subspaces in V such that $V_1 \cap V_2 = \{0\}$, $V_1 + V_2 = V$, where $V_1 + V_2 = \{x_1 + x_2 | x_1 \in V_1, x_2 \in V_2\}$.) Moreover, $P \mid \operatorname{Im} P = I \mid \operatorname{Im} P$ and $P \mid \operatorname{Ker} P = 0 \mid \operatorname{Ker} P$. Besides $\operatorname{Im}(I - P) = \operatorname{Ker} P$, $\operatorname{Ker}(I - P) = \operatorname{Im} P$.

Vice versa if $V = V_1 \dot{+} V_2$ and a linear operator $P \in \mathcal{L}(V)$ is given by $P(x_1 + x_2) = x_1$ for any $x_i \in V_i$, $i = 1, 2$, then P is a projection with $V_1 = \operatorname{Im} P = \operatorname{Ker}(I - P)$, $V_2 = \operatorname{Ker} P = \operatorname{Im}(I - P)$, and $(I - P)(x_1 + x_2) = x_2$ for any $x_i \in V_i$, $i = 1, 2$.

(c) If P is a projection then

$$\operatorname{Tr} P = \dim \operatorname{Im} P.$$

1.3. Inner product

Definition 1. An *inner product* (or *scalar product*) in a finite dimensional vector space V is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ which satisfies the following conditions:

(i) *It is linear with respect to the 1st variable:*

$$(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 (x_1, y) + \lambda_2 (x_2, y), \ \ x_1, x_2, y \in V, \lambda_1, \lambda_2 \in \mathbb{C};$$

(ii) *It is hermitian:* $(x, y) = \overline{(y, x)}$ $(x, y \in V)$
(Hence, it is antilinear with respect to the 2nd variable:
$(x, \lambda_1 y_1 + \lambda_2 y_2) = \bar{\lambda}_1 (x, y_1) + \bar{\lambda}_2 (x, y_2)$, $x, y_1, y_2 \in V$, $\lambda_1, \lambda_2 \in \mathbb{C}$);

(iii) *It is positive definite:* $(x, x) \geq 0$, $x \in V$; $(x, x) = 0$ if and only if $x = 0$.

The pair $(V, (.,.))$ is called a finite-dimensional Hilbert space.

Definition 2. (a) For $A \in \mathcal{L}(V)$ the *adjoint operator* $A^* \in \mathcal{L}(V)$ is defined by the formula

$$(Ax, y) = (x, A^* y), \ \ x, y \in V.$$

Obviously $(A^*)^* = A$, so $A \mapsto A^*$ is an involution on $\mathcal{L}(V)$.

(b) $A \in \mathcal{L}(V)$ is called *self-adjoint* if $A^* = A$.

(c) $U \in \mathcal{L}(V)$ is called *unitary* if $U^* U = U U^* = I$ (or $U^* = U^{-1}$).

(d) $x, y \in V$ are called *orthogonal* (we write $x \perp y$) if $(x, y) = 0$.
Subsets $X, Y \subset V$ are called *orthogonal* (denoted again $X \perp Y$) if $(x, y) = 0$ for every $x \in X$, $y \in Y$.

Definition 3. A projection P is called *orthogonal* if $P^* = P$ (that is, if P is self-adjoint).

Exercise. (a) A projection P in V is orthogonal if and only if $\operatorname{Im} P \perp \operatorname{Ker} P$, or equivalently, if V can be decomposed into an orthogonal direct sum $V = V_1 \oplus V_2$ so that $P \mid V_1 = I \mid V_1$, $P \mid V_2 = 0$

(b) There is a one-one correspondence

$$\underset{\substack{\text{subspaces}\\ \text{of } V\\ \text{(grassmanian of } V)}}{\operatorname{Gr}(V)} \cong \underset{\substack{\text{orthogonal}\\ \text{projections}\\ \text{in } V}}{\operatorname{Proj}(\mathcal{L}(V))}$$

given by

$$\operatorname{Im} P \leftrightarrow P.$$

1.4. Spectral theorem for self-adjoint operators

Assume that we are given an inner product on V. Let $A \in \mathcal{L}(V)$ be self-adjoint, that is, $A^* = A$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ in V such that $Ae_j = \lambda_j e_j$, $\lambda_j \in \mathbb{R}$, i.e. e_j are normalized eigenvectors of A and λ_j are its eigenvalues.

In the correspondence $\mathbb{C}^n \cong V$

$$(x_1, \ldots, x_n) \leftrightarrow x_1 e_1 + \ldots x_n e_n \quad (Ae_j = \lambda_j e),$$

the operator A corresponds to the matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

or, equivalently, to the operator $(x_1, \ldots, x_n) \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n)$. If we present $\mathbb{C}^n$ as complex valued functions on the finite set $\{1, 2, \ldots, n\}$, then A becomes a multiplication operator by $\lambda(i) = \lambda_i$.

Note that we will always consider the following standard inner product in $\mathbb{C}^n$:

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 \bar{y}_1 + \ldots x_n \bar{y}_n,$$

where the bar means the complex conjugation in $\mathbb{C}$.

Let $V_\lambda = \operatorname{span}\{e_i | \lambda_i \leq \lambda\}$, and E_λ denote the corresponding orthogonal projection (which has V_λ as its range).

Generally if $\Delta \subset \mathbb{R}$ then denote $V(\Delta) = \operatorname{span}\{e_i | \lambda_i \in \Delta\}$, and let $E(\Delta)$ denotes the corresponding orthogonal projection. Then $E_\lambda = E((-\infty, \lambda])$.

Exercise. Check that

$$\begin{aligned} A &= \sum_{i=1}^{n} \lambda_i E(\{\lambda_i\}) = \sum_{i=1}^{n} \lambda_i \left(E_{\lambda_i} - E_{\lambda_{i-1}}\right) \\ &= \int_{-\infty}^{\infty} \lambda \, dE_\lambda \qquad \text{(Stieltjes integral)} \end{aligned}$$

where $\lambda_0 < \min\limits_{1 \leq i \leq n} \lambda_i$.

1.5. Spectral theorem for unitary operators

Exercise 1. The following conditions are equivalent:

(i) U is unitary, that is $U^* = U^{-1}$.
(ii) There is an orthonormal basis of eigenvectors with eigenvalues μ_i, such that $|\mu_i| = 1$.
(iii) U can be represented as the multiplication operator by μ, $\mu(i) = \mu_i$, $|\mu(i)| = 1$, in the space of functions on $\{1, 2, \ldots, n\}$.
(iv) $U = \int_0^{2\pi} e^{i\lambda} \, dE_\lambda$, where E_λ is the orthogonal projection on the linear span of eigenvectors of U with eigenvalues $\mu_j = e^{i\lambda_j}$ with $\lambda_j \in [0, \lambda]$, $0 \leq \lambda < 2\pi$.

Exercise 2. If $f : \mathcal{L}(V) \to \mathbb{C}$ is a linear functional such that $f(U^{-1}AU) = f(A)$ for any $A \in \mathcal{L}(V)$, U unitary, then $f = c\,\mathrm{Tr}$ with a constant $c \in \mathbb{C}$.

Hint. This can be proved directly by decomposing A into a sum of two self-adjoint operators and then using the spectral theorem, or by using the result of the following Exercise.

EXERCISE 3. Every linear operator $C \in \mathcal{L}(V)$ can be represented as a linear combination of 4 unitary ones.

Hint. Present $C = C_1 + iC_2$, where $C_1^* = C_1$, $C_2^* = C_2$, $C_1 = \frac{1}{2}(C + C^*)$, $C_2 = \frac{1}{2i}(C - C^*)$ and then present any self-adjoint operator as a linear combination of two unitary operators by using the Spectral Theorem (see 1.4).

1.6. Algebras and commutant

Assume that $\mathcal{M} \subset \mathcal{L}(V)$ is a subalgebra.

DEFINITION 1. $\mathcal{M}$ is called *$*$-algebra* if $A \in \mathcal{M} \Rightarrow A^* \in \mathcal{M}$.

DEFINITION 2. The *commutant* of $\mathcal{M}$ is

$$\mathcal{M}' = \{B \in \mathcal{L}(V) | BA = AB \text{ for every } A \in \mathcal{M}\}.$$

We will define the commutant $\mathcal{M}'$ by the same formula for any subset $\mathcal{M} \subset \mathcal{L}(V)$ (and not only a subalgebra).

DEFINITION 3. The *center* of a subalgebra $\mathcal{M}$ is

$$Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' = \{B \in \mathcal{M} | AB = BA \text{ for every } A \in \mathcal{M}\}.$$

EXERCISE 1. Prove that $Z(\mathcal{L}(V)) = \{\text{scalar operators}\} = \{\lambda I | \lambda \in \mathbb{C}\} = \mathcal{L}(V)'$.

EXAMPLES OF SUBALGEBRAS AND COMMUTANTS.
Let $V = V_1 \oplus V_2$. We will give several examples of subalgebras in $\mathcal{L}(V)$.

$$\begin{aligned} \mathcal{M}_1 &= \{A \in \mathcal{L}(V) | AV_1 \subset V_1, AV_2 \subset V_2\} \\ &= \{A \in \mathcal{L}(V) | V_1, V_2 \text{ are invariant subspaces for } A\}. \end{aligned} \tag{1}$$

In block-matrix terms.

$$\mathcal{M}_1 = \left\{ \begin{array}{cc} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} & \begin{array}{c} V_2 \\ V_1 \end{array} \\ \begin{array}{cc} V_1 & V_2 \end{array} & \end{array} \text{ where } A_j \in \mathcal{L}(V_j) \right\}.$$

Hence $\mathcal{M}_1 = \mathcal{L}(V_1) \oplus \mathcal{L}(V_2)$.

EXERCISE 2. Check that

$$\mathcal{M}_1' = \left\{ \begin{pmatrix} \lambda_1 I_1 & 0 \\ 0 & \lambda_2 I_2 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{C} \right\} = \{\lambda_1 I_1 \oplus \lambda_2 I_2 | \lambda_j \in \mathbb{C}\},$$

where I_j is the identity operator on V_j, $j = 1, 2$. In particular $\mathcal{M}_1' \subset \mathcal{M}_1$, hence $Z(\mathcal{M}_1) = \mathcal{M}_1'$.

$$\mathcal{M}_2 = \left\{ \begin{array}{cc} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} V_2 \\ V_1 \end{array} \\ \begin{array}{cc} V_1 & V_2 \end{array} & \end{array}, A_1 \in \mathcal{L}(V_1) \right\} = \{A \in \mathcal{L}(V) | AV_1 \subset V_1, AV_2 = \{0\}\}. \tag{2}$$

Clearly, $\mathcal{M}_2 \not\ni I$.

EXERCISE 3. Check that $\mathcal{M}_2' = \left\{ \begin{pmatrix} \lambda I_1 & 0 \\ 0 & A_2 \end{pmatrix} | \lambda \in \mathbb{C},\ A_2 \in \mathcal{L}(V_2) \right\}$ and $Z(\mathcal{M}_2) = \left\{ \begin{pmatrix} \lambda I_1 & 0 \\ 0 & 0 \end{pmatrix} | \lambda \in \mathbb{C} \right\}$.

Note that $\mathcal{M}_1, \mathcal{M}_2$ are both $*$-algebras.

(3)
$$\mathcal{M}_3 = \left\{ \begin{array}{cc} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} & \begin{array}{c} V_1 \\ V_2 \end{array} \\ \begin{array}{cc} V_1 & V_2 \end{array} & \end{array} \right\} = \{A \in \mathcal{L}(V) | AV_1 \subset V_1\}.$$

This is not a $*$-algebra.

EXERCISE 4. Check that $\mathcal{M}_3' = \{\lambda I | \lambda \in \mathbb{C}\}$, that is, $\mathcal{M}_3' = (\mathcal{L}(V))'$ (but at the same time $\mathcal{M}_3 \neq \mathcal{L}(V)$).

(4) Let $V_1 = V_2$, so $V = V_1 \oplus V_1$. For $A_1 \in \mathcal{L}(V_1)$ define $\hat{A}_1 = A_1 \oplus A_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$ and $\mathcal{M}_4 = \{\hat{A}_1 | A_1 \in \mathcal{L}(V_1)\}$. Clearly $\mathcal{M}_4$ is a $*$-algebra, $I \in \mathcal{M}_4$.

EXERCISE 5. Prove that
$$\mathcal{M}_4' = \left\{ \begin{pmatrix} \lambda_{11} I & \lambda_{12} I \\ \lambda_{21} I & \lambda_{22} I \end{pmatrix}, \lambda_{ij} \in \mathbb{C} \right\} \text{ or } \mathcal{M}_4' = \{\lambda \otimes I | \lambda \in M_2(\mathbb{C})\}$$
in the representation $V = \mathbb{C}^2 \otimes V_1$. Hence $Z(\mathcal{M}_4) = \{\lambda I | \lambda \in \mathbb{C}\}$.

1.7. Double commutant

Note that $\mathcal{M}'$ is always a subalgebra with $I \in \mathcal{M}$. Define $\mathcal{M}'' = (\mathcal{M}')'$. Clearly $\mathcal{M} \subset \mathcal{M}''$. What are conditions of coincidence of $\mathcal{M}$ and $\mathcal{M}''$?

THEOREM (Double Commutant Theorem). *Let $\mathcal{M}$ be a $*$-algebra, $\mathcal{M} \subset \mathcal{L}(V)$ and $\mathcal{M} \ni I$. Then $\mathcal{M}'' = \mathcal{M}$.*

LEMMA. *Define $\mathcal{M}''' = (\mathcal{M}'')'$, $\mathcal{M}^{IV} = (\mathcal{M}''')'$ etc. Then*
$$\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^V = \dots$$
and
$$\mathcal{M}'' = \mathcal{M}^{IV} = \dots$$
for any subset $\mathcal{M} \subset \mathcal{L}(V)$.

PROOF. It is sufficient to prove that $\mathcal{M}''' = \mathcal{M}'$; then we can apply this result to $\mathcal{M}', \mathcal{M}'', \dots$ to get all other relations. Obviously $\mathcal{M}' \subset \mathcal{M}'''$. On the other hand $\mathcal{M} \subset \mathcal{M}''$ implies $\mathcal{M}' \supset \mathcal{M}'''$, hence $\mathcal{M}' = \mathcal{M}'''$. □

PROOF OF THEOREM. 1) Choose $x \in V$ and consider $\mathcal{M}x = \{Ax | A \in \mathcal{M}\}$, which is a linear subspace in V. Consider the orthogonal projection P with $\operatorname{Im} P = \mathcal{M}x$. Clearly $\mathcal{M}x$ is $\mathcal{M}$-invariant, that is, it is A-invariant for every $A \in \mathcal{M}$. Therefore it is also A^*-invariant for every $A \in \mathcal{M}$, because $\mathcal{M}$ is a $*$-algebra. It follows that $AP = PA$ for every $A \in \mathcal{M}$. This means that $P \in \mathcal{M}' = \mathcal{M}'''$, hence $PB = BP$ for all $B \in \mathcal{M}''$. Therefore $\mathcal{M}x$ is $\mathcal{M}''$-invariant. This implies that $\mathcal{M}''x \subset \mathcal{M}x$, hence $\mathcal{M}''x = \mathcal{M}x$.

2) We have proved that $\forall x \in V$, $\forall B \in \mathcal{M}''$, $\exists A \in \mathcal{M}$ such that $Bx = Ax$. It would be good to do it not only for a single vector x but for a N-tuple $x_1, \dots, x_N$ (e.g. for a basis). This is done by the following trick.

Consider the algebra $\mathcal{M} \otimes I$ in $V \otimes \mathbb{C}^N$, so

$$\mathcal{M} \otimes I = \left\{ \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}, A \in \mathcal{M} \right\}.$$

This is again a $*$-algebra with I. Now take

$$\xi = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = x_1 \oplus \cdots \oplus x_N \in V \otimes \mathbb{C}^N = \underbrace{V \oplus \cdots \oplus V}_{N\text{times}}.$$

Exercise 1. Check that $(\mathcal{M} \otimes I)' = \mathcal{M}' \otimes M_n(\mathbb{C})$, $(\mathcal{M} \otimes I)'' = \mathcal{M}'' \otimes I$. (Compare Exercise 5 in 1.6)

Applying the argument in the first part of the proof, we obtain that for every $B \in \mathcal{M}''$ there exist $A \in \mathcal{M}$ such that $(B \otimes I)\xi = (A \otimes I)\xi$, i.e.

$$Bx_1 = Ax_1,\ Bx_2 = Ax_2, \ldots,\ Bx_N = Ax_N.$$

If $x_1, \ldots, x_N$ span V, then we obtain $B = A$. Hence $\mathcal{M}'' = \mathcal{M}$. $\square$

Remark. Example $\mathcal{M}_3$ in 1.6 shows that the symmetry condition ($A \in \mathcal{M} \Rightarrow A^* \in \mathcal{M}$) is essential ($\mathcal{M}_3$ is an algebra, $\mathcal{M}_3 \ni I$ but $\mathcal{M}_3'' \neq \mathcal{M}_3$).

Exercise 2. Find an example of an algebra $\mathcal{M} \subset \mathcal{L}(V)$, such that $\mathcal{M}'' = \mathcal{M}$ but $\mathcal{M}$ is not a $*$-algebra.

Hint. The condition $\mathcal{M}'' = \mathcal{M}$ is purely algebraic and does not depend on the choice of the inner product in V, but the symmetry condition (of being a $*$-algebra) does depend on the choice of an inner product.

1.8. Factors

Definition. A *factor* is a $*$-algebra $\mathcal{M} \subset \mathcal{L}(V)$ such that $\mathcal{M} \ni I$ and $Z(\mathcal{M}) = \{\lambda I : \lambda \in \mathbb{C}\}$.

In other words a factor is a subalgebra in $\mathcal{L}(V)$ which contains the identity operator and has a trivial center.

Example. $\mathcal{L}(V)$ and $\mathcal{M}_4$ are factors; $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are not. $\mathcal{M}_1$ is a direct sum of 2 factors.

Exercise. If $\mathcal{M}$ is a factor, then $\mathcal{M}'$ is a factor too.

1.9. Generation of $*$-algebras by projections

Definition. Denote

$$\operatorname{Proj}(\mathcal{M}) = \{P \in \mathcal{M} | P^2 = P = P^*\}.$$

This means that $\operatorname{Proj}(\mathcal{M})$ is the set of all orthogonal projections in $\mathcal{M}$.

Proposition. *Let $\mathcal{M}$ be a $*$-algebra in V (possibly $I \notin \mathcal{M}$). Then $\mathcal{M}$ is spanned by* $\operatorname{Proj}(\mathcal{M})$, *that is, every element $A \in \mathcal{M}$ can be written as $A = \sum_{i=1}^N c_i P_i$, where $c_i \in \mathbb{C}$, $P_i \in \operatorname{Proj}(\mathcal{M})$.*

PROOF. We can write $A = A_1 + iA_2$, where $A_i^* = A_i$, $A_1 = (A + A^*)/2$, $A_2 = (2i)^{-1}(A - A^*)$. Hence it is sufficient to consider a self-adjoint $A = A^* \in \mathcal{L}(V)$. Then we can use its spectral decomposition

$$A = \sum_i \lambda_i E(\{\lambda_i\})$$

(sum over all eigenvalues of A). It is sufficient to prove that $E(\{\lambda_i\}) \in \mathcal{M}$ provided $\lambda_i \neq 0$. We can find a polynomial $f_i = f_i(x) \in \mathbb{R}[x]$ such that $f_i(\lambda_i) = 1$, $f_i(0) = 0$ and $f_i(\lambda_j) = 0$ if $j \neq i$. Then obviously $E(\{\lambda_i\}) = f_i(A) \in \mathcal{M}$. □

1.10. Cancellation property

LEMMA (Cancellation property). *Let $\mathcal{M}$ be a factor in V, $A \in \mathcal{M}$, $A' \in \mathcal{M}'$, $AA' = 0$. Then either $A = 0$ or $A' = 0$.*

PROOF. Consider $\operatorname{Ker} A$. It is sufficient to prove that either $\operatorname{Ker} A = \{0\}$ or $\operatorname{Ker} A = V$. (If $\operatorname{Ker} A = \{0\}$ then A is invertible and $AA' = 0$ implies $A' = 0$; if $\operatorname{Ker} A = V$ then $A = 0$.) Obviously $\operatorname{Ker} A$ is $\mathcal{M}'$-invariant. ($Ax = 0$, $B \in \mathcal{M}' \Rightarrow A(Bx) = BAx = 0$.) Hence $P_{\operatorname{Ker} A} \in \mathcal{M}$, where $P_{\operatorname{Ker} A}$ is the orthogonal projection on $\operatorname{Ker} A$. (Another proof: $\operatorname{Ker} A = \operatorname{Ker} A^*A$ and $P_{\operatorname{Ker} A} = E_0(A^*A)$ is a spectral projection for A^*A.)

Consider a maximal $\mathcal{M}$-invariant subspace in $\operatorname{Ker} A$:

$$M = \{f \in V | ABf = 0 \text{ for every } B \in \mathcal{M}\} \subset \operatorname{Ker} A.$$

Obviously $P_M \in \mathcal{M}'$ since M is $\mathcal{M}$-invariant. But M is also $\mathcal{M}'$-invariant since $f \in M$, $C \in \mathcal{M}'$ implies $ABCf = CABf = 0$ for every $B \in \mathcal{M}$. So $P_M \in \mathcal{M} \cap \mathcal{M}' = Z(\mathcal{M}) = \{\lambda I : \lambda \in \mathbb{C}\}$. Therefore either $M = \{0\}$ or $M = V$. If $M = V$, then $\operatorname{Ker} A = V$ and Lemma is proved.

Now let us assume that $M = \{0\}$, and use the fact that $AA' = 0$. Choosing $f \in V$ let us prove that $A'f \in M$ (and hence $A'f = 0$). Indeed, $ABA'f = AA'Bf = 0$ for every $B \in \mathcal{M}$. Therefore $A' = 0$. □

REMARK. The Cancellation property in the Lemma is characteristic for factors. Namely, if $\mathcal{M}$ is a $*$-algebra with $I \in \mathcal{M}$ and $\mathcal{M}$ is not a factor, then there exist $A \in \mathcal{M}$, $A' \in \mathcal{M}'$ such that $A \neq 0$, $A' \neq 0$ but $AA' = 0$. Indeed, let C be a non-scalar self-adjoint operator in $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$. Let $\lambda \in \mathbb{R}$ lies between minimal and maximal eigenvalues of C (which do not coincide because C is not scalar). Denote $A = E_\lambda$, where E_λ is the spectral projection of C, corresponding to the chosen λ. Then $E_\lambda \in Z(\mathcal{M})$. Now take $A' = I - E_\lambda \in Z(\mathcal{M})$ and then $A \neq 0$, $A' \neq 0$ but $AA' = 0$.

1.11. Structure of factors.

LEMMA 1 (on minimal subspaces). *Let $\mathcal{M}$ be a factor in V, $f \in V$, $f \neq 0$. The subspace $\mathcal{M}f$ is a minimal $\mathcal{M}$-invariant subspace if and only if*

$$\mathcal{M}f \cap \mathcal{M}'f = \mathbb{C}f = \{\lambda f | \lambda \in \mathbb{C}\}.$$

COROLLARY. *$\mathcal{M}f$ and $\mathcal{M}'f$ are minimal simultaneously.*

PROOF OF LEMMA. 1. Let E, E' be the orthogonal projections onto $\mathcal{M}'f$, $\mathcal{M}f$ respectively, so $E \in \mathcal{M}$ and $E' \in \mathcal{M}'$. Hence $EE' = E'E$. Moreover $Ef = E'f = f$.

It follows that

$$\mathcal{M}f \cap \mathcal{M}'f = E\mathcal{M}f = E\mathcal{M}Ef.$$

Indeed, since $E\mathcal{M}f \subset \mathcal{M}f$ and, on the other hand, $E\mathcal{M}f \subset EV = \mathcal{M}'f$, we have $E\mathcal{M}f \subset \mathcal{M}f \cap \mathcal{M}'f$; vice versa $\mathcal{M}f \cap \mathcal{M}'f = \mathcal{M}f \cap E\mathcal{M}'f \subset E\mathcal{M}f \cap E\mathcal{M}'f \subset E\mathcal{M}f$.

Suppose that $\mathcal{M}'f$ is a minimal $\mathcal{M}'$-invariant subspace in V. Consider the algebra of operators $E\mathcal{M}E = \{EAE|A \in \mathcal{M}\}$. This is a $*$-algebra and in EV it has the identity operator $E|_{EV} = I|_{EV}$. Now take a projection $P \in \mathrm{Proj}(E\mathcal{M}E)$ (extended by 0 to $(EV)^{\perp}$). Obviously, if $P \neq 0$ and $P \neq E$ then $\mathrm{Im}\, P$ is a $\mathcal{M}'$-invariant subspace because $P \in \mathcal{M}$. Besides $\mathrm{Im}\, P \subset EV = \mathcal{M}'f$ and $\mathrm{Im}\, P \neq EV$, which contradicts minimality of $\mathcal{M}'f$. Hence the algebra $E\mathcal{M}E$ does not contain any non-trivial projection. Therefore it consists only of scalar operators in EV. Hence

$$\mathcal{M}f \cap \mathcal{M}'f = E\mathcal{M}Ef = \{\lambda f | \lambda \in \mathbb{C}\}.$$

Replacing $\mathcal{M}'$ by $\mathcal{M}$, we come to the same conclusion, provided $\mathcal{M}f$ is a minimal $\mathcal{M}$-invariant subspace in V.

2. Conversely suppose that $\mathcal{M}f \cap \mathcal{M}'f = \mathbb{C}f$ and the space $\mathcal{M}f$ is not minimal. Then there exists $\mathcal{M}$-invariant subspace $L \subset \mathcal{M}f$, $L \neq \mathcal{M}f$ and $L \neq \{0\}$. If P is the orthogonal projection onto L, then $P \in \mathcal{M}'$. Let $L_1 = \mathcal{M}f \ominus L = \mathcal{M}f \cap L^{\perp}$. Again if P_1 is the orthogonal projection on L_1, then $P_1 \in \mathcal{M}'$, $P_1 \neq 0$.

Now $PE \neq 0$, $P_1E \neq 0$ because of the Cancellation Lemma. Since P and E commute, this means that $L \cap \mathcal{M}'f \neq \{0\}$ and $L_1 \cap \mathcal{M}'f \neq \{0\}$.

On the other hand

$$L \cap \mathcal{M}'f \subset \mathcal{M}f \cap \mathcal{M}'f = \mathbb{C}f \quad \text{and} \quad L_1 \cap \mathcal{M}'f \subset \mathcal{M}f \cap \mathcal{M}'f = \mathbb{C}f,$$

so $f \in L \cap \mathcal{M}'f$ and $f \in L_1 \cap \mathcal{M}'f$ which contradicts the orthogonality of L and L_1. □

EXAMPLE 1. Let $V = V_0 \otimes V_1$, $\mathcal{M} = \mathcal{L}(V_0) \otimes I_{V_1}$; then $\mathcal{M}' = I_{V_0} \otimes \mathcal{L}(V_1)$. (Here I_{V_0}, I_{V_1} are the identity operators in V_0, V_1 respectively.) Choosing a basis $\{e_1, \dots, e_N\}$ in V_1 we can present V as the direct sum $V = V_0 \oplus \dots \oplus V_0$ (with N copies of V_0). This is an orthogonal decomposition if the inner product in V is chosen naturally compatible with the tensor product structure, which means that it is induced by the inner products $(\cdot,\cdot)_0$, $(\cdot,\cdot)_1$ in V_0, V_1 so that

$$(x_0 \otimes x_1, y_0 \otimes y_1) = (x_0, y_0)_0 (x_1, y_1)_1, \quad x_0, y_0 \in V_0, x_1, y_1 \in V_1.$$

Now $\mathcal{M}$ in this direct decomposition consists of all operators which have a block-matrix form

$$A = \begin{pmatrix} A_0 & & & 0 \\ & A_0 & & \\ & & \ddots & \\ 0 & & & A_0 \end{pmatrix}, \quad A_0 \in \mathcal{L}(V_0).$$

In the same decomposition $\mathcal{M}'$ consists of the operators of the form

$$A' = \begin{pmatrix} \lambda_{11} I_0 & \lambda_{12} I_0 & \dots & \lambda_{1N} I_0 \\ \lambda_{21} I_0 & \lambda_{22} I_0 & \dots & \lambda_{2N} I_0 \\ \dots & & & \dots \\ \lambda_{N1} I_0 & \lambda_{N2} I_0 & \dots & \lambda_{NN} I_0 \end{pmatrix}, \quad \lambda_{ij} \in \mathbb{C},$$

where $I_0 = I_{V_0}$. Taking $f = f_0 \otimes e_1$, where $f_0 \in V_0$, $f_0 \neq 0$, we see that $\mathcal{M}f = V_0 \otimes e_1$ is a minimal $\mathcal{M}$-invariant subspace. Obviously here $\mathcal{M}f \cap \mathcal{M}'f = \mathbb{C}f$.

LEMMA 2 (On irreducible algebras). *Let $\mathcal{M}$ be a $*$-algebra with $I \in \mathcal{M}$ in V and $\mathcal{M}$ does not have any non-trivial $\mathcal{M}$-invariant subspace in V. Then $\mathcal{M} = \mathcal{L}(V)$.*

PROOF. By the Double Commutant Theorem, it is sufficient to prove that $\mathcal{M}' = \{\lambda I | \lambda \in \mathbb{C}\}$. But if this is not true then $\mathcal{M}'$ contains a non-trivial projection whose image will be a non-trivial $\mathcal{M}$-invariant subspace which is a contradiction. □

DEFINITION. Let $\mathcal{M}_1, \mathcal{M}_2$ be $*$-algebras. An algebra homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ is called a $*$-*homomorphism* if $f(A^*) = f(A)^*$ for every $A \in \mathcal{M}$.

EXAMPLE 2. If U is a unitary operator in V then it defines a $*$-automorphism of $\mathcal{L}(V)$ by the formula $f(A) = UAU^{-1} = UAU^*$.

LEMMA 3 (On automorphisms of $\mathcal{L}(V)$). *Every $*$-automorphism of $\mathcal{L}(V)$ is an inner unitary equivalence, i.e. if $f : \mathcal{L}(V) \to \mathcal{L}(V)$ is a $*$-automorphism of $\mathcal{L}(V)$ then there exists a unitary operator $U \in \mathcal{L}(V)$ such that $f(A) = UAU^{-1}$ for every $A \in \mathcal{L}(V)$.*

PROOF. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis in V and $\{E_{ij} | 1 \leq i, j \leq n\}$ is the standard basis in $\mathcal{L}(V)$, where $E_{ij} \in \mathcal{L}(V)$ is defined as follows: $E_{ij}e_k = \delta_{kj}e_i$ (so the corresponding matrix has 1 in the ith row and the jth column and all others elements are 0). Obviously

$$E_{ik}E_{lj} = \delta_{kl}E_{ij}, \qquad E_{ij}^* = E_{ji}.$$

In particular $P_i = E_{ii}$ are commuting orthogonal projections, $P_i^2 = P_i = P_i^*$, $P_iP_j = 0$ if $i \neq j$, $P_1 + P_2 + \cdots + P_n = I$.

Now define $F_{ij} = f(E_{ij})$. Then F_{ij} satisfy the same relations as E_{ij}. In particular $Q_i = F_{ii} = f(P_i)$, $i = 1, \dots, n$, are mutually orthogonal projections such that $Q_1 + Q_2 + \cdots + Q_n = I$. It follows that $\dim_{\mathbb{C}} \operatorname{Im} Q_i = 1$, $i = 1, \dots, n$.

Choose a vector $h_1 \in \operatorname{Im} Q_1$, $(h_1, h_1) = 1$, and define then $h_i = F_{i1}h_1$, $i = 2, \dots, n$. Let us check that $\{h_1, h_2, \dots, h_n\}$ is an orthonormal basis in V. Indeed

$$(h_i, h_j) = (F_{i1}h_1, F_{j1}h_1) = (F_{j1}^*F_{i1}h_1, h_1) = (F_{1j}F_{i1}h_1, h_1) = \delta_{ij}(F_{11}h_1, h_1) = \delta_{ij}$$

Moreover $F_{ij}h_k = F_{ij}F_{k1}h_1 = \delta_{jk}F_{i1}h_1 = \delta_{jk}h_i$, that is, we get the same description of the action of F_{ij} as for E_{ij} but with the basis $\{h_1, \dots, h_n\}$ instead of $\{e_1, \dots, e_n\}$.

Now define a unitary operator U in V by the formula $Ue_i = h_i$, $i = 1, \dots, n$. Then obviously $f(E_{ij}) = F_{ij} = UE_{ij}U^{-1}$. It follows that $f(A) = UAU^{-1}$ for every $A \in \mathcal{M}$. □

THEOREM. *Let $\mathcal{M}$ be a factor in V. Then V can be represented in the form $V = V_0 \otimes V_1$ so that the inner product in V is compatible with the decomposition into the tensor product and $\mathcal{M} = \mathcal{L}(V_0) \otimes I_{V_1}$.*

PROOF. Let V_0 be a minimal $\mathcal{M}$-invariant subspace in V. Taking the restriction of the operators $A \in \mathcal{M}$ to V_0 we obviously obtain an algebra $*$-homomorphism $\pi : \mathcal{M} \to \mathcal{L}(V_0)$. Minimality of V_0 implies due to Lemma 2, that π is surjective. We shall prove that π is injective too.

Obviously $\mathcal{J} = \operatorname{Ker} \pi$ is a $*$-algebra (but $I \notin \mathcal{J}$ unless $V_0 = V$). Consider the action of $\mathcal{J}$ on $V_0^\perp$ which is also a $\mathcal{M}$-invariant subspace. Now suppose that

$\mathcal{J} \neq \{0\}$. Then $\mathcal{J}$ contains a non-trivial operator $P \neq 0$. Clearly $P \in \mathcal{M}$ but $P|_{V_0} = 0$. On the other hand if P' is the orthogonal projection on V_0, then $P' \in \mathcal{M}'$ since V_0 is $\mathcal{M}$-invariant. But $PP' = 0$ which contradicts the Cancellation property Lemma in 1.10, thus proving injectivity of π.

We have proved that π is an isomorphism. Now let us consider the action of $\mathcal{M}$ on $V_0^\perp$. Let $V_0^{(1)} = V_0$ and $V_0^{(2)}$ be a minimal $\mathcal{M}$-invariant subspace in $V_0^\perp$. Denoting $\pi^{(1)} = \pi$ and reasoning as in the previous part of the proof we obtain another restriction map $\pi^{(2)} : \mathcal{M} \to \mathcal{L}(V_0^{(2)})$ which is also an isomorphism. In particular $\dim_{\mathbb{C}} V_0^{(1)} = \dim_{\mathbb{C}} V_0^{(2)}$.

By induction we get an orthogonal decomposition

$$V = V_0^{(1)} \oplus V_0^{(2)} \oplus \cdots \oplus V_0^{(N)}$$

where all $V_0^{(j)}$, $j = 1, 2, \ldots, N$, are copies of the same space V_0, and $V_0^{(j)}$ are $\mathcal{M}$-invariant subspaces such that all restrictions of the action of $\mathcal{M}$ to $V_0^{(j)}$ are isomorphisms of $\mathcal{M}$ onto $\mathcal{L}(V_0^{(j)})$. Due to the description of automorphisms of $\mathcal{L}(V_0)$ (see Lemma 3), we may by changing orthonormal bases in all $V_0^{(j)}$ obtain the desired representation $V = V_0 \otimes V_1$ (with $\dim_{\mathbb{C}} V_1 = N$) and $\mathcal{M} = \mathcal{L}(V_0) \otimes I_{V_1}$. $\square$

1.12. Commutative algebras

THEOREM. *Let $\mathcal{M}$ be a commutative $*$-algebra with $I \in \mathcal{M}$ in V. Then there exists an orthogonal decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_N$ such that*

$$\mathcal{M} = \{\lambda_1 I_1 \oplus \lambda_2 I_2 \oplus \cdots \oplus \lambda_N I_N | \lambda_i \in \mathbb{C}\},$$

where I_j is the identity operator in V_j. This decomposition is unique up to the order of terms. Moreover in this decomposition $N = \dim_{\mathbb{C}} \mathcal{M}$.

In particular there is an algebra isomorphism $\mathcal{M} = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (N copies of $\mathbb{C}$).

PROOF. Let us choose a basis $\{A_1, \ldots, A_k\}$ in $\mathcal{M}$. Since $\mathcal{M}$ is commutative, all operators A_j are normal i.e. $A_j^* A_j = A_j A_j^*$. Hence we can choose orthonormal basis of eigenvectors for A_1 and take the orthogonal decomposition of V into eigenspaces of A_1:

$$V = V^{(1)} \oplus \cdots \oplus V^{(l)}, \qquad A_1|_{V^{(j)}} = \mu_j I_j, \qquad j = 1, \ldots, l,$$

$\mu_i \neq \mu_j$ for $i \neq j$. Now every subspace $V^{(j)}$ is invariant with respect to A_2, hence we can similarly decompose every $V^{(j)}$ into an orthogonal sum of eigenspaces of A_2. Proceeding this way we shall come to an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ in V, such that every vector e_j is an eigenvector for every $A \in \mathcal{M}$. Then obviously $Ae_j = \mu_j(A) e_j$ where $\mu_j : \mathcal{M} \to \mathbb{C}$ is an algebra $*$-homomorphism. There might be equal homomorphisms among eigenvalue-homomorphisms μ_j. Gathering the equal ones together we will arrive to the orthogonal decomposition

$$V = V_1 \oplus \cdots \oplus V_N, \qquad A|_{V_j} = \lambda_j(A) I_j, \qquad j = 1, \ldots, N, \qquad A \in \mathcal{M},$$

where $\lambda_j : \mathcal{M} \to \mathbb{C}$ are algebra $*$-homomorphisms, $\lambda_i \neq \lambda_j$ if $i \neq j$, I_j is the identity operator in V_j.

Now we have to prove that the vector $(\lambda_1(A), \ldots, \lambda_N(A))$ takes all the values in $\mathbb{C}^N$ when A runs through $\mathcal{M}$. Since all functions $\lambda_1, \ldots, \lambda_N$ are linear (and different) on $\mathcal{M}$, we can obviously find $A_0 \in \mathcal{M}$ such that $\lambda_i(A_0) \neq \lambda_j(A_0)$ for $i \neq j$. But then the orthogonal projection P_j on V_j will be a polynomial of A_0, hence $P_j \in \mathcal{M}$, $j = 1, \dots, N$, and the desired statement becomes obvious, because the vectors $(\lambda_1(P_j), \lambda_2(P_j), \dots, \lambda_N(P_j)) = (0, \dots, 0, 1, 0, \dots, 0)$ (1 on jth place) constitute the standard basis in $\mathbb{C}^N$ and we can provide linear combinations by taking corresponding linear combinations in $\mathcal{M}$.

Now the equality $N = \dim_{\mathbb{C}} \mathcal{M}$ is obvious, and the uniqueness of the decomposition is obvious too because subspaces V_j can be defined as maximal common eigenspaces of all operators $A \in \mathcal{M}$. □

1.13. Structure of arbitrary $*$-algebras

THEOREM. *Let $\mathcal{M}$ be a $*$-algebra in V, $I \in \mathcal{M}$. Then there exists an orthogonal decomposition $V = V_1 \oplus V_2 \oplus \dots \oplus V_N$ such that each V_j is an $\mathcal{M}$-invariant subspace and $\mathcal{M}_j = \mathcal{M}|V_j$ is a factor in V_j. Besides $\mathcal{M} = \mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_N$ in this decomposition, and $N = \dim_{\mathbb{C}} Z(\mathcal{M})$. The decomposition is unique up to the order of terms.*

PROOF. Decompose V into the direct sum of eigenspaces for $Z(\mathcal{M})$ in the theorem on the structure of commutative $*$-algebras:

$$V = V_1 \oplus \dots \oplus V_N$$

(here $N = \dim_{\mathbb{C}} Z(\mathcal{M})$). All the subspaces V_j will be invariant with respect to $\mathcal{M}$ because they are defined as eigenspaces of $Z(\mathcal{M})$. The projections π_j with $\operatorname{Im} \pi_j = V_j$ (and $\operatorname{Ker} \pi_j = \oplus_{k \neq j} V_k$) belong to $Z(\mathcal{M})$. Hence $\mathcal{M}_j = \pi_j \mathcal{M} \pi_j$ are $*$-algebras in V_j and $\mathcal{M} = \mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_N$.

Each algebra $\mathcal{M}_j$ is a factor because $\pi_j Z(\mathcal{M}_j) \pi_j \subset Z(\mathcal{M})$, hence $Z(\mathcal{M}_j)$ contains only scalar operators due to the construction of the decomposition. Now the uniqueness of the decomposition is obvious due to the corresponding uniqueness statement for the commutative algebras. □

1.14. Traces on $*$-algebras

DEFINITION. Let $\mathcal{M}$ be an algebra (over $\mathbb{C}$). A *trace* on $\mathcal{M}$ is a linear map $\tau : \mathcal{M} \to \mathbb{C}$ such that $\tau(AB) = \tau(BA)$ for every $A, B \in \mathcal{M}$.

THEOREM.
1. *If $\mathcal{M}$ is a factor then trace on $\mathcal{M}$ is unique up to a constant factor i.e. every such trace has the form $\tau = c \operatorname{Tr}$ where $c \in \mathbb{C}$ and Tr is the usual matrix trace.*
2. *If $\mathcal{M}$ is a $*$-algebra then every linear map $\tau_0 : Z(\mathcal{M}) \to \mathbb{C}$ can be uniquely extended to a trace $\tau : \mathcal{M} \to \mathbb{C}$.*

PROOF. The result obviously follows from the uniqueness of the trace on $\mathcal{L}(V)$ and from the description of structure of factors and $*$-algebras given in previous theorems. □

REMARK 1. Theorems on the structure of commutative and general $*$-algebras as well as the last Theorem about the structure of traces on factors and general $*$-algebras have analogues in infinite-dimensional case, though more complicated phenomena may appear there. For example, in infinite-dimensional case a factor can have no more than one trace (up to a constant factor and with all definitions properly modified), but there exist factors with no trace at all.

REMARK 2. The requirement on the algebra to be a $*$-algebra is crucial in all the classification results. It is believed that a classification of general algebras of operators is impossible even in the finite dimensional case.