CHAPTER 3

Index theory and Γ-index

3.1. Fredholm operators and index

Let $\mathcal{H}_1$, $\mathcal{H}_2$, be Hilbert spaces, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the (Banach) space of all bounded linear operators $A : \mathcal{H}_1 \to \mathcal{H}_2$. For any $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we denote

$$\begin{aligned} \operatorname{Ker} A &= \{x \mid x \in \mathcal{H}_1, Ax = 0\}, \\ \operatorname{Im} A &= \{y = Ax \mid x \in \mathcal{H}_1\}, \\ \operatorname{Coker} A &= \mathcal{H}_2 / \operatorname{Im} A. \end{aligned}$$

So $\operatorname{Ker} A$ is a closed subspace in $\mathcal{H}_1$, $\operatorname{Im} A$ is a (not necessarily closed) subspace in $\mathcal{H}_2$, $\operatorname{Coker} A$ is a linear space. If $\operatorname{Im} A$ is closed, then we have a natural isomorphism

$$\operatorname{Coker} A \cong (\operatorname{Im} A)^{\perp} = \operatorname{Ker} A^*.$$

DEFINITION 1. An operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *Fredholm operator* if $\dim \operatorname{Ker} A < \infty$ and $\dim \operatorname{Coker} A < \infty$. Then its index is defined as

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A.$$

(Here dim means $\dim_{\mathbb{C}}$, and all linear spaces are considered over $\mathbb{C}$). Let $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all Fredholm operators $A : \mathcal{H}_1 \to \mathcal{H}_2$. We shall write $\mathcal{F}(\mathcal{H})$ instead of $\mathcal{F}(\mathcal{H}, \mathcal{H})$.

LEMMA 1. *If $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$, then $\operatorname{Im} A$ is closed in $\mathcal{H}_2$.*

PROOF. Let $L \subset \mathcal{H}_2$ be a linear subspace such that $\operatorname{Im} A \dot{+} L = \mathcal{H}_2$ (and $\operatorname{Im} A \cap L = \{0\}$). Consider a bounded linear map $\hat{A} : (\operatorname{Ker} A)^{\perp} \oplus L \to \mathcal{H}_2$ defined by $\hat{A}(x \oplus y) = Ax + y$, $x \in (\operatorname{Ker} A)^{\perp}$, $y \in L$. Obviously $\operatorname{Im} \hat{A} = \mathcal{H}_2$ and $\operatorname{Ker} \hat{A} = 0$. Hence due to the Inverse Mapping Theorem (alternatively called also Open Mapping Theorem) $\hat{A}$ is a linear topological isomorphism (see e.g. theorem III.11 in [**ReSi**]). But then $\operatorname{Im} A = \hat{A}((\operatorname{Ker} A)^{\perp})$ should be closed. □

Let us introduce the following notations:

(a) $\mathcal{B}_{\text{fin}}(\mathcal{H})$ is the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$ i.e. operators $K \in \mathcal{B}(\mathcal{H})$ such that $\dim \operatorname{Im} K < \infty$. Similarly $\mathcal{B}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all finite rank operators $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

(b) $\mathcal{L}_{\text{fin}}(\mathcal{H})$ is the set of all (not necessarily bounded) linear operators of finite rank in $\mathcal{H}$. Similarly $\mathcal{L}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all finite rank operators $K : \mathcal{H}_1 \to \mathcal{H}_2$.

(c) $\mathcal{C}(\mathcal{H})$ is the set of all compact operators in $\mathcal{H}$ i.e. operators $K \in \mathcal{B}(\mathcal{H})$ such that K maps the unit ball $\{x \mid \|x\| \le 1\} \subset \mathcal{H}$ into a precompact set. Similarly $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all compact operators $K : \mathcal{H}_1 \to \mathcal{H}_2$.

It is well known that $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ are closures of $\mathcal{B}_{\text{fin}}(\mathcal{H})$ and $\mathcal{B}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_2)$ respectively with respect to the topology induced by the operator norm (see e.g. Theorem VI.13 in [**ReSi**]). (This is true for Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ but not necessarily true if $\mathcal{H}_1$, $\mathcal{H}_2$ are Banach spaces.)

It is easy to see that $\mathcal{B}_{\text{fin}}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ are two-sided ideals in $\mathcal{B}(\mathcal{H})$; more generally $\mathcal{B}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ are left $\mathcal{B}(\mathcal{H}_2)$-modules and right $\mathcal{B}(\mathcal{H}_1)$-modules (all actions given by operator multiplication). Now if $\mathcal{L}(\mathcal{H})$, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote sets of all linear (not necessarily bounded) linear operators in $\mathcal{H}$ and from $\mathcal{H}_1$ to $\mathcal{H}_2$ respectively, then $\mathcal{L}_{\text{fin}}(\mathcal{H})$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$, and $\mathcal{L}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_2)$ is a left $\mathcal{L}(\mathcal{H}_2)$-module and a right $\mathcal{L}(\mathcal{H}_1)$-module.

Note that if $K \in \mathcal{L}_{\text{fin}}(\mathcal{H})$, then

$$\dim \operatorname{Ker}(I - K) < \infty \quad \text{and} \quad \dim \operatorname{Coker}(I - K) < \infty,$$

because clearly

$$\operatorname{Ker}(I - K) \subset \operatorname{Im} K, \qquad \operatorname{Im}(I - K) \supset \operatorname{Ker} K,$$

and besides

$$\operatorname{codim} \operatorname{Ker} K = \dim \operatorname{Im} K.$$

In particular, if $K \in \mathcal{B}_{\text{fin}}(\mathcal{H})$ then the operator $I - K$ is Fredholm in $\mathcal{H}$.

Lemma 2. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the following conditions are equivalent:*

(i) *A is a Fredholm operator.*
(ii) *A has an inverse operator modulo $\mathcal{L}_{\text{fin}}$ in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ i.e. there exists $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $BA = I - K_1$, $AB = I - K_2$, where $K_j \in \mathcal{L}_{\text{fin}}(\mathcal{H}_j)$ $j = 1, 2$, and I is the identity operator in the corresponding space.*
(ii)$'$ *A has left and right inverse operators modulo $\mathcal{L}_{\text{fin}}$ in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ i.e. there exist $B_1, B_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $B_1 A = I - K_1$, $AB_2 = I - K_2$ where $K_j \in \mathcal{L}_{\text{fin}}(\mathcal{H}_j)$, $j = 1, 2$.*
(iii) *A has an inverse operator modulo $\mathcal{B}_{\text{fin}}$ in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*
(iii)$'$ *A has left and right inverse operators modulo $\mathcal{B}_{\text{fin}}$ in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*
(iv) *A has an inverse operator modulo $\mathcal{S}_1$ (i.e. modulo trace class operators) in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*
(iv)$'$ *A has left and right inverses operators modulo $\mathcal{S}_1$ in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*
(v) *A has an inverse operator modulo $\mathcal{C}$ (i.e. modulo compact operators) in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*
(v)$'$ *A has left and right inverse operators modulo $\mathcal{C}$ in $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$.*

Proof. The following implications are obvious:

$$\begin{array}{ccc}
\text{(ii)} & \Rightarrow & \text{(ii)}' \\
\Uparrow & & \Uparrow \\
\text{(iii)} & \Rightarrow & \text{(iii)}' \\
\Downarrow & & \Downarrow \\
\text{(iv)} & \Rightarrow & \text{(iv)}' \\
\Downarrow & & \Downarrow \\
\text{(v)} & \Rightarrow & \text{(v)}'
\end{array}$$

Now let us check the implications (ii)$'$ $\Rightarrow$ (i) $\Rightarrow$ (iii) and (v)$'$ $\Rightarrow$ (iii)$'$. It will follow that all statements in Lemma are equivalent.

1. (ii)$'$ $\Rightarrow$ (i). Assuming that B_1, B_2, K_1, K_2 are as in (ii)$'$ we get
$$\operatorname{Ker} A \subset \operatorname{Ker}(I - K_1), \quad \operatorname{Im} A \supset \operatorname{Im}(I - K_2),$$
hence $\dim \operatorname{Ker} A < \infty$ and $\dim \operatorname{Coker} A < \infty$.
2. (i) $\Rightarrow$ (iii). Let A_1 be the restriction of A to $(\operatorname{Ker} A)^\perp$ considered as an operator from $(\operatorname{Ker} A)^\perp$ to $\operatorname{Im} A$. Since $\operatorname{Im} A$ is closed due to Lemma 1 and $\operatorname{Ker} A_1 = 0$, the operator A_1 has a bounded inverse $B_1 = A_1^{-1}$, $B_1 \in \mathcal{B}(\operatorname{Im} A, (\operatorname{Ker} A)^\perp)$. Now define $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $B = B_1$, on $\operatorname{Im} A$, $B = 0$ on $(\operatorname{Im} A)^\perp$. Then $BA = I - P_1$, $AB = I - P_2$, where P_1, P_2 are orthogonal projections on $\operatorname{Ker} A$ and $(\operatorname{Im} A)^\perp$ respectively.
3. (v)$'$ $\Rightarrow$ (iii)$'$. Let $B_1 A = I - C_1$, where $B_1 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C_1 \in \mathcal{C}(\mathcal{H}_1)$. Let us find $K_1 \in \mathcal{B}_{\text{fin}}(\mathcal{H}_1)$ such that $\|K_1 - C_1\| < 1/2$. Now
$$I - C_1 = I - (C_1 - K_1) - K_1 = (I - (C_1 - K_1))[I - (I - (C_1 - K_1))^{-1} K_1]$$
which makes sense because
$$(I - (C_1 - K_1))^{-1} = I + (C_1 - K_1) + (C_1 - K_1)^2 + \cdots,$$
where the series converges in the operator norm. Hence taking
$$\tilde{B}_1 = (I - (C_1 - K_1))^{-1} B_1, \quad \tilde{K}_1 = (I - (C_1 - K_1))^{-1} K_1,$$
we obtain $\tilde{B}_1 A = I - \tilde{K}_1$, $\tilde{B}_1 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $\tilde{K}_1 \in \mathcal{B}_{\text{fin}}(\mathcal{H}_1)$. Similarly we can construct $\tilde{B}_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $\tilde{K}_2 \in \mathcal{B}_{\text{fin}}(\mathcal{H}_2)$ such that $A\tilde{B}_2 = I - \tilde{K}_2$, which proves Lemma.

□

Now we shall prove an important algebraic property of the index and then its stability under norm-continuous deformations. We shall need an important fact from homological algebra that will be stated first. Suppose that
$$0 \longrightarrow E \xrightarrow{\ i\ } F \xrightarrow{\ r\ } G \longrightarrow 0$$
is an exact sequence of complexes of vector spaces. This means that we have the following diagram with exact rows, such that the columns are complexes (i.e. $d'_k \circ d'_{k-1} = 0$, $d_k \circ d_{k-1} = 0$, $d''_k \circ d''_{k-1} = 0$):

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{k-1} & \xrightarrow{i_{k-1}} & F_{k-1} & \xrightarrow{r_{k-1}} & G_{k-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_{k-1}} & & \downarrow{\scriptstyle d_{k-1}} & & \downarrow{\scriptstyle d''_{k-1}} & & \\
0 & \longrightarrow & E_k & \xrightarrow{i_k} & F_k & \xrightarrow{r_k} & G_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_k} & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d''_k} & & \\
0 & \longrightarrow & E_{k+1} & \xrightarrow{i_{k+1}} & F_{k+1} & \xrightarrow{r_{k+1}} & G_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Now let $H^k(E), H^k(F), H^k(G)$ be the cohomologies of the corresponding complexes, i.e. $H^k(E) = \operatorname{Ker} d'_k / \operatorname{Im} d'_{k-1}$ etc.

LEMMA 3. *There is an exact sequence*

$$\begin{aligned} \dots \longrightarrow H^k(E) \xrightarrow{i_k^*} H^k(F) \xrightarrow{r_k^*} H^k(G) \xrightarrow{\delta_k} H^{k+1}(E) \\ \xrightarrow{i_{k+1}^*} H^{k+1}(F) \xrightarrow{r_{k+1}^*} H^{k+1}(G) \xrightarrow{\delta_{k+1}} \dots, \end{aligned}$$

where i_k^*, r_k^* are the maps induced by i_k, r_k on cohomology. Here δ_k is constructed as follows: if $\overline{\omega''} \in H^k(G)$, $\overline{\omega''} = \omega'' \mod \operatorname{Im} d''_{k-1}$, where $\omega'' \in \operatorname{Ker} d''_k$, and $\omega \in F_k$ is chosen so that $r_k\omega = \omega''$, then consider $d_k\omega$ and check that $d_k\omega = i_{k+1}\omega'$ with $\omega' \in E_{k+1}$ such that $d'_{k+1}\omega' = 0$; then define $\delta_k\overline{\omega''} = \omega' \mod \operatorname{Im} d'_k$.

EXERCISE 1. Check that the map δ_k in Lemma 3 is well defined and prove the statement of Lemma.

COROLLARY 1. *Suppose that all complexes E, F, G are finite (i.e. E_k, F_k, G_k are zero spaces if $|k|$ is large enough) and that two of them have finite dimensional cohomologies. Then the third one also has finite dimensional cohomologies, and there is the following relation between the Euler characteristics:*

$$\chi(F) = \chi(E) + \chi(G).$$

PROOF. It suffices to apply Corollary from 2.27 (with the trivial group Γ) to the exact sequence of cohomologies above. □

PROPOSITION 1. *Let E, F, G be Hilbert spaces, $A \in \mathcal{F}(E, F)$, $B \in \mathcal{F}(F, G)$. Then $BA \in \mathcal{F}(E, G)$ and $\operatorname{ind}(BA) = \operatorname{ind} A + \operatorname{ind} B$.*

PROOF. Consider the following commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & E \arrow[r, "{(I,A)}"] \arrow[d, "A"'] & E \oplus F \arrow[r, "(-A)\oplus I"] \arrow[d, "BA\oplus I"'] & F \arrow[r] \arrow[d, "B"'] & 0 \\ 0 \arrow[r] & F \arrow[r, "{(B,I)}"] \arrow[d] & G \oplus F \arrow[r, "(-I)\oplus B"] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where columns should be considered as complexes and rows are exact. Now it remains to apply Corollary 1. □

LEMMA 4. *If $K \in \mathcal{L}_{fin}(\mathcal{H})$ then* $\operatorname{ind}(I - K) = 0$.

PROOF. Let $L = \operatorname{Im} K$, so $\dim L < \infty$. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "(I-K)|_L"'] & \mathcal{H} \arrow[r, "r"] \arrow[d, "I-K"] & \mathcal{H}/L \arrow[r] \arrow[d, "I"] & 0 \\
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & \mathcal{H} \arrow[r, "r"] \arrow[d] & \mathcal{H}/L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns should be considered as complexes and the rows are exact. Obviously $\operatorname{ind}(I-K)|_L = 0$, $\operatorname{ind} I = 0$, and we may again apply Corollary 1 which gives the desired result. □

Now we can prove the stability result.

PROPOSITION 2. *$\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ is an open subset in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ (with the norm topology) and* $\operatorname{ind} : \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ *is locally constant.*

PROOF. We have to check that for any $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ there exists $\epsilon > 0$ such that if $\tilde{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\|\tilde{A} - A\| < \epsilon$ then $\tilde{A} \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ and $\operatorname{ind} \tilde{A} = \operatorname{ind} A$. To do this let us find $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $BA = I - K_1$, $AB = I - K_2$ where $K_j \in \mathcal{B}_{\text{fin}}(\mathcal{H}_j)$, $j = 1, 2$. Now if $\epsilon = \|B\|^{-1}$ and $\|\tilde{A} - A\| < \epsilon$ then we can write

$$\begin{aligned} B\tilde{A} &= B[A + (\tilde{A} - A)] = I + B(\tilde{A} - A) - K_1 \\ &= [I + B(\tilde{A} - A)][I - (I + B(\tilde{A} - A))^{-1} K_1], \end{aligned}$$

so if $\tilde{B}_1 = [I + B(\tilde{A} - A)]^{-1} B$, $\tilde{K}_1 = [I + B(\tilde{A} - A)]^{-1} K_1$, then $\tilde{B}_1 \tilde{A} = I - \tilde{K}_1$ and $\tilde{B}_1 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $\tilde{K}_1 \in \mathcal{B}_{\text{fin}}(\mathcal{H}_1)$. Similarly we can construct $\tilde{B}_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $\tilde{A}\tilde{B}_2 = I - \tilde{K}_2$, where $\tilde{K}_2 \in \mathcal{B}_{\text{fin}}(\mathcal{H}_2)$. Hence $\tilde{A} \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$. Now due to Proposition 1 and Lemma 4 we have

$$\operatorname{ind} \tilde{A} = -\operatorname{ind} \tilde{B}_1 = -\operatorname{ind} B = \operatorname{ind} A.$$

□

COROLLARY 2. *Let $t \mapsto A_t$, $t \in [0, 1]$, be a continuous function of t with values in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, such that all its values A_t are Fredholm operators. Then* $\operatorname{ind} A_t$ *is a constant function; in particular,* $\operatorname{ind} A_0 = \operatorname{ind} A_1$.

PROPOSITION 3. *If C is a compact operator in $\mathcal{H}$, then* $\operatorname{ind}(I - C) = 0$.

PROOF. Consider the family $t \mapsto I - tC$ and apply Corollary 2. □

PROPOSITION 4. *Assume that $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$, $C \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. Then $A + C \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ and* $\operatorname{ind}(A + C) = \operatorname{ind} A$.

PROOF. Consider an operator $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $BA = I - C_1$ and $AB = I - C_2$ where $C_j \in \mathcal{C}(\mathcal{H}_j)$, $j = 1, 2$. Then the same B gives similar identities for $A + C$ (with other compact operators C_1, C_2), hence $A + C \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ and $\operatorname{ind}(A + C) = -\operatorname{ind} B = \operatorname{ind} A$. □

REMARK. The inclusion $A \in \mathcal{F}(\mathcal{H})$ means that $A \mod \mathcal{C}(\mathcal{H})$ is an invertible element of the Banach algebra $\mathrm{Ca}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ which is called the *Calkin algebra*. So if $\mathrm{Ca}^\times(\mathcal{H})$ is the set of all invertible elements in $\mathrm{Ca}(\mathcal{H})$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathrm{Ca}(\mathcal{H})$ is the canonical projection, then $\mathcal{F}(\mathcal{H}) = \pi^{-1}(\mathrm{Ca}^\times(\mathcal{H}))$. Hence ind becomes a locally constant function $\mathrm{ind} : \mathrm{Ca}^\times(\mathcal{H}) \to \mathbb{Z}$, which is in fact a group homomorphism.

Now note that if an inverse element exists, then all left and right inverses coincide. This implies that for any fixed $A \in \mathcal{F}(\mathcal{H})$ all elements $B \in \mathcal{B}(\mathcal{H})$ such that $BA - I \in \mathcal{C}(\mathcal{H})$, coincide modulo $\mathcal{C}(\mathcal{H})$, and they also coincide modulo $\mathcal{C}(\mathcal{H})$ with all B such that $AB - I \in \mathcal{C}(\mathcal{H})$.

EXERCISE 2. Fix an operator $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ and prove that if $B_1, B_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $B_1 A - I \in \mathcal{C}(\mathcal{H}_1)$, $AB_2 - I \in \mathcal{C}(\mathcal{H}_2)$ then $B_1 - B_2 \in \mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$. The same is true if we replace $\mathcal{C}$ by $\mathcal{B}_{\mathrm{fin}}$.

EXERCISE 3. 1) Let $S = S^* \in \mathcal{B}(\mathcal{H})$. Prove that $S \in \mathcal{F}(\mathcal{H})$ if and only if $\dim \mathrm{Ker}\, S < \infty$ and 0 is either an isolated point in the spectrum of S, or it does not belong to the spectrum of S.
2) Prove that if $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $A = US$ is its polar decomposition then $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $S \in \mathcal{F}(\mathcal{H}_1)$ and $U \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$.

3.2. Elliptic differential operators

One of the main results in the theory of elliptic operators states that an elliptic operator on a compact closed manifold (i.e. a compact manifold without boundary) is Fredholm (in appropriate spaces). The Hilbert spaces that naturally appear here are Sobolev spaces and the technique that allows one to construct operators which are inverses modulo compact operators, is the pseudo-differential operators technique. All this will be briefly explained later.

First consider an open set $U \subset \mathbb{R}^n$. A (linear) *differential operator* in U is an operator

$$A : C^\infty(U) \to C^\infty(U)$$

of the form

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where $m \in \mathbb{Z}_+$ (i.e. m is a non-negative integer); $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $a_\alpha \in C^\infty(U)$; $D = (D_1, \dots, D_n)$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, where $i = \sqrt{-1}$; $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$ i.e. $D^\alpha = i^{-|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{|\alpha_n|}}$. We shall write in this case that $\mathrm{ord}\, A \le m$ (and say that A is a differential operator of order $\le m$).

Define $a : U \times \mathbb{R}^n \to \mathbb{C}$ by

$$a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}$. Then $a \in C^\infty(U \times \mathbb{R}^n)$ is called the *symbol* of A. It is a polynomial in ξ with coefficients in $C^\infty(U)$. The function

$$a_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is called the *principal symbol* of A. It is a homogeneous polynomial in ξ of degree m with coefficients in $C^\infty(U)$.

Obviously there is a one-one correspondence between differential operators and their symbols. The operator A is naturally written through its symbol as

$$A = a(x, D) = a(x, D_x).$$

Now let B be another differential operator in U, $b = b(x, \xi)$ its symbol. Consider the composition $C = AB$, which is also a differential operator in U, and let $c = c(x, \xi)$ be its symbol.

LEMMA 1. *The symbol c of the composition $C = AB$ can be written in terms of the symbols a, b of the operators A, B by the formula*

$$c(x, \xi) = \sum_\alpha \frac{1}{\alpha!} \left[\partial_\xi^\alpha a(x, \xi)\right] \cdot \left[D_x^\alpha b(x, \xi)\right],$$

where the (finite) sum is taken over all multi-indices α, $\partial_\xi = \left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right)$, $\partial_\xi^\alpha = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \dots \partial \xi_n^{|\alpha_n|}}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$.

PROOF. For any $u \in C^\infty(U)$ due to the Leibniz and Taylor formulas we obviously obtain

$$\begin{aligned}
(AB)u(x) &= a(x, D_x)\left[b(x, D_x)u(x)\right] \\
&= a(x, D_x + D_y)\left[b(y, D_x)u(x)\right]\big|_{y=x} \\
&= \sum_\alpha \left[(\partial_\xi^\alpha a)(x, D_x)\right] \frac{D_y^\alpha}{\alpha!} \left[b(y, D_x)u(x)\right]\Big|_{y=x} \\
&= \left[\sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a \cdot D_x^\alpha b\right](x, D_x)u(x) \\
&= c(x, D_x)u(x),
\end{aligned}$$

where c is given by the required formula. $\square$

COROLLARY 1. *Let* $\operatorname{ord} A \le m$, $\operatorname{ord} B \le m_1$ *and let* a_m, b_{m_1} *be the principal symbols of* A, B *respectively. Then* $C = AB$ *has the principal symbol*

$$c_{m+m_1}(x, \xi) = a_m(x, \xi) b_{m_1}(x, \xi).$$

DEFINITION 1. A differential operator A is called an *elliptic operator* (of order m) if $\operatorname{ord} A \le m$ and $a_m(x, \xi) \neq 0$ provided $\xi \neq 0$. (Here a_m is the principal symbol of A).

Obviously, the composition of two elliptic operators is again elliptic.

Now let us make a first generalization of notions just introduced. Let p, q be positive integers. An operator

$$A : C^\infty(U)^p \to C^\infty(U)^q$$

is called a *differential operator* if it is again can be written in the same form but with matrix C^∞-functions a_α (its values should be complex $q \times p$-matrices). Then the symbol a and the principal symbol a_m are also matrix-valued functions on $U \times \mathbb{R}^n$. Such an operator is called elliptic if $p = q$ and the matrix $a_m(x, \xi)$ is invertible for every $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ or, equivalently, if $\det a_m(x, \xi) \neq 0$ provided $\xi \neq 0$. Lemma 1 and Corollary 1 are obviously true for the matrix case too.

Another necessary generalization is that to differential operators on manifolds. Let X be a C^∞-manifold. An operator

$$A : C^\infty(X) \to C^\infty(X)$$

is called a *differential operator* on X if for any coordinate neighborhood $U \subset X$ and any function $f \in C^\infty(X)$ the restriction $(Af)|_U$ depends only on $f|_U$, and in local coordinates on U, $f|_U \mapsto (Af)|_U$ is a differential operator on U. This fact does not depend on the choice of local coordinates on U. The operator A is called *elliptic* if all its restrictions to coordinate neighborhoods are elliptic. To check that the ellipticity does not depend on the choice of local coordinates, it is useful to investigate the invariant meaning of the principal symbol.

To do this let us choose an open set $U \subset \mathbb{R}^n$ and then consider a real-valued function $\varphi \in C^\infty(U)$. Apply A to $e^{i\lambda\varphi(x)}$ where λ is a real parameter. Clearly $e^{-i\lambda\varphi(x)}A(e^{i\lambda\varphi(x)})$ is a polynomial with respect to λ (with coefficients in $C^\infty(U)$). If $\operatorname{ord} A \leq m$, then this polynomial has the form

$$e^{-i\lambda\varphi(x)}A(e^{i\lambda\varphi(x)}) = \lambda^m a_m(x, \varphi_x) + O(\lambda^{m-1}),$$

where a_m is the principal symbol of A, $\varphi_x = \frac{\partial\varphi}{\partial x} = \left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_n}\right)$ is the gradient (or the differential) of φ, $O(\lambda^{m-1})$ means the sum of all terms of degree $\leq m-1$ with respect to λ (they can be estimated by $C|\lambda|^{m-1}$ when $|\lambda| \to \infty$ and x varies on a compact subset of U). The left hand side of this identity does not depend on the choice of the (local) coordinates on U. Hence the right hand side and in particular $a_m(x, \varphi_x)$ does not depend on the choice of coordinates too. But it is well known that if we change coordinates then φ_x transforms as a cotangent vector. Hence a_m is actually a well defined function on the cotangent bundle T^*U.

Patching a_m defined on different open sets U together, we obtain

LEMMA 2. *The principal symbol of a differential operator $A : C^\infty(X) \to C^\infty(X)$ on a manifold X is a well-defined function on T^*X. If $\operatorname{ord} A \leq m$ then the principal symbol a_m is homogeneous of order m with respect to the multiplication of cotangent vectors by scalars : $a_m(\lambda\nu) = \lambda^m a_m(\nu)$, $\lambda \in \mathbb{R}, \nu \in T^*X$.*

Now let E, F be complex C^∞ vector bundles on a C^∞-manifold X. Consider an operator $A : C^\infty(X, E) \to C^\infty(X, F)$ where $C^\infty(X, E), C^\infty(X, F)$ are linear spaces of C^∞-sections of E, F respectively. Such an operator is called *(linear) differential operator* if it becomes a (matrix) differential operator $A : C^\infty(U)^p \to C^\infty(U)^q$ over any coordinate neighborhood $U \subset X$ where trivializations of $E|_U$ and $F|_U$ are given. Here p, q are the dimensions of the fibers of E, F respectively, and given trivializations define isomorphisms $C^\infty(U, E) \cong C^\infty(U)^p, C^\infty(U, F) \cong C^\infty(U)^q$. Now in local coordinates the principal symbol becomes a matrix-valued function. Its invariant meaning is easily recovered from the formula

$$e^{-i\lambda\varphi(x)}A(e^{i\lambda\varphi(x)}f(x)) = \lambda^m a_m(x, \varphi_x(x))f(x) + O(\lambda^{m-1}),$$

where $\operatorname{ord} A \leq m$, λ, φ are chosen as before, $f \in C^\infty(U, E)$. It follows that for any $\nu \in T^*X$ the principal symbol gives a well-defined linear map

$$a_m(\nu) : E_x \to F_x,$$

where E_x, F_x are fibers of E, F at $x \in X$, $x = \pi(\nu)$, $\pi : T^*X \to X$ is the canonical projection. The operator A is called *elliptic* if $a_m(\nu)$ is an isomorphism of linear spaces for every non-zero cotangent vector ν.

Some simple examples of elliptic operators:

1. The Laplacian in $\mathbb{R}^n$: $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.
2. The Laplace-Beltrami operator $\Delta = -\delta d : C^\infty(X) \to C^\infty(X)$ on a Riemannian manifold X (here $\delta = d^*$ is a formally adjoint operator to the usual differential $d : C^\infty(X) \to \Lambda^1(X)$, $\Lambda^1(X)$ is the space of all smooth 1-forms on X, the inner products are induced by the Riemannian metric).
3. The Cauchy-Riemann operator $\overline{\partial}$ on a Riemannian surface $X : \bar{\partial} u = \frac{\partial u}{\partial \bar{z}} d\bar{z}$, where z is a local complex parameter on X, so $\overline{\partial} : C^\infty(X) \to \Lambda^{0,1}(X)$.
4. The Hodge Laplacian on differential forms of degree p on a Riemannian manifold X

$$\Delta_p = -(\delta d + d\delta) : \Lambda^p(X) \to \Lambda^p(X).$$

Here again δ is formally adjoint to d, where $d : \Lambda^p(X) \to \Lambda^{p+1}(X)$ is the de Rham exterior differential acting on the the differential k-forms on X, $k = 0, 1, \ldots \dim X - 1$.

Exercise. Calculate the principal symbols and verify the ellipticity for all these operators.

3.3. Sobolev spaces

Let $U \subset \mathbb{R}^n$ be an open set, $s \in \mathbb{Z}_+$.

Definition 2. The *Sobolev space* $H^s(U)$ is the space of all $f \in L^2(U)$ such that $D^\alpha f \in L^2(U)$ for all multi-indices α with $|\alpha| \leq s$. Here $L^2(U)$ is taken with respect to the Lebesgue measure on U, and $D^\alpha f$ is understood in the sense of distributions, i.e. $D^\alpha f = g$ means that

$$(-1)^{|\alpha|} \int_U f D^\alpha \varphi \, dx = \int_U g \varphi \, dx$$

for every $\varphi \in C_c^\infty(U)$.

Obviously $H^0(U) = L^2(U)$ and $H^s(U) \subset H^t(U)$ if $s > t$.

Now we shall formulate without proofs the following important properties of Sobolev spaces.

1° $H^s(U)$ is a Hilbert space with the inner product

$$(f, g) = \sum_{|\alpha| \leq s} \int_U D^\alpha f \cdot \overline{D^\alpha g} \, dx.$$

2° (*Sobolev embedding theorem*) If $s > n/2 + k$ where $s, k \in \mathbb{Z}_+$, then $H^s(U) \subset C^k(U)$. In particular,

$$\bigcap_{s \in \mathbb{Z}_+} H^s(U) \subset C^\infty(U).$$

3° (*Compactness theorem*) If $s > t$ and U is bounded, then the inclusion $H^s(U) \subset H^t(U)$ is a compact operator.

If $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is a differential operator in U such that all derivatives $D^\beta a_\alpha$ are bounded in U, then A defines a bounded linear operator

$$A : H^{s+m}(U) \to H^s(U)$$

for any $s \in \mathbb{Z}_+$. It can be actually considered as a bounded linear operator $A : H^{s+m}(U) \to H^{s-l}(U)$ for any $l \leq s, l \in \mathbb{Z}_+$. If $l > 0$ and U is also bounded, then it is actually a compact operator due to the Compactness theorem because the composition of a bounded operator $A : H^{s+m}(U) \to H^s(U)$ and the compact inclusion operator $H^s(U) \subset H^{s-l}(U)$ is a compact operator.

We shall also need extensions of these definitions and facts to compact closed manifolds and their normal coverings.

Let X be a compact closed C^∞-manifold, $X = \bigcup_{1 \leq j \leq N} U_j$ its covering by coordinate neighborhoods, $1 = \sum_{j=1}^N \varphi_j$ a partition of unity which is subordinate to the covering, i.e. $\varphi_j \geq 0$, $\varphi_j \in C_c^\infty(U_j)$. Let us fix local coordinates in U_j. Then we can define the Sobolev spaces $H^s(U_j)$; denote the corresponding Sobolev norm by $\|\cdot\|_{s,U_j}$ and the inner product $(\cdot,\cdot)_{s,U_j}$. Now we are ready for

Definition 3. The *Sobolev space* $H^s(X)$ is given by

$$H^s(X) = \{f \mid f \in L^2(X), \varphi_j f|_{U_j} \in H^s(U_j) \text{ for every } j = 1, \dots, N\}$$

with the norm given by

$$\|f\|_s^2 = \sum_{j=1}^N \|\varphi_j f\|_{s,U_j}^2$$

and the corresponding inner product.

It is easy to see that the space $H^s(X)$ does not depend on the arbitrary choices made above (the choice of coordinate neighborhoods, local coordinates, partition of unity) in its definition.

Namely, first note that if A is a differential operator on X and $\operatorname{ord} A \leq m$, then A defines a bounded linear operator

$$A : H^{s+m}(X) \to H^s(X), \quad s \in \mathbb{Z}_+.$$

So $H^s(X)$ can be defined as

$$H^s(X) = \{u \mid u \in L^2(X), Au \in L^2(X) \text{ for any differential operator } A \text{ on } X \text{ such that } \operatorname{ord} A \leq s\}.$$

The norm (and inner product) depends on the choice of the arbitrary elements but different choices lead to equivalent norms. (Recall that two norms $\|\cdot\|, \|\cdot\|'$ on the same linear space V are called equivalent if there exists $C > 0$ such that $C^{-1}\|u\| \leq \|u\|' \leq C\|u\|$ for every $u \in V$).

It is easy to check that $H^s(X)$ is a Hilbert space.

Assume that a complex C^∞ vector bundle E over X is given. Then if U is a coordinate neighborhood in X and a trivialization of $E|_U$ is given, then sections of $E|_U$ are identified with vector-functions, so Sobolev spaces $H^s(U, E)$ of such sections are defined. If we have $X = \bigcup_{j=1}^N U_j$ where U_j are such coordinate neighborhoods with given trivializations, then we define (using the same partition of unity as before)

$$H^s(X, E) = \{f : f \in L^2(X, E), \varphi_j f|_{U_j} \in H^s(U_j, E), j = 1, \dots, N\}.$$

It is easy to see that $H^s(X, E)$ does not depend on the arbitrary choices involved, and it is a Hilbert space. (The norm and the inner product are defined as in the case of trivial bundles, and different choices lead to equivalent norms.)

Now if $A : C^\infty(X,E) \to C^\infty(X,F)$ is a differential operator (E, F are vector bundles over X which is a compact C^∞-manifold), and $\operatorname{ord} A \le m$ then A can be extended to a bounded linear operator

$$A : H^{s+m}(X,E) \to H^s(X,F), \quad s \in \mathbb{Z}_+.$$

The Sobolev Embedding and Compactness theorems hold for Sobolev spaces in sections of vector bundles over a compact closed manifold X.

Now we can formulate the main theorem of the theory of elliptic operators on compact manifolds.

THEOREM. *Let $A : C^\infty(X,E) \to C^\infty(X,F)$ be an elliptic differential operator of order m in sections of vector bundles over a compact manifold X without boundary. Then the following statements are true:*

1) $\dim \operatorname{Ker} A < \infty$, $\dim \operatorname{Coker} A < \infty$, *so the index*

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$$

is well defined.

2) *For any $s \in \mathbb{Z}_+$, the extension of A by continuity to a bounded linear operator $A_s : H^{s+m}(X,E) \to H^s(X,F)$ in the Sobolev spaces is a Fredholm operator. Besides* $\operatorname{Ker} A_s = \operatorname{Ker} A$ *and* $\dim \operatorname{Coker} A_s = \dim \operatorname{Coker} A$, *so in particular* $\operatorname{ind} A_s = \operatorname{ind} A$ *does not depend on s.*

3) *Let us choose a positive smooth density on X and smooth Hermitian metrics in fibers of E and F and consider the induced inner products in $C^\infty(X,E)$ and $C^\infty(X,F)$, so the formally adjoint operator A^* is well defined. Then $A^* : C^\infty(X,F) \to C^\infty(X,E)$ is an elliptic operator and*

$$\dim \operatorname{Ker} A^* = \dim \operatorname{Coker} A.$$

Moreover,

$$\operatorname{Im} A_s = \{f \mid f \in H^s(X,F), f \perp \operatorname{Ker} A^*\},$$

where orthogonality is understood in the sense of the inner product in $L^2(X,F)$.

4) $\operatorname{ind} A$ *depends only on the principal symbol a_m of the operator A. In particular, if $B : C^\infty(X,E) \to C^\infty(X,F)$ is any differential operator of order $\le m-1$ then* $\operatorname{ind}(A+B) = \operatorname{ind} A$. *Moreover,* $\operatorname{ind} A$ *is locally constant with respect to the uniform topology on the space of the principal symbols restricted to the cosphere bundle S^*X (taken with respect to any Riemannian metric).*

COMMENTS TO THEOREM

1°. The notion of adjoint operator can be made somewhat more natural by use of the density bundle $\Omega(X)$ over X. This is a (complex) vector bundle with a 1-dimensional fiber which has local sections of the form $f(x)dx$ over a coordinate neighborhood (the choice of local coordinates defines a trivialization of $\Omega(X)$ over the coordinate neighborhood); if we switch to other local coordinates x' then the section $f(x)dx$ transforms to $f(x)\left|\det \frac{\partial x}{\partial x'}\right| dx'$. Here dx, dx' can be understood as Lebesgue measures. If $\omega \in C_c^\infty(X, \Omega(X))$ i.e. ω is a C^∞-section of $\Omega(X)$ with a compact support then the integral $\int_X \omega \in \mathbb{C}$ is well defined.

Now if E is a vector bundle over X, then let us define the dual vector bundle as $E^* = \mathrm{Hom}(E, \Omega(X))$. Then there is a natural Hermitian pairing

$$C^\infty(X, E) \times C_c^\infty(X, E^*) \to \mathbb{C}, \quad \{u, v\} \mapsto \int_X u\bar{v}$$

which is obtained by taking the natural duality pairing at every point $x \in X$ and then integrating over X. Hence we can define the formally adjoint operator to a differential operator $A : C^\infty(X, E) \to C^\infty(X, F)$, as a differential operator $A^* : C^\infty(X, F^*) \to C^\infty(X, E^*)$ such that

$$(Au, v) = (u, A^*v), \quad u \in C_c^\infty(X, E), \quad v \in C^\infty(X, F^*).$$

Also the Hermitian pairing can be replaced by a similar bilinear pairing.

2°. The cosphere bundle in part 4) of the Theorem can be understood as the set $\{\nu \mid a(\nu) = 1\} \subset T^*X$, where a is any continuous function $a : T^*X \to [0, \infty)$, such that $a(\nu) > 0$ for any $\nu \neq 0$ and a is homogeneous of a positive order $\alpha > 0$ i.e. $a(t\nu) = t^\alpha a(\nu)$, $\nu \in T^*X, t > 0$.

3°. The equality $\mathrm{ind}(A + B) = \mathrm{ind}\, A$ in part 4) of Theorem follows from the compactness of $B : H^{s+m}(X, E) \to H^s(X, F)$.

4°. The index of A may be calculated through a_m by the famous Atiyah-Singer index formula. The structure of this formula is as follows: a special differential form on S^*X is defined algebraically in terms of a_m and a form defined by X itself, then this form is integrated over S^*X.

5°. To prove that A_s is Fredholm, it is convenient to construct an operator $B : C^\infty(X, F) \to C^\infty(X, E)$ such that $BA - I$ and $AB - I$ are operators with a C^∞ Schwartz kernel. Such an operator B will not be a differential operator but a pseudo-differential operator and we shall briefly describe some basic facts about pseudo-differential operators now.

3.4. Pseudo-differential operators

Let us consider a function $u \in C_c^\infty(U)$ where U is an open set in $\mathbb{R}^n$. We can represent it as a continuous linear combination of exponents by the Fourier transform

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \tilde{u}(\xi)\, d\xi,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$, $x, \xi \in \mathbb{R}^n$, and

$$\tilde{u}(\xi) = \int_U e^{-iy\cdot\xi} u(y)\, dy$$

Now suppose that $A = a(x, D_x)$ is a differential operator in U with the symbol $a = a(x, \xi)$. Since $D_x e^{ix\cdot\xi} = e^{ix\cdot\xi}\xi$, we obviously have

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi)\tilde{u}(\xi)\, d\xi \tag{1}$$

or

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x, \xi) u(y)\, dy d\xi. \tag{2}$$

The first formula makes sense because $\tilde{u}$ is in $S(\mathbb{R}^n)$, where $S(\mathbb{R}^n)$ is the L. Schwartz space which consists of all functions $\psi : \mathbb{R}^n \to \mathbb{C}$, such that

$\psi \in C^\infty(\mathbb{R}^n)$ and $(1+|x|)^N|D^\alpha\psi(x)| \le C_{\alpha N}$ for every multiindex α and every $N > 0$. In the formula (2) the integral should be understood as repeated one (we should first integrate with respect to dy and then with respect to $d\xi$).

Now the natural idea is to extend the class of symbols in this formulas from polynomials with respect to ξ to more general functions. We will describe a class of these functions that leads to one of the possible versions of the theory of pseudo-differential operators.

Definition 1. For any $m \in \mathbb{R}$ define $S^m(U \times \mathbb{R}^n)$ as the set of functions $a \in C^\infty(U \times \mathbb{R}^n)$ satisfying estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha\beta K}(1+|\xi|)^{m-|\alpha|}, \quad x \in K,\ \xi \in \mathbb{R}^n,$$

where K is a compact subset in U, α, β are multi-indices (and the estimates should be fulfilled for all α, β, K). Define also $S^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(U \times \mathbb{R}^n)$.

Note that if $a \in S^m(U \times \mathbb{R}^n)$ then the operator $A = a(x, D_x)$ is still well defined by the integral formulas. Moreover if $m < -n$ then the integral in the formula (2) is absolutely convergent and can be written in the form

$$Au(x) = \int_U K_A(x,y)u(y)\,dy$$

with $K_A \in C(U \times U)$ given by the formula

$$K_A(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\,d\xi.$$

If $a \in S^{-\infty}(U \times \mathbb{R}^n)$, then in fact $K_A \in C^\infty(U \times U)$ because we can differentiate under the integral sign in this integral as many times as we want.

The definition of the class of all operators above is not symmetric with respect to the transposition of x and y. And it is not only the specific definition which is to blame. For example, it is easy to see that if U is bounded and $a \in S^{-\infty}(U\times\mathbb{R}^n)$ then $y \mapsto K_A(x,y)$ extends to a function from $C^\infty(\mathbb{R}^n)$ for any fixed $x \in U$. On the other hand in the same situation the function $x \mapsto K_A(x,y)$ (for a fixed $y \in U$) can be arbitrarily singular when x approaches the boundary of U. This can be easily seen from the fact that multiplying $a(x,\xi)$ by an arbitrary function $\phi = \phi(x) \in C^\infty(U)$ does not change the form of the operator (and $\phi a \in S^{-\infty}(U \times \mathbb{R}^n)$ if is true for a).

To avoid this asymmetry we will add all operators with $K_A \in C^\infty(U \times U)$ (not only the ones that have the form $a(x, D_x)$ with $a \in S^{-\infty}(U \times \mathbb{R}^n)$), and we will consider them as negligible ones.

Definition 2. $\Psi^m(U)$ is the class of all operators $A : C_c^\infty(U) \to C^\infty(U)$ that have the form $A = a(x, D_x) + T$, where $a \in S^m(U \times \mathbb{R}^n)$ and T is an operator with a C^∞ Schwartz kernel (i.e. $K_T \in C^\infty(U \times U)$). Define also $\Psi^{-\infty}(U) = \bigcap_{m\in\mathbb{R}} \Psi^m(U)$ which is the class of all operators T with a C^∞ Schwartz kernel in U.

Now we will describe a narrower class of pseudo-differential operators (and symbols) which still includes all differential operators. These operators are called *classical* or *polyhomogeneous* pseudo-differential operators.

Definition 3. $S_{\mathrm{cl}}(U \times \mathbb{R}^n)$ is the class of functions $a \in C^\infty(U \times \mathbb{R}^n)$ such that there exist functions $a_{m-j} = a_{m-j}(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \setminus 0))$, $j = 0, 1, 2, \dots$ which are (positively) homogeneous of order $m - j$ with respect to ξ i.e.

$$a_{m-j}(x, t\xi) = t^{m-j}a_{m-j}(x,\xi), \quad (x,\xi) \in U \times (\mathbb{R}^n \setminus 0),\ t > 0,$$

and $a(x,\xi)$ is presented as an asymptotic sum

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$$

in the following sense: if $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ near 0 and $\chi(\xi) = 1$ near ∞, then

$$a(x,\xi) - \sum_{j=0}^{N-1} \chi(\xi) a_{m-j}(x,\xi) \in S^{m-N}(U \times \mathbb{R}^n)$$

for every $N = 1, 2, 3, \ldots$.

Note that obviously $\chi(\xi)a_{m-j}(x,\xi) \in S^{m-j}(U \times \mathbb{R}^n)$, hence

$$S^m_{\mathrm{cl}}(U \times \mathbb{R}^n) \subset S^m(U \times \mathbb{R}^n).$$

Definition 4. $\Psi^m_{\mathrm{cl}}(U)$ is the class of all operators $A \in \Psi^m(U)$ that have the form $A = a(x, D_x) + T$ with $a \in S^m_{\mathrm{cl}}(U \times \mathbb{R}^n)$ and $K_T \in C^\infty(U \times U)$. Then the function a_m is called the *principal symbol* of A.

An important fact is that every sequence of homogeneous functions a_{m-j}, $j = 0, 1, \ldots$, can be present in the asymptotic expansion for $a \in S^m_{\mathrm{cl}}$.

Lemma 1. *For every sequence of functions $a_{m-j} \in C^\infty(U \times (\mathbb{R}^n \setminus 0))$, which are homogeneous of order $m - j$ with respect to ξ, $j = 0, 1, 2, \ldots$, there exists $a \in S^m(U \times \mathbb{R}^n)$ such that*

$$a \sim \sum_{j=0}^{\infty} a_{m-j}$$

in the sense described earlier.

The idea of the proof is as follows. Let us choose a function $\chi \in C^\infty(\mathbb{R}^n)$ which vanishes near 0 and is equal to 1 near ∞, and a sequence $\epsilon_j > 0$, $j = 0, 1, \ldots$, such that $\epsilon_j \to 0$ sufficiently fast. Then we may put

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_{m-j}(x,\xi), \quad (x,\xi) \in U \times \mathbb{R}^n.$$

Exercise 1. Supply all the details for the proof of this Lemma.

Now we would like to discuss composition of pseudo-differential operators. Unfortunately the composition AB of two operators $A \in \Psi^m(U)$ and $B \in \Psi^{m_1}(U)$ is not generally defined because the operator A is not necessarily defined on $C^\infty(U)$, and on the other hand the operator B does not necessarily maps $C^\infty_c(U)$ to $C^\infty_c(U)$. However all this can be overcome by adding to A (or to B) an operator with a smooth kernel. To make it more precise note first that every operator $A \in \Psi^m(U)$ has a kernel $K_A \in \mathcal{D}'(U \times U)$ i.e. K_A is a distribution on $U \times U$. Formally it can be defined for any operator $A = a(x, D_x)$ by the same integral as in the case $m < -n$; more precisely K_A can be defined by the formula

$$\langle K_A, \varphi \rangle = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)\varphi(x,y)\, dy d\xi dx, \quad \varphi \in C^\infty_c(U \times U).$$

One of the most important properties of pseudo-differential operators is given by the following

LEMMA 2. *For any $A \in \Psi^m(U)$, its Schwartz kernel is C^∞ in the complement of the diagonal Δ in $U \times U$. In other words, $K_A \in C^\infty(U \times U \setminus \Delta)$, where $\Delta = \{(x,x) \mid x \in U\} \subset U \times U$ is the diagonal in $U \times U$.*

PROOF. It is sufficient to consider the case when $A = a(x, D_x)$ with $a \in S^m(U \times \mathbb{R}^n)$. Let us take an arbitrary function $\varphi \in C_c^\infty(U \times U)$ such that $\varphi = 0$ in a neighborhood of Δ. We have to prove then that $\langle K_A, \varphi\rangle = \langle \tilde{K}_A, \varphi\rangle$ where $\tilde{K}_A \in C^\infty(U \times U \setminus \Delta)$ and $\tilde{K}_A$ does not depend on φ. This can be done by integration by parts with the use of the equality

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N}(-\Delta_\xi)^N e^{i(x-y)\cdot\xi}, \quad N \in \mathbb{Z}_+,$$

(here Δ_ξ is the Laplacian with respect to ξ), which leads to the formula

$$\langle K_A, \varphi\rangle = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \left[(-\Delta_\xi)^N a(x,\xi)\right] |x-y|^{-2N} \varphi(x,y)\, dyd\xi dx.$$

Obviously $(-\Delta_\xi)^N a(x,\xi) \in S^{m-2N}(U \times \mathbb{R}^n)$, $|x-y|^{-2N}\varphi(x,y) \in C_c^\infty(U \times U)$, so the integration by parts is easily validated. Now we can take

$$\tilde{K}_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \left[(-\Delta_\xi)^N a(x,\xi)\right] |x-y|^{-2N}\, d\xi.$$

The result does not depend on N (provided $m - 2N < -n$) and therefore we obtain $\tilde{K}_A \in C^\infty(U \times U \setminus \Delta)$. □

REMARK. The property of A given in Lemma 2 is often formulated by saying that A is *pseudolocal*, which means that

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp} u$$

where $u \in \mathcal{E}'(U)$ (i.e. u is a distribution with a compact support in U), A is extended to $\mathcal{E}'(U)$ by continuity or by duality and $\operatorname{sing\,supp} u$ is the complement (in U) of the maximal open subset $V \subset U$ such that $u|_V \in C^\infty(V)$.

LEMMA 3. *If $A \in \Psi^m(U)$ and V is a neighborhood of the diagonal in $U \times U$ then A can be represented in the form $A = A_1 + T$ where $T \in \Psi^{-\infty}(U)$ and $\operatorname{supp} K_{A_1} \subset V$.*

PROOF. Let $\chi \in C^\infty(U \times U)$ be chosen so that $\operatorname{supp}\chi \subset V$ and $\chi = 1$ in a smaller neighborhood of the diagonal. Then take A_1 with the kernel $K_{A_1}(x,y) = \chi(x,y)K_A(x,y)$ and put $T = A - A_1$. Lemma 2 implies that $K_T \in C^\infty$ and the result follows. □

Now we can choose V with the following special property: if we take the closure $\bar{V}$ of V in $U \times U$, then for any compact $K \subset U$ both $\pi_1^{-1}(K) \cap \bar{V}$ and $\pi_2^{-1}(K) \cap \bar{V}$ are compact subsets in $U \times U$. (Here $\pi_1, \pi_2 : U \times U \to U$ are projections of $U \times U$ on the first and the second factor respectively.) Then we say that both projections of $\bar{V}$ to U are proper maps. Hence the same will be true for $\operatorname{supp} K_{A_1}$ in the construction given by the previous proof.

DEFINITION 5. An operator $A : C_c^\infty(U) \to C^\infty(U)$ with the Schwartz kernel K_A is called *properly supported* if both projections $\pi_1, \pi_2 : \operatorname{supp} K_A \to U$ are proper maps.

In this case $\operatorname{supp}(Af) \subset \pi_1(\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} f))$ and it easily follows that for every compact set $K \subset U$ there exists a compact set $\tilde{K} \subset U$ such that

$$\{f \in C_c^\infty(U), \operatorname{supp} f \subset K\} \Rightarrow \{\operatorname{supp}(Af) \subset \tilde{K}\}.$$

Hence A in fact maps $C_c^\infty(U)$ to $C_c^\infty(U)$. Moreover, for every compact set $K \subset U$ there exists a compact set $\tilde{K}' \subset U$ such that $Au|_K$ depends on $u|_{\tilde{K}'}$ only. Therefore A naturally extends to an operator $A : C^\infty(U) \to C^\infty(U)$.

Now let $A \in \Psi^m(U)$, $B \in \Psi^{m_1}(U)$ and one of the operators A, B is properly supported. Then the compositions AB and BA are well defined. In this case we have

THEOREM 1 (Composition Formula). *Consider the composition $C = AB$ of two operators $A \in \Psi^m(U)$, $B \in \Psi^{m_1}(U)$, one of which is properly supported. Then $C \in \Psi^{m+m_1}(U)$. Moreover, if $A = a(x, D_x) + T_1$, $B = b(x, D_x) + T_2$ where $a \in S^m(U \times \mathbb{R}^n)$, $b \in S^{m_1}(U \times \mathbb{R}^n)$, $T_j \in \Psi^{-\infty}(U)$, $j = 1, 2$, then $C = c(x, D_x) + T$ where $c \in S^{m+m_1}(U \times \mathbb{R}^n)$, $T \in \Psi^{-\infty}(U)$ and c has the asymptotic expansion*

$$c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \left[\partial_\xi^\alpha a(x, \xi)\right] \cdot \left[D_x^\alpha b(x, \xi)\right]$$

in the sense that for every $N \in \mathbb{Z}_+$

$$c(x, \xi) - \sum_{|\alpha| \le N-1} \frac{1}{\alpha!} \left[\partial_\xi^\alpha a(x, \xi)\right] \cdot \left[D_x^\alpha b(x, \xi)\right] \in S^{m+m_1-N}(U \times \mathbb{R}^n).$$

COROLLARY 1. *If $A \in \Psi_{\mathrm{cl}}^m(U)$, $B \in \Psi_{\mathrm{cl}}^{m_1}(U)$ and one of the operators A, B is properly supported then $AB \in \Psi_{\mathrm{cl}}^{m+m_1}(U)$. Besides if*

$$a \sim \sum_{j=0}^\infty a_{m-j}, \qquad b \sim \sum_{l=0}^\infty b_{m_1-l}$$

are asymptotic expansions of the symbols, then for the symbol c of the composition $C = AB$ we have $c \sim \sum_{k=0}^\infty c_{m+m_1-k}$ where

$$c_{m+m_1-k}(x, \xi) = \sum_{|\alpha|+j+l=k} \frac{1}{\alpha!} [\partial_\xi^\alpha a_{m-j}(x, \xi)][D_x^\alpha b_{m_1-l}(x, \xi)].$$

In particular $c_{m+m_1}(x, \xi) = a_m(x, \xi) b_{m_1}(x, \xi)$ i.e. the principal symbol of the composition is the product of the principal symbols of the factors.

The idea of the proof of Theorem 1 is the same as for differential operators except that the Taylor formula with a remainder should be used and the remainder has to be properly estimated.

Another important operation is taking the formally adjoint operator i.e. operator A^* such that

$$(Au, v) = (u, A^* v), \quad u, v \in C_c^\infty(U).$$

(Here $(\cdot, \cdot)$ means the inner product in the usual Hilbert space $L^2(U)$.)

THEOREM 2. *Let $A \in \Psi^m(U)$. Then there exists a unique formally adjoint operator $A^* \in \Psi^m(U)$. If $A = a(x, D_x) + T$ with $a \in S^m(U \times \mathbb{R}^n)$, $T \in \Psi^{-\infty}(U)$,* *then* $A^* = \tilde{a}(x, D_x) + T_1$, *where* $T_1 \in \Psi^{-\infty}(U)$ *and* $\tilde{a} \in S^m(U \times \mathbb{R}^n)$ *has the following asymptotic expansion:*

$$\tilde{a}(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a(x,\xi)},$$

where the bar over a means the complex conjugation, and the asymptotic expansion is understood as in the composition formula:

$$\tilde{a}(x,\xi) - \sum_{|\alpha| \le N-1} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a(x,\xi)} \in S^{m-N}(U \times \mathbb{R}^n)$$

for every $N \in \mathbb{Z}_+$.

Corollary 2. *If* $A \in \Psi^m_{\mathrm{cl}}(U)$ *then* $A^* \in \Psi^m_{\mathrm{cl}}(U)$.

Exercise 2. Prove Theorem 2 for differential operators. The idea of the proof in the general case is similar to the idea described for the composition formula earlier.

Definition 6. An operator $A \in \Psi^m_{\mathrm{cl}}(U)$ is called *elliptic* if $a_m(x,\xi) \neq 0$ for all $(x,\xi) \in U \times (\mathbb{R}^n \setminus 0)$. (Here a_m is the principal symbol of A.)

The following theorem is the first one that shows why pseudo-differential operators are important and why it is not sufficient to work with differential operators only.

Theorem 3. *Let* $A \in \Psi^m_{\mathrm{cl}}(U)$ *be an elliptic operator. Then there exists an operator* $B \in \Psi^{-m}_{\mathrm{cl}}(U)$ *such that* B *is properly supported and*

$$BA = I - T_1, \qquad AB = I - T_2, \qquad T_j \in \Psi^{-\infty}(U),\ j = 1, 2.$$

This operator B *is elliptic too. It is unique modulo* $\Psi^{-\infty}(U)$. *Moreover, if* $\tilde{B} \in \Psi^k(U)$ *is properly supported (with any* $k \in \mathbb{R}$*) and either* $\tilde{B}A - I \in \Psi^{-\infty}(U)$ *or* $A\tilde{B} - I \in \Psi^{-\infty}(U)$, *then* $\tilde{B} - B \in \Psi^{-\infty}(U)$.

Proof. We may assume without loss of generality that A is properly supported. Consider an arbitrary operator $B_0 \in \Psi^{-m}_{\mathrm{cl}}(U)$ which is properly supported and has the principal symbol $b_{-m}(x,\xi) = (a_m(x,\xi))^{-1}$. Then $B_0 A = I - L$, where $L \in \Psi^{-1}_{\mathrm{cl}}(U)$ due to the composition formula and L is properly supported. Now let us construct $B_1 \in \Psi^0_{\mathrm{cl}}(U)$ such that B_1 is properly supported and

$$B_1 \sim I + L + L^2 + \dots.$$

This means that we should take the formal sum of all homogeneous components of the symbols of the operators $I, L, L^2, \dots$ (this makes sense because $L^k \in \Psi^{-k}_{\mathrm{cl}}(U)$) and then construct the symbol of B_1 by the asymptotic summation of the series as described above. Now we can take $B = B_1 B_0$ to obtain that $BA = I - T_1$ with $T_1 \in \Psi^{-\infty}(U)$. Similarly we can construct $\tilde{B} \in \Psi^{-m}_{\mathrm{cl}}(U)$ such that $\tilde{B}$ is properly supported and $A\tilde{B} = I - \tilde{T}_2$, where $\tilde{T}_2 \in \Psi^{-\infty}(U)$. Multiplying this equality by B from the left, we obtain that $B - \tilde{B} \in \Psi^{-\infty}(U)$. So all left and right inverse (modulo $\Psi^{-\infty}$) operators coincide modulo $\Psi^{-\infty}$. Hence B satisfies all required conditions. $\square$

The following terminology is usually used: B is called a *parametrix* of A. So parametrix is an inverse (modulo $\Psi^{-\infty}$) operator to A. It is uniquely defined modulo $\Psi^{-\infty}(U)$.

Now let us consider the action of pseudo-differential operators in Sobolev spaces. We shall formulate the results without proofs. To avoid considering the influence of the boundary we shall introduce cut-off functions. Also it is convenient to introduce Sobolev spaces $H^s(U)$, where s is a negative integer.

As before U denotes an open subset in $\mathbb{R}^n$.

DEFINITION 7. $H^s(U)$ for an integer $s<0$ is the space dual to $H^{-s}(U)$ in the following sense:
$H^s(U) = \{f \mid f \in \mathcal{D}'(U), \varphi \mapsto \langle f, \varphi\rangle$ can be extended from $C_c^\infty(U)$ to a linear continuous functional on $H^{-s}(U)\}$.

Another (equivalent) definition is:
$H^s(U) = \{f \mid f \in \mathcal{D}'(U)$, there exist $f_\alpha \in L^2(U)$, $|\alpha| \le |s|$, such that $f = \sum_{|\alpha|\le -s} D^\alpha f_\alpha\}$.

It follows from the Sobolev imbedding theorem that the delta function $\delta(x-x_0)$ belongs to $H^s(U)$ provided $s < -n/2$.

THEOREM 4. *Let $A \in \Psi^m(U)$, $\varphi, \psi \in C_c^\infty(U)$. Then the operator $u \mapsto \varphi A(\psi u)$ extends from $C_c^\infty(U)$ to a bounded linear operator $H^{s'}(U) \to H^s(U)$ provided $s' \ge s+m$. In particular, if $m \in \mathbb{Z}$ then $\varphi A \psi$ extends to a bounded linear operator $H^{s+m}(U) \to H^s(U)$ for every $s \in \mathbb{Z}$.*

We will not prove this theorem. But we will show how it can be applied to prove local regularity results for elliptic equations, when combined with the existence of the parametrix.

For simplicity of formulation, let us introduce

DEFINITION 8. The *local Sobolev spaces* are

$$H^s_{\text{loc}}(U) = \{u \mid u \in \mathcal{D}'(U),\ \varphi u \in H^s(U) \text{ for every } \varphi \in C_c^\infty(U)\}.$$

THEOREM 5. *Let us consider an equation $Au = f$ where $u, f \in \mathcal{D}'(U)$, A is an elliptic differential operator of order m. Assume that $f \in H^s_{\text{loc}}(U_0)$ for an open set $U_0 \subset U$. Then $u \in H^{s+m}_{\text{loc}}(U_0)$. In particular, if $f \in C^\infty(U_0)$ then $u \in C^\infty(U_0)$.*

PROOF. Let us choose a properly supported parametrix B of A so that $BA = I - T$, $T \in \Psi^{-\infty}(U)$. It follows that T is also properly supported. Applying B to both sides of the equation $Au = f$ we obtain $u = Bf + Tu$. Obviously $Tu \in C^\infty(U)$. Now let us choose an open set $U_1 \subset U_0$ such that $\bar U_1$ is a compact subset in U_0. It is sufficient to prove that $Bf \in H^{s+m}_{\text{loc}}(U_1)$ for every such U_1. Let us take $\chi \in C_c^\infty(U_0)$ such that $\chi = 1$ in a neighborhood of $\bar U_1$. Then $B(\chi f) \in H^{s+m}_{\text{loc}}(U)$ due to Theorem 4 because $\chi f \in H^s(U)$. Also $B((1-\chi)f)|_{U_1} \in C^\infty(U_1)$ due to the pseudolocal property of B. Therefore $Bf = B(\chi f) + B((1-\chi)f) \in H^{s+m}_{\text{loc}}(U_1)$. $\square$

The missing proofs and more details concerning results in this section can be found e.g. in [**Sh6**], Chapter 1.

3.5. Pseudo-differential operators on manifolds

Let X be a C^∞-manifold, $A : C_c^\infty(X) \to C^\infty(X)$ a linear operator. For any open $U \subset X$ consider a restriction $A_U : C_c^\infty(U) \to C^\infty(U)$ defined as the composition $A_U = r_U A i_U$, where $i_U : C_c^\infty(U) \to C_c^\infty(X)$ is the natural inclusion and $r_U : C^\infty(X) \to C^\infty(U)$ is the restriction operator.

DEFINITION 1. $A \in \Psi^m(X)$ (respectively $\Psi^m_{\mathrm{cl}}(X)$) if for every coordinate neighborhood $U \subset X$ and for every choice of local coordinates in U we have $A_U \in \Psi^m(U)$ (resp. $\Psi^m_{\mathrm{cl}}(U)$).

In fact the choice of local coordinates does not matter because it can be proved that both classes $\Psi^m(U)$ and $\Psi^m_{\mathrm{cl}}(U)$ are invariant under the change of local coordinates (see e.g. Sect.4 in [**Sh6**]). The principal symbol a_m of $A \in \Psi^m_{\mathrm{cl}}(X)$ is a well-defined function on T^*X. The operator A is called *elliptic* if $a_m(\nu) \neq 0$ for every $\nu \in T^*X$, $\nu \neq 0$. The Schwartz kernel K_A of any operator $A \in \Psi^m(X)$ is C^∞ outside the diagonal in $X \times X$. This follows from the corresponding local fact because any two points $x, y \in X$ can be included in a coordinate neighborhood (which is not required to be connected). So the pseudolocality property extends to manifolds.

Now let us consider the case of a compact closed manifold X. It is easily deduced from the corresponding local fact that every operator $A \in \Psi^m(X)$ can be extended to a bounded linear operator $A : H^{s'}(X) \to H^s(X)$ provided $s' \geq s + m$. If $s' > s + m$ then this operator is compact. The following theorem globalizes the parametrix construction.

THEOREM 1. *Let $A \in \Psi^m_{\mathrm{cl}}(X)$ be an elliptic pseudo-differential operator of order m on a compact closed manifold X. Then there exists $B \in \Psi^{-m}_{\mathrm{cl}}(X)$ such that*

$$BA = I - T_1, \qquad AB = I - T_2, \qquad T_j \in \Psi^{-\infty}(X),\ j = 1, 2,$$

where $\Psi^{-\infty}(X)$ is the space of all operators A with a C^∞ Schwartz kernel. Such an operator B is unique modulo $\Psi^{-\infty}(X)$.

COROLLARY 1 (Elliptic Regularity). *Let $A \in \Psi^m_{\mathrm{cl}}(X)$ be an elliptic pseudo-differential operator of order m on a compact closed manifold X. If $u \in \mathcal{D}'(X)$ and $Au = f \in H^s(X)$, then $u \in H^{s'}(X)$ for any s' such that $s' \leq s + m$. (In particular, if $m \in \mathbb{Z}$ then $u \in H^{s+m}(X)$).*

PROOF OF COROLLARY. Let B be as in Theorem 1. Then $BAu = Bf$ implies $u = Bf + T_1 u$ and it remains to notice that $Bf \in H^{s'}(X)$, $T_1 u \in C^\infty(X)$. □

SKETCH OF PROOF OF THEOREM 1. Choose a covering of X by coordinate neighborhoods $X = \bigcup_{j=1}^N U_j$, construct a parametrix B_j for the restriction of A to U_j using the local theorem and then patch them together defining

$$Bu = \sum_{j=1}^N \psi_j B_j(\varphi_j u),$$

where $\{\varphi_j \mid 1 \leq j \leq N\}$ is a partition of unity subordinated to the covering, i.e. $\varphi \geq 0$, $\varphi \in C_c^\infty(U_j)$, $\sum_{j=1}^N \varphi_j = 1$ and $\psi_j \in C_c^\infty(U_j)$ is chosen so that $\psi_j \equiv 1$ in a neighborhood of $\operatorname{supp} \varphi_j$. □

EXERCISE 1. Check that this construction really provides a parametrix for A.

REMARK. It is often convenient to have Sobolev spaces $H^s(X)$ for non-integer $s \in \mathbb{R}$ as well. The most convenient way to construct them is by use of pseudo-differential operators. Namely, choose an elliptic pseudo-differential operator

$\Lambda_s \in \Psi^s_{\mathrm{cl}}(X)$ for any $s \in \mathbb{R}$. (For example, we may take any operator with the principal symbol $|\nu|^s$, $\nu \in T^*X$, where $|\nu|$ is given as the length of the cotangent vector ν with respect to a chosen Riemannian metric.) Then we can define

$$H^s(X) = \{u \mid u \in \mathcal{D}'(X),\ \Lambda_s u \in L^2(X)\}.$$

EXERCISE 2. (a) Prove that $H^s(X)$ does not depend of the choice of Λ_s. *Hint:* use a parametrix of Λ_s.

(b) Prove that if $A \in \Psi^m(X)$, $m \in \mathbb{R}$, then for any $s \in \mathbb{R}$ the operator A defines a map $A : H^{s+m}(X) \to H^s(X)$, i.e. the restriction of A (defined on distributions) maps $H^{s+m}(X)$ to $H^s(X)$.

(c) Prove that $H^s(X)$ is a Hilbert space with the norm given by

$$\|u\|_s^2 = \|\Lambda_s u\|^2_{L^2(X)} + \|u\|^2_{H^{-N}(X)},$$

where $N \in \mathbb{Z}$ is chosen so that $-N < s$.

(d) Prove that $A \in \Psi^m(X)$ defines a bounded linear map $A : H^{s+m}(X) \to H^s(X)$ for every $m \in \mathbb{R}, s \in \mathbb{R}$.

The last generalization that we will mention is that pseudo-differential operators of the same classes Ψ^m, Ψ^m_{cl} can be defined naturally in sections of C^∞ vector bundles over C^∞-manifolds. These classes of operators $A : C_c^\infty(X, E) \to C^\infty(X, F)$ will be denoted $\Psi^m(X; E, F)$, $\Psi^m_{\mathrm{cl}}(X; E, F)$ respectively. (We shall also write $\Psi^m(X; E)$, $\Psi^m_{\mathrm{cl}}(X; E)$ instead of $\Psi^m(X; E, E)$, $\Psi^m_{cl}(X; E, E)$.) The composition formula is written as before after the choice of local coordinates and trivializations of bundles. If we choose Hermitian inner products in fibers of E and F and a C^∞ positive density on X, then for any $A \in \Psi^m(X; E, F)$, the formally adjoint operator $A^* \in \Psi^m(X; F, E)$ is well defined. If, moreover, $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ and a_m is its principal symbol, so $a_m(\nu) : E_x \to F_x$, where $\nu \in T^*X$, $\nu \neq 0$, $\pi(\nu) = x$ (here $\pi : T^*X \to X$ is the canonical projection), then the principal symbol of the adjoint operator A^* is $a_m^*(\nu) : F_x \to E_x$ where a_m^* is the Hermitian adjoint linear map to a_m.

More details about the material of this section can be found e.g. in **[Sh6]**, Sect.7.

3.6. Trace class operators (sufficient conditions)

Earlier (in 2.21), we gave formulas for the trace of trace class operators. Now we will give several sufficient conditions for operators to be in the trace class. We will concentrate on the case when operators are considered on a compact closed manifold but all the results will be valid on non-compact manifolds provided the operators under consideration have Schwartz kernels with compact supports. Also for simplicity we will consider only operators in scalar functions; all results will be automatically true for operators acting in sections of vector bundles.

So let X be a compact closed C^∞-manifold, $n = \dim X$. We will suppose that a positive smooth density is fixed on X (though all conditions actually do not depend on this density), so the Hilbert space $L^2(X)$ and the Sobolev spaces $H^s(X)$ are well defined.

LEMMA 1. *Let $A \in \Psi^l(X)$ where $l < -n/2$. Then $A \in \mathcal{S}_2(L^2(X))$, i.e. A is a Hilbert-Schmidt operator in $L^2(X)$.*

PROOF. Using a partition of unity, we can represent A as a sum of operators with C^∞ Schwartz kernels and operators of the form $\varphi A \psi$ where $\varphi, \psi \in C_c^\infty(U)$, U is a coordinate neighborhood. Now $\varphi A \psi$ has the Schwartz kernel supported in $U \times U$ and having (modulo operators with the kernel in $C_c^\infty(U \times U)$) the form

$$K(x,y) = (2\pi)^{-n} \int \varphi(x) e^{i(x-y)\cdot\xi} a(x,\xi) \psi(y)\, d\xi$$

where $a \in S^l(U \times \mathbb{R}^n)$. But $l < -n/2$ implies that $a(x,\cdot) \in L^2(\mathbb{R}^n)$, uniformly with respect to $x \in \operatorname{supp} \varphi$. Integrating (in the sense of the Fourier-Plancherel transform) with respect to ξ we obtain

$$K(x,y) = \varphi(x) L(x, x-y) \psi(y)$$

where $L(x,\cdot) \in L^2(\mathbb{R}^n)$, uniformly with respect to $x \in \operatorname{supp} \varphi$. It follows that $K \in L^2(U \times U)$, hence the statement of Lemma follows. $\square$

THEOREM 1. *Let $C \in \Psi^\ell(X)$, $\ell < -n$. Then $C \in \mathcal{S}_1(L^2(X))$ i.e. C is a trace class operator in $L^2(X)$.*

PROOF. Let $A \in \Psi_{\mathrm{cl}}^{-\ell/2}(X)$ be an elliptic operator (of order $-\ell/2$), $B \in \Psi_{\mathrm{cl}}^{\ell/2}(X)$ its parametrix. In particular, $BA = I - T$ where $T \in \Psi^{-\infty}(X)$. Multiplying by C on the right, we obtain $C = B(AC) + TC$. But $AC \in \Psi^{\ell/2}(X)$, hence B, AC, T, C are Hilbert-Schmidt operators due to Lemma 1. It follows that $C \in \mathcal{S}_1(L^2(X))$. $\square$

COROLLARY 1. *If C is an operator in $L^2(X)$ with a C^∞ Schwartz kernel, then C is a trace class operator.*

Now let us give some conditions which do not require for the operator in question to be a pseudo-differential operator.

LEMMA 2. *Let C be a bounded operator in $L^2(X)$, such that there exists an elliptic (pseudo-differential) operator A of order $m > n$ such that AC is a bounded operator in $L^2(X)$. Then C is a trace class operator.*

PROOF. Let B be a parametrix of A, so $B \in \Psi^{-m}(X)$, hence B is in the trace class. We have $BA = I - T$ with $T \in \Psi^{-\infty}(X)$. Therefore again $C = B(AC) + TC$, and now it remains to notice that both B and T are trace class operators due to Theorem 1. $\square$

COROLLARY 2. *If C is an integral operator in $L^2(X)$ with the Schwartz kernel $K_C \in C^{n+2}(X \times X)$, then T is in the trace class.*

PROOF. Let $\ell \in \mathbb{Z}_+$ is chosen so that $2\ell = n+1$ or $n+2$. Consider an elliptic differential operator A of order 2ℓ on X (e.g. Δ^ℓ where Δ is the Laplacian of a Riemannian metric) and apply Lemma 2, which gives the result because AT has a continuous kernel, hence is a Hilbert-Schmidt operator, hence is a bounded operator. $\square$

This Corollary can be considerably improved, especially if we use Sobolev spaces. We will need a Hilbert space

$$H^s(X) \otimes L^2(X) \subset L^2(X) \otimes L^2(X) = L^2(X \times X),$$

where $s \geq 0$. It is obtained as a Hilbert space tensor product (see 2.12) of $H^s(X)$ and $L^2(X)$ and has the norm defined by

$$\|K\|^2 = \int_M \|K(\cdot, y)\|_s^2 dy,$$

where $\|\cdot\|_s$ is the norm in $H^s(X)$.

Now we have the following

PROPOSITION 1. *Let C be an operator in $L^2(X)$ with the Schwartz kernel $K_C \in H^s(X) \otimes L^2(X)$, where $s > n/2$. Then C is in the trace class.*

PROOF. Let $\epsilon > 0$ is chosen so that $s > n/2 + \epsilon$. Let A be an elliptic (pseudo-differential) operator of order $n/2 + \epsilon$, B its parametrix, $BA = I - T$, where $T \in \Psi^{-\infty}(X)$. Then the order of B is $-n/2 - \epsilon$, so B is a Hilbert-Schmidt operator in $L^2(X)$. Clearly AC has a Schwartz kernel

$$K_{AC} = (A \otimes I)K_C.$$

It follows that $K_{AC} \in L^2(X \times X)$, so AC is a Hilbert-Schmidt operator too.

Now multiplying AC by B from the left, we obtain $C = B(AC) + TC$, and it becomes clear that C is in the trace class because all four operators B, AC, T, C are Hilbert-Schmidt operators. □

COROLLARY 3. *Let C be an operator in $L^2(X)$ such that its Schwartz kernel K_C in any local coordinates has derivatives of the form $D_x^\alpha K_C(x, y)$ with $|\alpha| \leq k$, where $k > n/2$, and these derivatives are continuous on $X \times X$. Then C is in the trace class in $L^2(X)$.*

THEOREM 2. *Let C be an operator in $L^2(X)$ that is represented as a composition $C = i \circ \tilde{C}$, where $\tilde{C} : L^2(X) \to H^{n+\epsilon}(X)$ is a bounded linear operator, $\epsilon > 0$, and $i : H^{n+\epsilon}(X) \to L^2(X)$ is the natural inclusion. Then C is a trace class operator.*

PROOF. Let A be an elliptic operator of order $n + \epsilon$, $B \in \Psi^{-n-\epsilon}(X)$ its parametrix, so that $BA = I - T$ where $T \in \Psi^{-\infty}(X)$. Then

$$C = BAC + TC = BAi\tilde{C} + TC$$

and it remains to notice that $Ai\tilde{C}$ is bounded in $L^2(X)$ and B, T are in the trace class. □

COROLLARY 4. *Suppose that $C \in \mathcal{B}(L^2(X))$ and $\operatorname{Im} C \subset H^{n+\epsilon}(X)$ with $\epsilon > 0$. Then C is a trace class operator in $L^2(X)$.*

PROOF. The operator C may be considered as a composition $C = i \circ \tilde{C}$ with i as in Theorem 2 and $\tilde{C} : L^2(X) \to H^{n+\epsilon}(X)$. But $\tilde{C}$ has a closed graph since C is bounded. Hence $\tilde{C}$ is bounded and we may apply Theorem 2. □

Now let us consider the case of a n-dimensional manifold X which is not necessarily compact. We will still assume that a positive C^∞-density on X is given, so the Hilbert space $L^2(X)$ is well defined. Let C be an operator in $L^2(X)$, K_C its Schwartz kernel which is a distribution on $X \times X$. It will be also convenient to use local Sobolev spaces.

DEFINITION 1. The *local Sobolev spaces* are defined as

$$H^s_{\text{loc}}(X) = \{u \mid u \in \mathcal{D}'(X), u|_U \in H^s(U)\}$$

for any coordinate neighborhood $U \subset X$ such that U is the compact in X and local coordinates in U can be extended to a neighborhood of $\overline{U}$. (Here $H^s(U)$ is defined using the local coordinates that are given on U.)

Also define the *compactly supported Sobolev spaces* as

$$H^s_{\text{comp}}(X) = \{u \mid u \in H^s_{\text{loc}}(X),\ \operatorname{supp} u \text{ is a compact subset in } X\},$$

i.e.

$$H^s_{\text{comp}}(X) = H^s_{\text{loc}}(X) \cap \mathcal{E}'(X).$$

THEOREM 3. *Suppose that K_C has a compact support in $X \times X$ and one of the following conditions is true:*

1) $C \in \Psi^\ell(X), \ell < -n$;
2) *There exists an elliptic (pseudo-differential) operator A of order $m > n$ such that AC is a bounded operator in $L^2(X)$;*
3) $K_C \in C^{n+2}(X \times X)$;
4) $K_C \in H^s(U) \otimes L^2(U)$ *with $s > n/2$ on any coordinate neighborhood U with a compact closure, such that the local coordinates can be extended to a neighborhood of $\bar{U}$;*
5) $\operatorname{Im} C \subset H^{n+\epsilon}_{\text{loc}}(X)$ *(hence $\operatorname{Im} C \subset H^{n+\epsilon}_{\text{comp}}(X)$) with $\epsilon > 0$.*

Then C is a trace class operator in $L^2(X)$.

PROOF. There exists a compact subset $K \subset X$ such that $\operatorname{supp} K_C \subset K \times K$. We can find a compact n-dimensional C^∞ submanifold (with boundary) $X_0 \subset X$ such that $K \subset X_0$. Obviously we can restrict everything to X_0. But now we can include X_0 into a compact closed n-dimensional manifold $\hat{X}$ (e.g. we may take the double of X_0). Then the statements of Theorem are reduced to the case when X is replaced by $\hat{X}$ and there they were already proved before. $\square$

3.7. The Atiyah-Bott formula

THEOREM 1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ are inverse to each other modulo trace class operators, i.e.*

$$I - BA \in \mathcal{S}_1(\mathcal{H}_1), \quad I - AB \in \mathcal{S}_1(\mathcal{H}_2)$$

Then A is a Fredholm operator and

$$\operatorname{ind} A = \operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB)$$

(abstract Atiyah-Bott formula).

If $\mathcal{H}_1 = \mathcal{H}_2$, then this formula can also be written in the form

$$\operatorname{ind} A = \operatorname{Tr}[A, B],$$

where $[A, B] = AB - BA$ is the commutator of A and B.

PROOF. It was proved earlier (see Lemma 2 in 3.1) that the existence of an inverse modulo trace class operators implies that A is Fredholm. The proof will be completed if we prove the following two statements:

(i) $\operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB)$ does not depend on the choice of B satisfying the conditions $I - BA \in \mathcal{S}_1(\mathcal{H}_1)$, $I - AB \in \mathcal{S}_1(\mathcal{H}_2)$;

(ii) there exists $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying these conditions such that the formula is true.

To prove (i) consider another operator B_1 satisfying the same conditions. Then

$$\begin{aligned}&[\operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB)] - [\operatorname{Tr}(I - B_1 A) - \operatorname{Tr}(I - AB_1)]\\&= \operatorname{Tr}((B_1 - B)A) - \operatorname{Tr}(A(B_1 - B)) = 0,\end{aligned}$$

because $B_1 - B$ is obviously a trace class operator, since an inverse modulo $\mathcal{S}_1$ is uniquely defined modulo $\mathcal{S}_1$. (Note that we argued as if $\mathcal{H}_1 = \mathcal{H}_2$, which we may as well assume because $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same dimension unless both are finite-dimensional, in which case the formula can be easily proved by simpler arguments.) So (i) is proved.

To prove (ii), let us use the special construction of B which was already applied earlier in the proof of Lemma 2 in 3.1. Namely, consider

$$A_1 = A|_{(\operatorname{Ker} A)^\perp} : (\operatorname{Ker} A)^\perp \to \operatorname{Im} A$$

and take $B = A_1^{-1}$ on $\operatorname{Im} A$ and $B = 0$ on $(\operatorname{Im} A)^\perp$. Then $I - BA = P_1$, $I - AB = P_2$ are orthogonal projections on $\operatorname{Ker} A$ and $\operatorname{Ker} A^* = (\operatorname{Im} A)^\perp$ respectively. Hence in this case

$$\operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB) = \operatorname{Tr} P_1 - \operatorname{Tr} P_2 = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A = \operatorname{ind} A,$$

which proves (ii). $\square$

EXERCISE. Let $\dim \mathcal{H}_j < \infty$, $j = 1, 2$. Prove that if $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_1$ are linear operators then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, hence

$$\operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB) = \dim \mathcal{H}_1 - \dim \mathcal{H}_2 = \operatorname{ind} A.$$

Hint. All eigenvalues of AB and BA except 0 coincide (with multiplicities counted).

THEOREM 2. *Let X be a compact closed manifold, E and F vector bundles over X, $A \in \Psi^m_{cl}(X; F, E)$ an elliptic operator, $B \in \Psi^{-m}_{cl}(X; F, E)$ its parametrix. Then $A \in \mathcal{F}(H^s(X, E), H^{s-m}(X, F))$ for every $s \in \mathbb{R}$ and*

$$\operatorname{ind} A = \operatorname{Tr}(I - BA) - \operatorname{Tr}(I - AB)$$

(the Atiyah-Bott formula), where Tr *is taken in the corresponding L^2-spaces of sections.*

PROOF. The fact that A is a Fredholm operator when considered as an operator $A_s : H^s(X, E) \to H^{s-m}(X, F)$, follows from Theorem 1 and the results about the trace class operators in 3.6.

It follows also that

$$\operatorname{ind} A_s = \operatorname{Tr}_s(I - BA) - \operatorname{Tr}_{s-m}(I - AB),$$

where Tr_s means operator trace in $H^s(X, E)$ and Tr_{s-m} is the trace in $H^{s-m}(X, F)$. Now the result will follow if we use the following

LEMMA. *Let C be an operator in $C^\infty(X, E)$ with a C^∞ Schwartz kernel K_C. Let us extend it by continuity to an operator $C_s : H^s(X, E) \to H^s(X, E)$. Then $\operatorname{Tr} C_s$ does not depend on s.*

SKETCH OF THE PROOF. We will use the Lidskii theorem (see [GK], Ch.III, Theorem 8.4), which states that for a trace class operator A in a Hilbert space

$$\operatorname{Tr} A = \sum_j \lambda_j$$

where λ_j are all (non-zero) eigenvalues of A, repeated according to their algebraic multiplicities. In our case, all generalized eigenfunctions of C_s, corresponding to non-zero eigenvalues (i.e. elements of $\operatorname{Ker}(C_s - \lambda I)^k$ for some $k \in \mathbb{Z}_+$, $\lambda \neq 0$), are obviously in $C^\infty(X, E)$. Therefore the non-zero eigenvalues and their algebraic multiplicities do not depend on s. Therefore $\operatorname{Tr} C_s$ does not depend on s. $\square$

REMARK. The right hand side of the Atiyah-Bott formula will not change if we add to B an operator of order $< -n - m$. So only n homogeneous terms in the asymptotic expansion of the symbol of B really matter. The cut-off functions involved also do not matter because they can only replace B by adding an operator with a C^∞ Schwartz kernel. This may be used to produce an index formula through the principal symbol of A in simplest situations (e.g. in case $X = S^n$ or $\mathbb{T}^n$, where S^n and $\mathbb{T}^n$ denote the n-dimensional sphere and torus respectively). The main difficulty is to understand why all lower-order terms of the symbol of A cancel in the right-hand side. In a similar context (with a use of heat kernels instead of parametrices) this can be done by using invariant theory or supersymmetry (see [**ABP, Gi, McSi**]).

Another advantage that we can get by changing B is that B can be made almost local i.e. we may choose B such that its Schwartz kernel is supported in a small neighborhood of the diagonal. If A is a differential operator (or an almost local pseudo-differential operator), then the same will be true for the operators $I - BA$, $I - AB$ which enter into the Atiyah-Bott formula. This will be important when we will consider covering spaces, because almost local operators can be lifted to any covering space, whereas this generally cannot be done for arbitrary operators with C^∞ kernels.

3.8. Unbounded linear operators in a Hilbert space

We shall give a brief description of necessary notions and facts of the theory of unbounded linear operators in a Hilbert space. This is important because sometimes it is necessary to consider differential operators as operators in a single space L^2 and not as operators acting from one Sobolev space to another one.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A *linear operator* A acting from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear map

$$A : D(A) \to \mathcal{H}_2,$$

where $D(A)$ is a linear (not necessarily closed) subspace in $\mathcal{H}_1$, which is called the *domain* of A.

For simplicity of notations we will only consider the case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ unless otherwise specified. Then we will say that A is a *linear operator* in $\mathcal{H}$.

As usual, define

$$\operatorname{Ker} A = \{f \mid f \in D(A), Af = 0\},$$
$$\operatorname{Im} A = \{g \mid g = Af \text{ for a } f \in D(A)\}.$$

If $D(A)$ is dense in $\mathcal{H}$, then we say that A is *densely defined*.

The *inverse operator* A^{-1} exists by definition if and only if $\operatorname{Ker} A = 0$, and in this case $D(A^{-1}) = \operatorname{Im} A$, $\operatorname{Im} A^{-1} = D(A)$, $A^{-1}g = f$ if and only if $g = Af$.

The product or *composition* BA of two linear operators is naturally defined and has the domain

$$D(BA) = \{f \mid f \in D(A), \text{ such that } Af \in D(B)\},$$

in this case $(BA)f = B(Af)$, $f \in D(BA)$.

The sum $A + B$ of two linear operators is a linear operator with the domain $D(A + B) = D(A) \cap D(B)$, $(A + B)f = Af + Bf$, $f \in D(A + B)$.

If there exists $C > 0$ such that $\|Af\| \le C\|f\|$, $f \in D(A)$, then A can be uniquely extended to a bounded linear operator $\overline{D(A)} \to \mathcal{H}$, in particular to an operator in $\mathcal{B}(\mathcal{H})$ if A was densely defined, so we may always suppose that bounded linear operators are defined on the whole Hilbert space $\mathcal{H}$.

The *graph* of a linear operator A is the subspace

$$G_A = \{\{f, Af\} \mid f \in D(A)\} \subset \mathcal{H} \oplus \mathcal{H}$$

(not necessarily closed). An important property of the graph is that it does not contain vectors of the form $\{0, g\}$ with $g \neq 0$. Conversely it is easy to see that if a subspace $G \subset \mathcal{H} \oplus \mathcal{H}$ has this property, then it is the graph of a linear operator.

We shall say that a linear operator A is *closed* if G_A is a closed subspace in $\mathcal{H} \oplus \mathcal{H}$, and *closable* if the closure $\overline{G_A}$ is the graph of a linear operator $\overline{A}$ (which is called then the *closure* of A). This means that $f_n \to 0$ and $Af_n \to g$ imply $g = 0$. Any continuous (bounded) operator is closable, and it is closed if and only if $D(A)$ is closed. Note that if A is closed then $\operatorname{Ker} A$ is a closed subspace. If A is closed and $\operatorname{Ker} A = 0$ then A^{-1} is closed too.

If A, B are two linear operators in $\mathcal{H}$ then we write $A \subset B$ (or $B \supset A$) and say that B is an *extension* of A if $D(A) \subset D(B)$ and $Bf = Af$ for all $f \in D(A)$. Clearly the closure $\overline{A}$ (if it exists) is an example of an extension of A.

Let us assume that A is densely defined in $\mathcal{H}$. Then the adjoint operator A^* is defined as follows: $D(A^*)$ is the set of all $g \in \mathcal{H}$, such that there exists $g^* \in \mathcal{H}$ satisfying

$$(Af, g) = (f, g^*), \quad f \in D(A),$$

Since $D(A)$ is dense in $\mathcal{H}$, the vector g^* is unique, and by this reason we can define $A^*g = g^*$. In particular,

$$(Af, g) = (f, A^*g), \quad f \in D(A), \quad g \in D(A^*).$$

Another description of $D(A^*)$ is the following: $g \in D(A^*)$ if and only if the linear functional $f \mapsto (Af, g)$ is continuous (or bounded) on $D(A)$, hence it can be extended by continuity to $\mathcal{H}$, and then by the Riesz theorem it can be written in the form $f \mapsto (f, g^*)$ with $g^* \in \mathcal{H}$.

EXERCISE 1. Check that if A_1 is densely defined and $A_1 \subset A_2$ then $A_1^* \supset A_2^*$.

EXERCISE 2. (a) Prove that A^* is always closed whatever A is.
(b) Prove that $\operatorname{Ker} A^* = (\operatorname{Im} A)^\perp$.

EXERCISE 3. Prove that operators $(A^*)^{-1}$ and $(A^{-1})^*$ exist if and only if A is densely defined, $\operatorname{Ker} A = 0$ and $\operatorname{Im} A$ is dense in $\mathcal{H}$. Then $(A^*)^{-1} = (A^{-1})^*$.

EXERCISE 4. Prove that $B^*A^* \subset (AB)^*$, provided A, B are linear operators in $\mathcal{H}$ such that A^*, B^* and $(AB)^*$ are well defined.

LEMMA 1. *A^* is densely defined if and only if A is closable; then*

$$\overline{A} = A^{**} = (A^*)^*.$$

PROOF. Let us introduce a skew-symmetric sesquilinear form on $\mathcal{H} \oplus \mathcal{H}$ given by

$$\langle \{f_1, f_2\}, \{g_1, g_2\} \rangle = (f_1, g_2) - (f_2, g_1).$$

Denote by $\perp_{\mathcal{J}}$ the orthogonality relation with respect to this form. Then $G_{A^*} = (G_A)^{\perp_{\mathcal{J}}}$ i.e. G_{A^*} is the orthogonal complement to G_A with respect to this form. (We can also use the usual inner product in $\mathcal{H} \oplus \mathcal{H}$ and write

$$G_{A^*} = (G_A)^{\perp_{\mathcal{J}}} = (\mathcal{J} G_A)^{\perp},$$

where $\mathcal{J} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is defined by $\mathcal{J}\{g_1, g_2\} = \{g_2, -g_1\}$.)

It follows that $(G_{A^*})^{\perp_{\mathcal{J}}} = \overline{G_A}$ since the form $\langle \cdot, \cdot \rangle$ is obviously non-degenerate in the same sense as the usual scalar product, because $\langle \alpha, \beta \rangle = (\alpha, \mathcal{J}\beta)$ for all $\alpha, \beta \in \mathcal{H} \oplus \mathcal{H}$ and $\mathcal{J}$ is an invertible operator in $\mathcal{H} \oplus \mathcal{H}$.

Note that $\overline{G_A}$ is a graph if and only if $\{0, h\} \in \overline{G_A}$ implies $h = 0$ or, equivalently, if $\{0, h\} \perp_{\mathcal{J}} G_{A^*}$ implies $h = 0$. But $\{0, h\} \perp_{\mathcal{J}} G_{A^*}$ means $h \perp D(A^*)$, hence $D(A^*)$ is dense if and only if A is closable.

Now $G_{A^{**}} = (G_{A^*})^{\perp_{\mathcal{J}}} = \overline{G_A}$ and the result follows. $\square$

DEFINITION 1. (a) A linear operator A in $\mathcal{H}$ is called *symmetric* if

$$(Af, g) = (f, Ag), \quad f, g \in D(A).$$

If A is densely defined, then this is obviously equivalent to the inclusion $A \subset A^*$.

(b) A linear operator A in $\mathcal{H}$ is called *self-adjoint* if it is densely defined and $A^* = A$.

EXERCISE 5. Prove that an operator A is symmetric if and only if $\langle \alpha, \beta \rangle = 0$ for every $\alpha, \beta \in G_A$, where the form $\langle \cdot, \cdot \rangle$ was introduced in the proof of Lemma 1.

Obviously a symmetric densely defined operator is always closable, and a self-adjoint operator A is always closed.

EXERCISE 6. Prove that if $A^* = A$ and $\operatorname{Ker} A = 0$ then $(A^{-1})^* = A^{-1}$ (i.e. A^{-1} is also self-adjoint provided A is).

DEFINITION 2. A linear operator A in $\mathcal{H}$ is called *essentially self-adjoint* if it has a closure $\overline{A}$ and $\overline{A}$ is self-adjoint.

EXERCISE 7. (a) Let A be a closed densely defined symmetric operator. Then $A^* = A$ if and only if A^* is symmetric.

(b) Let A be an arbitrary densely defined symmetric operator. Then A is essentially self-adjoint if and only if A^* is symmetric.

(c) A linear operator A is essentially self-adjoint if and only if A^* and A^{**} are well defined and $A^* = A^{**}$.

DEFINITION 3. Let A be a closed linear operator in $\mathcal{H}$, $z \in \mathbb{C}$. We shall say that z is a *regular point* for A if $(A - zI)^{-1}$ exists and is everywhere defined (hence bounded due to the Closed Graph Theorem). The *spectrum* $\sigma(A)$ (or $\operatorname{Spec}(A)$) is the complement in $\mathbb{C}$ of the set of all regular points.

EXERCISE 8. Prove that if $A = A^*$ then $\sigma(A) \subset \mathbb{R}$.

The difference between symmetric and self-adjoint operators is illustrated by the following example.

EXAMPLE. Consider the operator $D_x = \frac{1}{i}\frac{d}{dx}$ in $L^2([0,1])$. There are different possible choices of the domain which lead to different unbounded operators in $L^2([0,1])$. First consider the simplest domain $D(A_{(0)}) = C_c^\infty((0,1))$, $A_{(0)} = D_x$ on $D(A_{(0)})$. Then $A_{(0)}$ is symmetric. Its closure is a symmetric operator which is called the *minimal operator* associated with the differential operator D_x, and it is denoted $A_{\min}$. Its domain is in fact

$$D(A_{\min}) = H_0^1((0,1)) = \{u \mid u \in H^1((0,1)), u(0) = u(1) = 0\}.$$

Here $H^1((0,1))$ is the Sobolev space (consisting of all $u \in L^2((0,1))$, such that $u' \in L^2((0,1))$, where the derivative is taken in distributional sense). Note that $H^1((0,1)) \subset C([0,1])$, so the conditions $u(0) = u(1) = 0$ make sense. Obviously $A_{\min}$ is symmetric. But it is not self-adjoint. In fact, consider the adjoint operator which is called the *maximal operator* associated with the differential operator D_x and denoted $A_{\max}$. Obviously $A_{\max} = A_{\min}^* = A_{(0)}^*$ because the adjoint operator does not change if we replace the operator by its closure. Hence

$$D(A_{\max}) = H^1((0,1)) = \{u \mid u \in L^2((0,1)), u' \in L^2((0,1))\}.$$

This operator can be also obtained by taking the closure of the operator $A_{(1)} = D_x$ with the domain $D(A_{(1)}) = C^\infty([0,1])$. Obviously $A_{(1)}$ is not symmetric, hence $A_{\max}$ is also not symmetric.

So we have two closed operators generated by D_x: $A_{\min}$ and $A_{\max}$, and clearly $A_{\min} \subset A_{\max}$. Obviously $\dim[D(A_{\max})/D(A_{\min})] = 2$. Let us now try to describe symmetric operators A such that $A_{\min} \subset A \subset A_{\max}$. An example is the operator $A_\theta = D_x$ ($\theta \in \mathbb{R}$) with the domain $D(A_\theta) = \{u \mid u \in H^1((0,1)), u(1) = e^{i\theta}u(0)\}$. It is easy to see that A_θ is self-adjoint. Moreover, every self-adjoint extension of $A_{\min}$ has the form A_θ for some $\theta \in \mathbb{R}$.

EXERCISE 9. (a) In the notations of the Example above prove that if A is a symmetric operator such that $A_{\min} \subset A$, then $A \subset A_{\max}$ and then either $A = A_{\min}$ or $A = A_\theta$ for some $\theta \in \mathbb{R}$.

(b) Prove all statements made in the exposition of the Example above.

EXERCISE 10. Consider the differential expression $D_x^2 = -d^2/dx^2$ on $L^2([0,1])$. Define $A_{\min}$ as the closure of D_x^2 defined on the domain $C_c^\infty((0,1))$ and $A_{\max} = A_{\min}^*$.

(a) Prove that $A_{\min}$ is symmetric and closed but not self-adjoint.
(b) Prove that $A_{\max}$ is not symmetric.
(c) Consider D_x^2 with the domain $\{u \mid u \in C^\infty([0,1]), u(0) = u(1) = 0\}$. Prove that its closure is a self-adjoint operator with the domain

$$\{u \mid u \in H^2((0,1)), u(0) = u(1) = 0\}.$$

(d) The same as (c) but with the boundary conditions

$$u'(0) - \alpha u(0) = 0, \quad u'(1) - \beta u(1) = 0$$

where $\alpha, \beta \in \mathbb{R}$.

EXERCISE 11. Consider the differential operator $D_x^2 = -d^2/dx^2$ acting on $L^2(\mathbb{R})$ with the domain $C_c^\infty(\mathbb{R})$. Prove that it is essentially self-adjoint and its closure has the domain $H^2(\mathbb{R})$.

EXERCISE 12. Consider the differential operator $D_x^2 = -d^2/dx^2$ acting on $L^2((0,\infty))$ with the domain $C_c^\infty((0,\infty))$, denote $A_{\min}$ its closure and $A_{\max} = A_{\min}^*$. Prove that $\dim D(A_{\max})/D(A_{\min}) = 2$ and describe $D(A_{\min})$ and $D(A_{\max})$ in terms of Sobolev spaces. Describe all self-adjoint extensions of $A_{\min}$.

SPECTRAL THEOREM FOR UNBOUNDED SELF-ADJOINT OPERATORS

Let (X,μ) be a measure space, $a : X \to \mathbb{R}$ a measurable real-valued function on X (it may be defined almost everywhere). Define in $L^2(X, d\mu)$ the multiplication operator M_a by $(M_a f)(x) = a(x)f(x)$ with the domain

$$D(M_a) = \{f \mid f \in L^2(X, d\mu), af \in L^2(X, d\mu)\}.$$

EXERCISE 13. Prove that M_a is self-adjoint.

In fact this is the general form of a self-adjoint operator up to unitary equivalence. The following theorem, which is called the Spectral Theorem for self-adjoint operators (see e.g. [BeSh], Theorem 1.1 in Appendix 1, or [ReSi], Theorem VIII.4), states this fact more precisely.

THEOREM 1. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then there exists a measure space (X,μ), a unitary operator $U : \mathcal{H} \to L^2(X, d\mu)$ and a measurable almost everywhere defined function $a : X \to \mathbb{R}$, such that $A = U^{-1}M_aU$. (Here the equality of operators includes the equality of domains).*

Now again we can use this Theorem (as in the case of bounded self-adjoint operators) to construct functional calculus for the self-adjoint operator A. Namely if $f : \mathbb{R} \to \mathbb{C}$ is a Borel function, we can define $f(A)$ by the formula $f(A) = U^{-1}M_{f\circ a}U$ and the result does not depend on the choice of the spectral representation of A. If f is real-valued then $f(A)$ is self-adjoint. If f is bounded, then $f(A)$ is also bounded (and everywhere defined). An important example is provided by $f = \chi_\Delta$ (the characteristic function of an interval $\Delta \subset \mathbb{R}$). Then we obtain an operator $E(\Delta) = \chi_\Delta(A)$ which is an orthogonal projection. Define in particular $E_\lambda = E((-\infty, \lambda])$. Then E_λ is an increasing family of orthogonal projections depending on $\lambda \in \mathbb{R}$, i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ provided $\lambda \le \mu$. Besides $E_{\lambda+0} = E_\lambda$ in the strong operator topology and also $s\text{-}\lim_{\lambda\to-\infty} E_\lambda = 0$, $s\text{-}\lim_{\lambda\to+\infty} E_\lambda = I$.

For any Borel function $f : \mathbb{R} \to \mathbb{C}$, we have

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)dE_\lambda$$

in the weak sense. More precisely, if $x \in \mathcal{H}$, then the inclusion $x \in D(f(\lambda))$ is equivalent to

$$\|f(A)x\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda x, x) = \int_{-\infty}^{\infty} |f(\lambda)|^2 d\|E_\lambda x\|^2 < \infty,$$

and in this case

$$(f(A)x, y) = \int_{-\infty}^{\infty} f(\lambda)d(E_\lambda x, y), \quad x \in D(A), y \in \mathcal{H}.$$

In particular,

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

in the same weak sense.

EXERCISE 14. (a) Prove that if $A = A^*$ and $A = U^{-1} M_a U$ is its spectral representation, then

$$\operatorname{Spec}(A) = \{\lambda \mid \lambda \in \mathbb{R}, \mu\{x \mid x \in X, |a(x) - \lambda| < \epsilon\} > 0 \text{ for every } \epsilon > 0\}.$$

(b) Prove that if $A = A^*$ and $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ is its spectral representation through the spectral family of projections, then

$$\operatorname{Spec}(A) = \{\lambda \mid \lambda \in \mathbb{R}, E_{\lambda+\epsilon} - E_{\lambda-\epsilon} \neq 0 \text{ for every } \epsilon > 0\}.$$

(c) Prove that if $A = A^*$ then

$$\operatorname{Spec}(A) = \{\lambda \mid \lambda \in \mathbb{R}, \text{ there exists a sequence } f_n, n = 1, 2, \dots, \text{ such that } f_n \in D(A), \|f_n\| = 1 \text{ and } \lim_{n\to\infty} \|(A - \lambda I) f_n\| = 0\}$$

(d) Find the spectrum of the operators $A_{\min}$, $A_{\max}$ and A_θ from the Example above.

(e) Find the spectra of the operators from Exercises 10 - 12.

POLAR DECOMPOSITION OF CLOSED UNBOUNDED OPERATORS

THEOREM 2. *Let A be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then A can be uniquely represented in the form $A = US$ where S is a self-adjoint non-negative operator in $\mathcal{H}$, U a partial isometry in $\mathcal{H}$ such that*

$$\operatorname{Ker} U = \operatorname{Ker} S = (\operatorname{Im} S)^\perp.$$

Before proving Theorem 2, we shall prove the following Lemma which is the most important step in the proof of the theorem.

LEMMA 2. *Let A be a closed densely defined operator in $\mathcal{H}$. Then A^*A is self-adjoint (and in particular closed and densely defined).*

PROOF. Consider in $\mathcal{H} \oplus \mathcal{H}$ an operator given by the matrix

$$M = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

with the natural domain $D(M) = D(A^*) \oplus D(A)$. It is easy to check that

$$M^* = \begin{pmatrix} 0 & A^{**} \\ A^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} = M,$$

so M is self-adjoint. Therefore M^2 is also self-adjoint (e.g. due to the Spectral Theorem). But obviously

$$M^2 = \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}, \quad (M^2)^* = \begin{pmatrix} (AA^*)^* & 0 \\ 0 & (A^*A)^* \end{pmatrix},$$

hence both A^*A and AA^* are self-adjoint. □

SKETCH OF THE PROOF OF THEOREM 2. After Lemma 2 the proof proceeds as in the case of bounded operators (see 2.3). We necessarily have $S^2 = A^*A$, hence $S = \sqrt{A^*A}$. (We omit the proof that the non-negative square root is uniquely defined; it exists due to the Spectral Theorem). Now U is defined on $\operatorname{Im} S$ by $U(Sf) = Af$ (it may be proved that $D(S) = D(A)$). It remains to define $U = 0$ on $\operatorname{Ker} S = (\operatorname{Im} S)^\perp$. □

More details about this proof can be found e.g. in [**ReSi**], Sect. VIII.9.

UNBOUNDED OPERATORS AFFILIATED WITH A VON NEUMANN ALGEBRA

Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$, and let A be a (unbounded) linear operator in $\mathcal{H}$.

DEFINITION 4. We shall say that A is *affiliated to* $\mathcal{M}$ and write $A\eta\mathcal{M}$ if $U^{-1}AU = A$ for every unitary $U \in \mathcal{M}'$.

So for $A \in \mathcal{B}(\mathcal{H})$ the relation $A\eta\mathcal{M}$ is equivalent to the inclusion $A \in \mathcal{M}$. If A is closed and densely defined, and $A = US$ its polar decomposition, then the relation $A\eta\mathcal{M}$ is equivalent to $\{U \in \mathcal{M}$ and $S\eta\mathcal{M}\}$. If A is self-adjoint, $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ are its spectral projections then $A\eta\mathcal{M}$ is equivalent to $E_\lambda \in \mathcal{M}$ for every $\lambda \in \mathbb{R}$. It follows that $A\eta\mathcal{M}$, $A^* = A$ imply $f(A) \in \mathcal{M}$ for every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$ and $f(A)\eta\mathcal{M}$ for any Borel function $f : \mathbb{R} \to \mathbb{C}$.

Let A be a closed densely defined linear operator with the domain $D(A)$ in a Hilbert space $\mathcal{H}$, $B \in \mathcal{B}(\mathcal{H})$. We will say that A *commutes with* B, or B *commutes with* A, or A and B *commute*, if $BA \subset AB$, i.e. AB is an extension of AB. This means that for any $u \in D(A)$ we have $Bu \in D(A)$ and $BAu = ABu$. So it is required in particular that $D(A)$ is invariant under B i.e. $B(D(A)) \subset D(A)$. If B in invertible (in $\mathcal{B}(\mathcal{H})$) (e.g. if B is unitary), then the relation $BA \subset AB$ is equivalent to $B^{-1}AB = A$.

LEMMA 3. *Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$, A a closed densely defined operator in $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *$A\eta\mathcal{M}$.*
(ii) *A commutes with every $B \in \mathcal{M}'$.*
(iii) *A commutes with every $B \in S$ where $S \subset \mathcal{M}'$ is symmetric (i.e. $B \in S$ implies $B^* \in S$) and generates $\mathcal{M}'$ (see Definition 2 in 2.5).*

PROOF. Clearly (ii)$\Longrightarrow$(iii) and (ii)$\Longrightarrow$(i). Also (i)$\Longrightarrow$(ii) because every operator $B \in \mathcal{M}'$ can be presented as a linear combination of four unitary operators from $\mathcal{M}'$ (see Theorem 3 in 2.5). So to finish the proof it suffices to establish the implication (iii)$\Longrightarrow$(ii).

Note that the operators which commute with A, form a $*$-algebra. Therefore we can assume from the start that S is a $*$-algebra, hence $\mathcal{M}'$ is a weak closure of S. Now it suffices to establish that the commuting with A is preserved after taking a weak limit. Let us take a net $\{B_\gamma\}$ in $\mathcal{B}(\mathcal{H})$ with w-lim $B_\gamma = B$, $B \in \mathcal{B}(\mathcal{H})$, and assume that every B_γ commutes with A. Then

$$(AB_\gamma u, v) = (B_\gamma Au, v) = (B_\gamma u, A^*v)$$

for every $u \in D(A)$, $v \in D(A^*)$. Taking the limit in the second and third terms we obtain $(BAu, v) = (Bu, A^*v)$ or

$$(Au, B^*v) = (Bu, A^*v), \quad u \in D(A),\ v \in D(A^*).$$

It follows from this equality that for any fixed $u \in D(A)$

$$|(Bu, A^*v)| = |(Au, B^*v)| \le C\|v\|, \quad v \in D(A^*),$$

where $C = C(A, B, u)$. This implies that $Bu \in D(A^{**}) = D(A)$, and $(ABu, v) = (BAu, v)$ for all $u \in D(A), v \in D(A^*)$. Since $D(A^*)$ is dense in $\mathcal{H}$, we obtain $ABu = BAu$ for all $u \in D(A)$, so B commutes with A. $\square$

Unbounded Γ-operators in Hilbert Γ-modules

Slightly modifying the definitions and results above we can extend them to unbounded operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert Γ-modules. Let $L_g, g \in \Gamma$, denote the operators which define the action of Γ both in $\mathcal{H}_1$ and $\mathcal{H}_2$ (the choice between them is easily made from the context). We will say that A is a Γ-*operator* if $AL_g = L_g A$ for all $g \in \Gamma$. This means in particular that $D(A)$ is invariant under the action of Γ.

The graph of A is a linear subspace $G_A \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ consisting of the pairs $\{f, Af\}$ with $f \in D(A)$. By definition the operator is *closed* if and only if its graph is a closed subspace in $\mathcal{H}_1 \oplus \mathcal{H}_2$. In this case G_A itself is a Hilbert space with respect to the Hilbert structure induced from $\mathcal{H}_1 \oplus \mathcal{H}_2$. The corresponding norm is given by

$$||\{u, v\}||^2 = ||u||^2 + ||v||^2.$$

Under the canonical projection $\pi_1 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_2$ the graph maps bijectively onto $D(A)$. So we can transfer the Hilbert structure from G_A to $D(A)$. The corresponding norm is called the *graph norm*. So with this norm $D(A)$ is a Hilbert space and $A : D(A) \to \mathcal{H}_2$ becomes a bounded linear operator.

If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert Γ-modules then there is a *diagonal action* of Γ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by $L_g\{u, v\} = \{L_g u, L_g v\}$ which makes $\mathcal{H}_1 \oplus \mathcal{H}_2$ a Hilbert Γ-module.

Lemma 4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert Γ-modules, A a closed densely defined operator acting from $\mathcal{H}_1$ to $\mathcal{H}_2$, $D(A)$ its domain, G_A its graph. Then*

(a) *A is a Γ-operator if and only if its graph G_A is Γ-invariant (hence a Hilbert Γ-module) under the diagonal action of Γ.*
(b) *$D(A)$ with the graph norm is a Hilbert Γ-module.*

Proof. The first statement is straightforward and the second immediately follows from the fact that the canonical projection $\pi_1 : G_A \to D(A)$ is a unitary isomorphism if we consider $D(A)$ with the graph norm. □

Self-adjoint elliptic operators on compact closed manifolds

Let X be a compact closed manifold, E a C^∞ vector bundle on X. Let us suppose that a positive C^∞-density on X is given and a Hermitian structure on the fibers of E is also given. Then an inner product $(\cdot, \cdot)$ on $C^\infty(X, E)$ naturally appears and the completion of $C^\infty(X, E)$ with respect to the corresponding norm is a Hilbert space $L^2(X, E)$.

Theorem 3. 1) *Let $A : C^\infty(X, E) \to C^\infty(X, E)$, $A \in \Psi^m_{\rm cl}(X; E)$ be an elliptic operator of order m. Suppose that $m > 0$ and A is symmetric on $C^\infty(X, E)$ (or, in other words, formally self-adjoint) i.e.*

$$(Au, v) = (u, Av), \quad u, v \in C^\infty(X, E).$$

Then A is essentially self-adjoint in $L^2(X, E)$ and its closure $\overline{A}$ is a self-adjoint operator in $L^2(X, E)$ with the domain

$$D(\bar{A}) = H^m(X, E).$$

2) *A has discrete spectrum in the following precise sense: there exists an orthonormal basis $\{\psi_j \mid j = 1, 2, \ldots\}$ in $L^2(X, E)$, such that $A\psi_j = \lambda_j \psi_j$ (i.e. ψ_j is an eigenfunction of A with the eigenvalue λ_j) and $|\lambda_j| \to \infty$ as $j \to \infty$. Moreover $\psi_j \in C^\infty(X, E)$, $j = 1, 2, \ldots$.*

PROOF. 1) Since $A : H^m(X, E) \to L^2(X, E)$ is continuous, $H^m(X, E) \subset D(\bar{A})$. Obviously $\bar{A} \subset A^*$, so it is sufficient to prove that $D(A^*) \subset H^m(X, E)$. But $D(A^*) = \{u \mid u \in L^2(X, E),\ Au \in L^2(X, E)\}$ where A is applied in the sense of distributions. Hence the result follows from the elliptic regularity (see Corollary in 3.5).
2) All operators $A - \lambda I$ are Fredholm. It follows (consider a multiplication operator!) that the spectrum consists of isolated eigenvalues of finite multiplicity. The eigenfunctions are in $C^\infty(X, E)$ due to the elliptic regularity. □

3.9. Pseudo-differential operators on a Γ-manifold

Let M be a Γ-manifold, such that $X = M/\Gamma$ is a compact closed manifold, $n = \dim M$. We shall fix a Riemannian metric of X and lift it to M, so M will be equipped with a Γ-invariant Riemannian metric. We shall also need a Γ-invariant metric or distance function $d : M \times M \to [0, \infty]$. It can be constructed e.g. as the Riemannian distance $d(x, y)$ which is equal to the infimum of the Riemannian lengths of all arcs connecting x and y with the agreement $d(x, y) = \infty$ if x, y are in different connected components.

Let us consider a Γ-invariant locally finite covering of M by small balls. It can be constructed if we take a finite covering of X by small balls, then lift all the balls to M and take connected components of the liftings. Such a covering can be written in the form

$$M = \bigcup_{\substack{1\le j\le k\\ g\in\Gamma}} gU_j,$$

where all U_j (and gU_j) are small balls. Let $\sum_{\substack{1\le j\le k\\ g\in\Gamma}} \varphi_{j,g} = 1$ be a Γ-invariant partition of unity subordinated to this covering, i.e. $\varphi_{j,g} \in C_c^\infty(gU_j)$, $\varphi_{j,g}(x) = \varphi_{j,e}(g^{-1}x)$, $\varphi_{j,g} \ge 0$. This will allow us to construct all necessary Γ-invariant objects (Sobolev spaces, classes of pseudo-differential operators etc.)

Let us suppose that local coordinates are chosen on all U_j and by the group translations (i.e. by operators L_g defined by $L_g f(x) = f(g^{-1}x)$) they are translated to all $gU_j, g \in \Gamma, j = 1, \dots, k$.

DEFINITION 1. The *Sobolev space* $H^s(M)$ consists of all $u \in H^s_{\text{loc}}(M)$ such that

$$\|u\|_s^2 = \sum_{\substack{i\le j\le k\\ g\in\Gamma}} \|\varphi_{j,g}u\|^2_{s,gU_j} < \infty,$$

where $\|\cdot\|_{s,gU_j}$ means the Sobolev norm in the chosen local coordinates on gU_j. Similarly if E is a vector Γ-bundle on M, then the Sobolev space of sections $H^s(M, E)$ is defined in the same way after a choice of a Γ-invariant set of trivializations of E over the neighborhoods gU_j.

It is easy to see that the spaces $H^s(M), H^s(M, E)$ do not depend on the arbitrary choices that were made in their definition. Only the norm depends on the choices but another choice leads to an equivalent norm. The group Γ can also be replaced by another discrete group Γ_1 such that M is a Γ_1-manifold with a compact quotient manifold M/Γ_1. This also does not change the Sobolev spaces.

If $s \in \mathbb{Z}_+$, then the equivalent description of $H^s(M)$ is

$$H^s(M) = \{u \mid u \in L^2(M),\ Au \in L^2(M) \text{ for any } \Gamma\text{-invariant differential operator } A \text{ of order} \le s\}.$$

Here $L^2(M)$ is defined with respect to a Γ-invariant positive measure with a positive C^∞ density. The Sobolev space $H^s(M, E)$ with $s \in \mathbb{Z}_+$ can be defined similarly.

If s is an integer, $s < 0$, then $H^s(M)$ can be equivalently described as

$$H^s(M) = \{u \mid u \in \mathcal{D}'(M), u = \sum_{k=1}^{N} A_k f_k \text{ where } f_k \in L^2(M) \text{ and } A_k \text{ are } \Gamma\text{-invariant differential operators of order} \le -s\}$$

Definition 2. $\Psi_\Gamma^m(M)$ is the class of all Γ-invariant operators $A \in \Psi^m(M)$, where Γ-invariant means that $AL_g = L_gA$ for every $g \in \Gamma$. Similarly the class $\Psi^m_{\mathrm{cl};\Gamma}(M)$ consists of all $A \in \Psi^m_{\mathrm{cl}}(M)$ such that $AL_g = L_gA$ for every $g \in \Gamma$ i.e. $\Psi^m_{\mathrm{cl};\Gamma}(M) = \Psi^m_{\mathrm{cl}}(M) \cap \Psi^m_\Gamma(M)$. We shall define $\Psi_\Gamma^{-\infty}(M) = \bigcap_{m\in R} \Psi^m_\Gamma(M)$ i.e. $\Psi_\Gamma^{-\infty}(M)$ is the set of all operators with C^∞ Schwartz kernels that commute with all the operators $L_g, g \in \Gamma$.

Similarly classes $\Psi^m_\Gamma(M; E, F)$, $\Psi^m_\Gamma(M; E)$ etc. are defined for any pair of vector Γ-bundles (E, F) over M.

Remark 1. Note that the Γ-invariance property of a pseudo-differential operator leaves a lot of freedom in adding operators with smooth kernels. (This freedom does not exist in case of differential operators.) In case of an infinite group Γ (or, equivalently, non-compact M) a Γ-invariant operator T with a C^∞ Schwartz kernel (i.e. $T \in \Psi_\Gamma^{-\infty}(M)$) may be even unbounded in $L^2(M)$. As an example, we can take $M = \mathbb{R}$, $\Gamma = \mathbb{Z}$ (acting by translations) and T a convolution with a function $f \in C^\infty(\mathbb{R})$ with an unbounded Fourier transform $\tilde{f}$. (Such a function f can be obtained e.g. by the inverse Fourier transform from an unbounded function $\tilde{f} \in L^1(\mathbb{R})$ with a compact support.)

To avoid this difficulty we need an appropriate modification of the property of the operators to be properly supported.

Recall that an operator $A : C_c^\infty(M) \to C^\infty(M)$ is called properly supported if both projection $\pi_j : \operatorname{supp} K_A \to M,\ j = 1, 2$, are proper maps. If A is Γ-invariant, this means that K_A has a compact support in $\frac{M\times M}{\Gamma}$.

For an operator A on M to be properly supported, it is sufficient that $K_A(x, y) = 0$ if $d(x, y) > C$, with a finite $C \ge 0$. This motivates the following

Definition 3. Let $A : C_c^\infty(M) \to C^\infty(M)$ be an operator with the Schwartz kernel K_A (which is a distribution or a distributional section of an appropriate vector bundle on $M \times M$ – see 3.11 for more detail). We will say that A is *ϵ-local*, where $\epsilon \ge 0$ if $K_A(x, y) = 0$ if $d(x, y) > \epsilon$.

So for an operator $A : C_c^\infty(M) \to C^\infty(M)$ to be properly supported it is sufficient that it is C-local with some $C \ge 0$. All differential operators are 0-local.

Vice versa, if A is properly supported and Γ-invariant, then A is C-local with a finite $C \ge 0$.

Note that if A_j is ϵ_j-local, $j = 1, 2$, then A_1A_2 is $(\epsilon_1 + \epsilon_2)$-local.

LEMMA 1. *If $A \in \Psi_\Gamma^m(M)$ (or $A \in \Psi_{\mathrm{cl};\Gamma}^m(M)$), then for every $\epsilon > 0$ there exists an ϵ-local operator $A_1 \in \Psi_\Gamma^m(M)$ (or $A_1 \in \Psi_{\mathrm{cl};\Gamma}^m(M)$ respectively) such that $A - A_1 \in \Psi_\Gamma^{-\infty}(M)$. In particular this is true on X (we can take $M = X$ and trivial Γ). The same statements are true for the operators in sections of vector Γ-bundles.*

PROOF. First we should find a function $\chi_\epsilon \in C^\infty(M \times M)$ satisfying the following conditions:

(i) $\operatorname{supp} \chi_\epsilon \subset \{\{x, y\} \mid d(x, y) < \epsilon\}$;
(ii) $\chi_\epsilon = 1$ in a neighborhood of the diagonal;
(iii) $\chi_\epsilon(gx, gy) = \chi_\epsilon(x, y)$ for every $x, y \in M,\ g \in \Gamma$.

The last condition means that χ_ϵ is lifted from $\frac{M \times M}{\Gamma}$. So we can start by constructing a function in $C^\infty\left(\frac{M\times M}{\Gamma}\right)$ satisfying natural conditions replacing (i), (ii) (take the distance on $\frac{M\times M}{\Gamma}$ induced by $d(\cdot, \cdot)$, instead of $d(\cdot, \cdot)$ in (i); take the image of the diagonal in $\frac{M\times M}{\Gamma}$ instead of the diagonal in (ii)) and then lift this function to $M \times M$.

Now it remains to choose A_1 having the Schwartz kernel $\chi_\epsilon K_A$ and to use the pseudolocality of A. □

COROLLARY 1. *Suppose that A is an elliptic Γ-invariant pseudo-differential operator which is ϵ_0-local with some $\epsilon_0 > 0$. Then for every $\epsilon > 0$ there exists an ϵ-local Γ-invariant parametrix B for A. For this parametrix*

$$BA = I - T_1, \qquad AB = I - T_2,$$

where both operators $T_j \in \Psi_\Gamma^{-\infty}$ are $(\epsilon_0 + \epsilon)$-local.

PROOF. The construction of parametrix can be made Γ-invariant by use of a Γ-invariant covering by coordinate neighborhoods and a Γ-invariant partition of unity. Then it remains to apply Lemma 1. □

PROPOSITION 1 (Lifting of ϵ-local operators). *Define*

$$r_0 = \inf\{d(x, gx) \mid x \in M, g \in \Gamma - \{e\}\} > 0.$$

Let $A_0 : C^\infty(X) \to C^\infty(X)$ be an ϵ-local operator where $\epsilon < r_0/2$. Then there exists a unique ϵ-local lifting of A_0 i.e. an ϵ-local operator $A : C^\infty(M) \to C^\infty(M)$ such that $A(\pi^* u) = \pi^* A_0 u$. Here $u \in C^\infty(X)$ and $\pi : M \to X$ is the canonical projection. Besides if $A_0 \in \Psi^m(X)$ or $\Psi_{\mathrm{cl}}^m(X)$, then $A \in \Psi_\Gamma^m(M)$ or $\Psi_{\mathrm{cl};\Gamma}^m(M)$ respectively, and if A_0 is a differential operator, then A is also a differential operator. The same statements are true for the operators in sections of vector bundles.

PROOF. The condition $\epsilon < r_0/2$ implies that on M the following property of the distance function is true: if $d(x, y) < \epsilon$ then $d(gx, y) > \epsilon$ for every $g \in \Gamma \setminus \{e\}$, because

$$d(gx, y) \geq d(gx, x) - d(x, y) > r_0 - \epsilon > \epsilon.$$

Now if K_0 is the Schwartz kernel of A_0, then A can be constructed as the operator with the Schwartz kernel K defined as follows: $K(x, y) = K_0(\pi x, \pi y)$ if $d(x, y) < \epsilon$ and $K(x, y) = 0$ if $d(x, y) \geq \epsilon$. □

EXERCISE. Give a direct construction of the lifting in terms of the action of A on functions with a small support.

THEOREM 1. *Let $A \in \Psi^m_\Gamma(M; E, F)$ be properly supported. Then for any $s \in \mathbb{R}$ the operator A extends to a bounded linear operator*

$$A : H^s(M, E) \to H^{s-m}(M, F)$$

which commutes with the action of Γ in Sobolev spaces, i.e. A is a morphism of Hilbert Γ-modules.

PROOF. The proof is routinely obtained from the corresponding local result. Let us assume for simplicity of notations that E, F are trivial bundles with one-dimensional fibers $\mathbb{C}$. Let us use a Γ-invariant partition of unity $\sum_{\substack{1\le j\le k\\ g\in\Gamma}} \varphi_{j,g} = 1$ subordinated to a covering of M by small balls as described at the beginning of this section. Then we can write

$$\|Au\|^2_{s-m} = \sum_{i,j,g,h} \left(A(\varphi_{i,g}u), A(\varphi_{j,h}u)\right)_{s-m},$$

where $(\cdot,\cdot)_s$ means the inner product in $H^s(M)$ corresponding to the norm $\|\cdot\|_s$. Since A is properly supported, there exists $L > 0$ such that $d(g,h) \le L$ in all non-vanishing terms in the sum. Therefore we obtain from the local boundedness theorem (Theorem 4 in 3.4) that

$$\begin{aligned}\|Au\|^2_{s-m} &\le \sum_{\substack{i,j,g,h\\ d(g,h)\le L}} \|A(\varphi_{i,g}u)\|_{s-m}\|A(\varphi_{j,h}u)\|_{s-m} \le C \sum_{\substack{i,j,g,h\\ d(g,h)\le L}} \|\varphi_{i,g}u\|_s\|\varphi_{j,h}u\|_s \\ &\le C \sum_{\substack{i,j,g,h\\ d(g,h)\le L}} \frac{1}{2}\left(\|\varphi_{i,g}u\|^2_s + \|\varphi_{j,h}u\|^2_s\right) \le C_1\|u\|^2_s,\end{aligned}$$

which proves the theorem. □

THEOREM 2 (Uniform Elliptic Regularity). *Let $A \in \Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ be an elliptic properly supported operator. Suppose that $u \in H^{-N}(M, E)$ for some $N \in \mathbb{R}$ and that $Au = f \in H^{s-m}(M, F)$. Then $u \in H^s(M, E)$.*

PROOF. If $B \in \Psi^{-m}_{\mathrm{cl};\Gamma}(M; F, E)$ is a properly supported parametrix of A, then $BA = I - T$ where $T \in \Psi^{-\infty}_\Gamma(M; E)$ is again properly supported. It follows that $u = Bf + Tu$ and it remains to use the previous boundedness Theorem □

REMARK 2. The a priori condition $u \in H^{-N}(M, E)$ is very important, which makes a difference if we compare with the case of a compact manifold, where this condition is automatically fulfilled for any distributional section of E.

COROLLARY 2. *Assume that M is a Γ-manifold (with a compact quotient M/Γ), E is a vector Γ-bundle over M. Then the Sobolev space $H^s(M, E)$ is a Hilbert Γ-module for any $s \in \mathbb{R}$.*

PROOF. The case $s = 0$ is obvious (see 2.16). Now assume that $s > 0$. Let $A \in A \in \Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ be an elliptic properly supported operator. Let us consider A as an unbounded linear operator from $L^2(M, E)$ to $L^2(M, F)$ with the domain $D(A)$ which consists of all $u \in L^2(M, E)$ such that $Au \in L^2(M, F)$ where Au is understood in the sense of distributions. This operator is obviously closed.

Then Theorem 2 implies that $D(A) = H^s(M,E)$. Due to Theorem 1 the identity operator $H^s(M,E) \to D(A)$ is bounded if $D(A)$ is considered with the graph norm. Therefore it has a closed graph, hence it is a topological isomorphism due to the Closed Graph Theorem. But A is a (unbounded) closed densely defined Γ-operator from $L^2(M,E)$ to $L^2(M,F)$. Hence with the graph norm $D(A)$ is a Hilbert Γ-module (see Lemma 4 in 3.8). Therefore $H^s(M,E)$ is a Hilbert Γ-module too due to Corollary 1 from 2.16.

The result in case $s < 0$ easily follows by duality. Namely, the space $H^s(M,E)$ is Γ-invariantly dual to the space $H^{|s|}(M,E)$ (in the standard sense of the duality of Banach spaces) with respect to the sesquilinear duality which extends the inner product in $L^2(M,E)$. On the other hand $H^{|s|}(M,E)$ is also dual to itself in the same sense (as any Hilbert space). These two dualities obviously induce a linear topological isomorphism between $H^s(M,E)$ and $H^{|s|}(M,E)$. Since we already proved that $H^{|s|}(M,E)$ is a Hilbert Γ-module, the same should hold for $H^s(M,E)$ due to Corollary 1 in 2.16. □

THEOREM 3 (Essential Self-adjointness of Elliptic Γ-invariant operators). *Let E be a Hermitian vector Γ-bundle over M and suppose that a positive Γ-invariant C^∞ density on M is given. Let A be an elliptic properly supported Γ-invariant operator $A \in \Psi^m_{\mathrm{cl}}(M;E)$, $m > 0$. If A is symmetric in $L^2(M,E)$, then A is essentially self-adjoint and the domain of its closure is $H^m(M,E)$.*

PROOF. Clearly $H^m(M,E) \subset D(\bar{A})$ since A extends to a bounded linear operator $H^m(M,E) \to L^2(M,E)$. One the other hand, $D(A^*) \subset H^m(M,E)$ due to Theorem 2 and $D(\bar{A}) \subset D(A^*)$. It follows that $D(\bar{A}) = D(A^*) = H^m(M,E)$, which gives the desired result. □

Γ-TRACE CLASS OPERATORS (SUFFICIENT CONDITIONS)

LEMMA 2. *Let $A \in \Psi^\ell_\Gamma(M)$ where $\ell < -n/2$, $n = \dim M$, and A is properly supported. Then A is a Γ-Hilbert-Schmidt operator.*

PROOF. Obviously $K_A \in L^2_{\mathrm{loc}}(M \times M)$ and $\operatorname{supp} K_A$ is compact in $\frac{M\times M}{\Gamma}$. Hence $K_A \in L^2\left(\frac{M\times M}{\Gamma}\right)$ and A is a Γ-Hilbert-Schmidt operator. □

Now let C be a bounded linear operator in $L^2(M)$ with a Schwartz kernel K_C.

THEOREM 4. *Suppose that C is properly supported, Γ-invariant, and one of the following conditions is true:*

1) $C \in \Psi^\ell_\Gamma(M)$, $\ell < -n$;
2) *There exists an operator $A \in \Psi^m_{\mathrm{cl};\Gamma}(M)$ which is elliptic of order $m > n$ and properly supported, such that AC is a bounded operator in $L^2(M)$;*
3) $K_C \in C^{n+2}(M \times M)$;
4) $K_C \in H^s(U) \otimes L^2(U)$ *with $s > n/2$ on any coordinate neighborhood U with a compact closure, such that the local coordinates can be extended to a neighborhood of $\bar{U}$;*
5) $\operatorname{Im} C \subset H^{n+\epsilon}(M)$ *with* $\epsilon > 0$.

Then C is a Γ-trace class operator.

PROOF. The proof is just a repetition of the proof of the corresponding facts for a compact manifold (see 3.6), and it works because we already developed the necessary facts about Γ-invariant pseudo-differential operators and Sobolev spaces.

Let us choose a properly supported Γ-invariant elliptic operator A or order $-\ell/2$; in particular $A \in \Psi^{-\ell/2}_{\mathrm{cl};\Gamma}(M)$. Let $B \in \Psi^{\ell/2}_{\mathrm{cl};\Gamma}(M)$ be its properly supported parametrix, so $BA = I - T$ with a properly supported $T \in \Psi^{-\infty}_{\Gamma}(M)$. Now if $C \in \Psi^{\ell}_{\Gamma}(M)$, $\ell < -n$, and C is properly supported then representing $C = B(AC) + TC$ we see that due to Lemma 2, all the operators B, AC, T, C are Γ-Hilbert-Schmidt, hence C is in Γ-trace class.

The same reasoning applies if A is elliptic of order $m > n$ and AC is bounded (then B and T will be in Γ-trace class due to the first part that is already proved).

Using an elliptic Γ-invariant differential operator A of order $2\ell = n+1$ or $n+2$ we get the result if $K_C \in C^{n+2}(M \times M)$.

If C satisfies the condition in 4), we can use a properly supported Γ-invariant elliptic operator A of order $n/2 + \epsilon$ with $\epsilon > 0$, $n/2 + \epsilon < s$, and apply the same arguments.

Now let us assume that the condition 5) is fulfilled. Then the operator $C : L^2(M) \to H^{n+\epsilon}(M)$ is bounded due to the Closed Graph Theorem. Let us choose a properly supported operator $A \in \Psi^{n+\epsilon}_{\mathrm{cl};\Gamma}(M)$, which is elliptic of order $n + \epsilon$. Then AC is bounded and the condition 2) is fulfilled, so we conclude that C is in the Γ-trace class. □

COROLLARY 3. *If C is a properly supported Γ-invariant operator in $L^2(M)$ with the Schwartz kernel $K_C \in C^\infty(M \times M)$ (in other words if $K_C \in C^\infty_c\left(\frac{M\times M}{\Gamma}\right)$) then C is a Γ-trace class operator.*

Of course all the statements about Γ-trace class operators here are true for operators in sections of Γ-bundles.

3.10. Index Theory in von Neumann algebras

We shall begin with a more general context which is purely algebraic and does not include any topology. Later we shall add topology and see what it leads to.

Let the following data be given:

(i) A unital associative algebra $\mathcal{M}$ over $\mathbb{C}$ (the coefficient field $\mathbb{C}$ can be replaced by any other one here but we shall keep $\mathbb{C}$ for simplicity of notations), the identity element with respect to the multiplication will be denoted I;

(ii) A two-sided ideal $\mathcal{J}$ in $\mathcal{M}$, $\mathcal{J} \neq \mathcal{M}$;

(iii) A trace defined on $\mathcal{J}$, that is, a linear map $\tau : \mathcal{J} \to V$ (where V is a vector space) satisfying

$$\tau(AB) = \tau(BA), \quad A \in \mathcal{M},\ B \in \mathcal{J}.$$

In particular, V might be equal $\mathbb{C}$, but sometimes more general V are convenient. (For most purposes V may be even an abelian group; the vector space structure on V is rarely used.)

A universal trace (for fixed $\mathcal{M}$ and $\mathcal{J}$) may be obtained if we take the canonical map $\mathcal{J} \to \mathcal{J}/[\mathcal{M}, \mathcal{J}]$, where $[\mathcal{M}, \mathcal{J}]$ is the linear space spanned by the commutators $[A, B] = AB - BA$ with $A \in \mathcal{M}$, $B \in \mathcal{J}$.

DEFINITION 1. An element $A \in \mathcal{M}$ is called *relatively Fredholm or τ-Fredholm* if there exists $B \in \mathcal{M}$ such that $BA - I$, $AB - I \in \mathcal{J}$. In other words a τ-Fredholm element is an element in $\mathcal{M}$ such that its image in $\mathcal{M}/\mathcal{J}$ is invertible.

Denote by $\mathcal{F}_\tau(\mathcal{M})$ the set of all τ-Fredholm elements in $\mathcal{M}$. Let $p : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the canonical projection, $(\mathcal{M}/\mathcal{J})^\times$ the set of all invertible elements in $\mathcal{M}/\mathcal{J}$. It follows that $\mathcal{F}_\tau(\mathcal{M}) = p^{-1}((\mathcal{M}/\mathcal{J})^\times)$.

DEFINITION 2. For any $A \in \mathcal{F}_\tau(\mathcal{M})$ define its *τ-index* by

$$\operatorname{ind}_\tau A = \tau(I - BA) - \tau(I - AB),$$

where $B \in \mathcal{M}$ is chosen so that $I - BA,\ I - AB \in \mathcal{J}$. So $\operatorname{ind}_\tau A \in V$.

We can also rewrite the formula above as

$$\operatorname{ind}_\tau A = \tau([A, B]).$$

LEMMA 1. *$\operatorname{ind}_\tau A$ is well defined i.e. it does not depend on the choice of B in its definition.*

PROOF. Let B_1, B_2 be two elements in $\mathcal{M}$ such that $I - B_jA,\ I - AB_j \in \mathcal{J}$, $j = 1, 2$. Then $p(B_j)$ are inverse elements to $p(A)$ in $A/\mathcal{J}$. Hence $p(B_1) = p(B_2)$ and $C = B_1 - B_2 \in \mathcal{J}$. Hence $\tau([A, C]) = 0$ and

$$\tau([A, B_1]) = \tau([A, B_2]) + \tau([A, C]) = \tau([A, B_2]),$$

which ends the proof. □

Note that $\operatorname{ind}_\tau A = 0$ for every invertible element A in $\mathcal{M}$.

REMARK 1. A glimpse to a stability of the τ-index can be seen from the fact that if $A \in \mathcal{F}_\tau(\mathcal{M})$ and $c \in \mathbf{C} \setminus \{0\}$, then $cA \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}(cA) = \operatorname{ind} A$. (Indeed, taking $B \in \mathcal{M}$ which is an inverse to A modulo $\mathcal{J}$, we see that $c^{-1}B$ is an inverse to cA modulo $\mathcal{J}$, and $[cA, c^{-1}B] = [A, B]$.)

LEMMA 2. *If $A \in \mathcal{F}_\tau(\mathcal{M})$ and $C \in \mathcal{J}$ then $A + C \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}_\tau(A + C) = \operatorname{ind}_\tau A$.*

PROOF. Choosing $B \in \mathcal{M}$ such that $I - BA,\ I - AB \in \mathcal{J}$ we obtain

$$\operatorname{ind}_\tau(A + C) = \tau([A + C, B]) = \tau([A, B]) + \tau([C, B]) = \tau([A, B]) = \operatorname{ind}_\tau A,$$

as required. □

So in fact ind_τ provides a well defined map $\operatorname{ind}_\tau : (\mathcal{M}/\mathcal{J})^\times \to V$. We shall check now that it is a group homomorphism (if V is considered as an additive group).

LEMMA 3. *If $A_1, A_2 \in \mathcal{F}_\tau(\mathcal{M})$ then $A_1A_2 \in \mathcal{F}_\tau(\mathcal{M})$ and*

$$\operatorname{ind}_\tau(A_1A_2) = \operatorname{ind}_\tau A_1 + \operatorname{ind}_\tau A_2.$$

PROOF. Let us take B_1, B_2 such that

$$\begin{aligned} B_1A_1 &= I - S_1, \qquad A_1B_1 = I - S_2; \quad S_1, S_2 \in \mathcal{J}; \\ B_2A_2 &= I - T_1, \qquad A_2B_2 = I - T_2; \quad T_1, T_2 \in \mathcal{J}. \end{aligned}$$

Now for $A = A_1A_2$ we may take $B = B_2B_1$ as an inverse modulo $\mathcal{J}$. Then we obtain

$$\begin{aligned} (B_2B_1)(A_1A_2) &= B_2(I - S_1)A_2 = I - T_1 - B_2S_1A_2, \\ (A_1A_2)(B_2B_1) &= A_1(I - T_2)B_1 = I - S_2 - A_1T_2B_1. \end{aligned}$$

Hence

$$\begin{aligned}\operatorname{ind}_\tau(A_1A_2) = \tau(T_1) + \tau(B_2S_1A_2) - \tau(S_2) - \tau(A_1T_2B_1) = \\ \tau(T_1) - \tau(S_2) + \tau(A_2B_2S_1) - \tau(B_1A_1T_2) = \\ \tau(T_1) - \tau(S_2) + \tau((I-T_2)S_1) - \tau((I-S_1)T_2) = \\ \tau(T_1) - \tau(S_2) + \tau(S_1) - \tau(T_2) + \tau(S_1T_2) - \tau(T_2S_1) = \\ \tau(S_1) - \tau(S_2) + \tau(T_1) - \tau(T_2) = \operatorname{ind}_\tau A_1 + \operatorname{ind}_\tau A_2,\end{aligned}$$

as required. □

Now let us add a topological element to the picture. Assume that the following condition is fulfilled:

(iv) $\mathcal{M}$ is a Banach algebra i.e. $\mathcal{M}$ is a Banach space with a norm $\|\cdot\|$ such that $\|AB\| \le \|A\|\|B\|$ for every $A, B \in \mathcal{M}$, $\|I\| = 1$.

Under this condition we can prove stability of the τ-index:

LEMMA 4. *$\mathcal{F}_\tau(\mathcal{M})$ is an open subset in $\mathcal{M}$ and $\operatorname{ind}_\tau : \mathcal{F}_\tau(\mathcal{M}) \to V$ is a locally constant function.*

PROOF. It is sufficient to prove that if $A \in \mathcal{F}_\tau(\mathcal{M})$, $B \in \mathcal{M}$ is chosen so that $BA = I - T_1$, $AB = I - T_2$, $T_j \in \mathcal{J}$, $j = 1, 2$, and $C \in \mathcal{M}$ satisfies the inequality $\|C\| < \|B\|^{-1}$ then $A + C \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}_\tau(A + C) = \operatorname{ind}_\tau A$. To do this write $B(A + C) = I + BC - T_1$, so if we take $B_1 = (I + BC)^{-1}B$ then $B_1(A + C) = I - \tilde{T}_1$, $\tilde{T}_1 \in \mathcal{J}$. Similarly we can construct B_2 such that $(A + C)B_2 = I - \tilde{T}_2$, $\tilde{T}_2 \in \mathcal{J}$. Therefore $A + C \in \mathcal{F}_\tau(\mathcal{M})$. But obviously

$$\operatorname{ind}_\tau B_1 = \operatorname{ind}_\tau(I + BC)^{-1} + \operatorname{ind}_\tau B = \operatorname{ind}_\tau B,$$

$\operatorname{ind}_\tau B = -\operatorname{ind}_\tau A$, $\operatorname{ind}_\tau B_1 = -\operatorname{ind}_\tau(A + C)$, hence $\operatorname{ind}_\tau(A + C) = \operatorname{ind}_\tau A$. □

Now let us introduce the norm closure $\bar{\mathcal{J}}$ of the ideal $\mathcal{J}$ in $\mathcal{M}$. Note that $\bar{\mathcal{J}}$ is still a (2-sided) ideal in $\mathcal{M}$.

LEMMA 5. *If $A \in \mathcal{M}$ then the condition $A \in \mathcal{F}_\tau(\mathcal{M})$ is equivalent to the existence of $B \in \mathcal{M}$ such that $BA = I - C_1$, $AB = I - C_2$, $C_j \in \bar{\mathcal{J}}, j = 1, 2$. If $A \in \mathcal{F}_\tau(\mathcal{M})$ and $C \in \bar{\mathcal{J}}$ then $A + C \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}_\tau(A + C) = \operatorname{ind}_\tau A$.*

PROOF. We can find $T_1 \in \mathcal{J}$ so that $\|T_1 - C_1\| < 1$. Then we can write $BA = I + (T_1 - C_1) - T_1$ and $B_1A = I - \tilde{T}_1$ with $B_1 = [I + (T_1 - C_1)]^{-1}B$, $\tilde{T}_1 = [I + (T_1 - C_1)]^{-1}T_1 \in \mathcal{J}$. Similarly we may construct $B_2 \in \mathcal{M}$ such that $AB_2 = I - \tilde{T}_2$, $\tilde{T}_2 \in \mathcal{J}$, hence $A \in \mathcal{F}_\tau(\mathcal{M})$. Moreover if $BA = I - C_1$ with $A \in \mathcal{F}_\tau(\mathcal{M})$, $B \in \mathcal{M}$ and $C_1 \in \bar{\mathcal{J}}$ then this argument shows that $\operatorname{ind}_\tau A = -\operatorname{ind} B$; hence $\operatorname{ind}_\tau(A + C) = \operatorname{ind}_\tau A$ for every $A \in \mathcal{F}_\tau(\mathcal{M})$ and $C \in \bar{\mathcal{J}}$. □

Let us add another element: an involution $* : \mathcal{M} \to \mathcal{M}$, $A \mapsto A^*$ and an involution $* : V \to V$ so that the following conditions are fulfilled:

$(\mathrm{v})_1$ $*$ is an antilinear map, $*^2 = \mathrm{id}$ on $\mathcal{M}$, $*(\mathcal{J}) \subset \mathcal{J}$ (hence $*(\mathcal{J}) = \mathcal{J}$);
$(\mathrm{v})_2$ $(AB)^* = B^*A^*$, $A, B \in \mathcal{M}$;
$(\mathrm{v})_3$ $\tau(A^*) = \tau(A)^*$.

LEMMA 6. *If $A \in \mathcal{F}_\tau(\mathcal{M})$ then $A^* \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}_\tau A^* = -(\operatorname{ind}_\tau A)^*$.*

PROOF. The proof is straightforward if we apply the involution to the equalities $BA = I - T_1$, $AB = I - T_2$, $T_j \in \mathcal{J}$. □

Now we shall switch to a situation when $\mathcal{M}$ is a von Neumann algebra in a Hilbert space $\mathcal{H}$ and τ is a (faithful, normal, semifinite) trace of $\mathcal{M}$. In this case we shall always take $\mathcal{J} = \mathcal{S}_1(\mathcal{M}) = \mathcal{S}_1(\mathcal{M}, \tau)$, i.e. $\mathcal{J}$ is the ideal of all τ-trace class operators. The values of τ will be in $\mathbb{C}$. Also we shall use the dimension function $\dim_\tau$ associated with the trace τ and defined on the set $\mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$, whose elements are all closed subspaces $L \subset \mathcal{H}$ which are affiliated to $\mathcal{M}$ (recall the corresponding notation : $L\eta\mathcal{M}$), which means that if $P_L \in \mathrm{Proj}(\mathcal{B}(\mathcal{H}))$, $\mathrm{Im}\, P_L = L$, then either $P_L \in \mathcal{M}$ or $P_L \in \mathrm{Proj}(\mathcal{M})$. The involution in $\mathcal{M}$ will always mean taking the adjoint operator (whereas the involution in $\mathbb{C}$ will be always the complex conjugation).

LEMMA 7. *Let $A \in \mathcal{M}$, $A = US$ its polar decomposition. Then $A \in \mathcal{F}_\tau(\mathcal{M})$ if and only if $U \in \mathcal{F}_\tau(\mathcal{M})$ and $S \in \mathcal{F}_\tau(\mathcal{M})$.*

PROOF. Obviously if B is an inverse to A modulo $\mathcal{J}$ then BU is an inverse to S modulo $\mathcal{J}$, and SB is a right inverse to U modulo $\mathcal{J}$. Applying the same argument to $A^* = SU^* \in \mathcal{F}_\tau(\mathcal{M})$ we see that $S \in \mathcal{F}_\tau(\mathcal{M})$. So $p(A)$ and $p(S)$ are invertible in $\mathcal{M}/\mathcal{J}$, hence $p(U)$ is also invertible in $\mathcal{M}/\mathcal{J}$. We have proved that $A \in \mathcal{F}_\tau(\mathcal{M})$ implies $U, S \in \mathcal{F}_\tau(\mathcal{M})$ and the inverse statement is obvious. □

LEMMA 8. *Let $A \in \mathcal{F}_\tau(\mathcal{M})$. Then $\dim_\tau \mathrm{Ker}\, A < \infty$ and $\dim_\tau \mathrm{Ker}\, A^* < \infty$.*

PROOF. Obviously $A^* \in \mathcal{F}_\tau(\mathcal{M})$, so it suffices to prove that $\dim_\tau \mathrm{Ker}\, A < \infty$. To do this find $B \in \mathcal{M}$ such that $BA = I - T$, $T \in \mathcal{J}$. Then

$$\mathrm{Ker}\, A \subset \mathrm{Ker}(I - T) = \mathrm{Ker}(I - T)^*(I - T) = \mathrm{Ker}(I - S),$$

where $S^* = S \in \mathcal{J}$. So it is sufficient to prove that $\dim_\tau \mathrm{Ker}(I - S) < \infty$.

Note that $S|_{\mathrm{Ker}(I-S)} = I|_{\mathrm{Ker}(I-S)}$. Denote $P = P_{\mathrm{Ker}(I-S)}$. Then P is a function of S (therefore P commutes with S) and by the Spectral Theorem we have $P \le |S|$. But $|S| \in \mathcal{J}$, hence due to Theorem from 2.18 we obtain $P \in \mathcal{J}$ as required. □

LEMMA 9. *Let $U \in \mathcal{M}$ be a partial isometry. Then $U \in \mathcal{F}_\tau(\mathcal{M})$ if and only if $\dim_\tau \mathrm{Ker}\, U < \infty$ and $\dim_\tau \mathrm{Ker}\, U^* < \infty$. Besides*

$$\mathrm{ind}_\tau\, U = \dim_\tau \mathrm{Ker}\, U - \dim_\tau \mathrm{Ker}\, U^*.$$

PROOF. Lemma 8 and the relations

$$U^*U = I - P_{\mathrm{Ker}\, U}, \qquad UU^* = I - P_{\mathrm{Ker}\, U^*}$$

clearly imply the desired result. □

We shall use the notation $\mathrm{codim}_\tau\, L = \dim_\tau L^\perp$ for closed subspaces $L \subset \mathcal{H}$, such that $L\eta\mathcal{M}$.

LEMMA 10. *Let $A \in \mathcal{M}$. Then $A \in \mathcal{F}_\tau(\mathcal{M})$ if and only if the following two conditions are fulfilled:*

(i) $\dim_\tau \mathrm{Ker}\, A < \infty$;
(ii) *there exists a closed subspace $L \subset \mathcal{H}$, such that $L\eta\mathcal{M}$, $L \subset \mathrm{Im}\, A$ and $\mathrm{codim}_\tau\, L < \infty$.*

PROOF. 1) Let $A \in \mathcal{F}_\tau(\mathcal{M})$. We have proved already that $\dim_\tau \mathrm{Ker}\, A < \infty$. Now find $B \in \mathcal{M}$ such that $AB = I - T$, $T \in \mathcal{J}$. Obviously $\mathrm{Im}\, A \supset \mathrm{Im}(I - T)$ so we have to prove that (ii) is true for $I - T$. Note that

$$\mathrm{Im}(I - T) \supset \mathrm{Im}(I - T)(I - T)^* = \mathrm{Im}(I - S), \; S = S^* \in \mathcal{J}.$$

Denote by $E(\Delta)$ the spectral projection of S corresponding to a Borel set $\Delta \subset \mathbb{R}$. Obviously $E((1/2, 3/2)) \in \mathcal{J}$ since $E((1/2, 3/2)) \le 2|S|$. Now let us denote $L = \operatorname{Im}(I - E((1/2, 3/2)))$. Then L is closed and $\operatorname{codim}_\tau L = \tau(E(1/2, 3/2))$. Also obviously $L \subset \operatorname{Im}(I - S)$, hence $L \subset \operatorname{Im} A$ which proves (ii).

2) Vice versa, suppose that (i), (ii) are fulfilled. Consider $A_1 = A|_{(\operatorname{Ker} A)^\perp}$ and take $L_1 = A_1^{-1}(L)$. Then $L_1 \eta \mathcal{M}$ and the map $A_1 : L_1 \to L$ is an algebraic isomorphism, hence it is a topological isomorphism due to the Closed Graph Theorem. Define $B_1 = A_1^{-1} : L \to L_1$ and take $B = B_1$ on L, $B = 0$ on $L^\perp$. Then $BA = I - P_1$, $AB = I - P_2$, where P_1, P_2 are projections on $L_1^\perp$ and $L^\perp$ respectively. So it remains to prove that $\operatorname{codim}_\tau L_1 < \infty$. To do it consider $L_2 = (\operatorname{Ker} A)^\perp \ominus L_1$. Obviously $\operatorname{codim}_\tau L_1 = \dim_\tau L_2 + \dim_\tau \operatorname{Ker} A$ so it is sufficient to prove that $\dim_\tau L_2 < \infty$. But $AL_2 \cap L = \{0\}$ so $(I - P_L)A$ defines an almost isomorphism of L_2 to a subspace in $L^\perp$, so $\dim_\tau L_2 < \infty$ as required. □

LEMMA 11. *Let $S = S^* \in \mathcal{M}$. Then $S \in \mathcal{F}_\tau(\mathcal{M})$ if and only if there exists $\epsilon > 0$ such that $E((-\epsilon, \epsilon)) \in \mathcal{J}$ (or $\tau(E((-\epsilon, \epsilon))) < \infty$), where $E(\Delta)$ is the spectral projection of S corresponding to a Borel set $\Delta \subset \mathbb{R}$. Besides if $S = S^* \in \mathcal{F}_\tau(\mathcal{M})$ then* $\operatorname{ind}_\tau S = 0$.

PROOF. 1) Obviously $E((-\epsilon, \epsilon)) \in \mathcal{J}$ implies $\dim_\tau \operatorname{Ker} S < \infty$, and if $L = \operatorname{Im}(I - E((-\epsilon, \epsilon)))$, then $L \subset \operatorname{Im} S$, $L \eta \mathcal{M}$ and $\operatorname{codim}_\tau L < \infty$, so $S \in \mathcal{F}_\tau(\mathcal{M})$. Also L is an invariant subspace with respect to S and $S_1 = S|_L : L \to L$ is a topological isomorphism. Hence if we define $B = S_1^{-1}$ on L, $B = 0$ on $L^\perp$ then $B \in \mathcal{M}$ and $BS = SB = I - P_{L^\perp}$. Hence $\operatorname{ind}_\tau S = 0$.

2) Vice versa, let $S \in \mathcal{F}_\tau(\mathcal{M})$. Let us find $L \eta \mathcal{M}$, $L \subset \operatorname{Im} S$, such that $\operatorname{codim}_\tau L < \infty$. Let $S_1 = S|_{(\operatorname{Ker} S)^\perp} : (\operatorname{Ker} S)^\perp \to \mathcal{H}$ and $L_1 = S_1^{-1}(L)$. Then $S_1 : L_1 \to L$ is a topological isomorphism. The same reasoning as in the proof of the previous Lemma shows that $\operatorname{codim}_\tau L_1 < \infty$. Also obviously there exists $\epsilon > 0$ such that $\|Su\| \ge \epsilon \|u\|$, $u \in L_1$. But on $\operatorname{Im} E((-\epsilon, \epsilon))$ the opposite inequality is true: $\|Su\| < \epsilon \|u\|$, $u \in \operatorname{Im} E((-\epsilon, \epsilon)) \setminus \{0\}$. Hence $I - P_{L_1}$ gives an almost isomorphism of $\operatorname{Im} E((-\epsilon, \epsilon))$ to a subspace in $L_1^\perp$. Therefore $\tau(E((-\epsilon, \epsilon))) = \dim_\tau \operatorname{Im} E((-\epsilon, \epsilon)) \le \operatorname{codim}_\tau L_1 < \infty$, which proves Lemma. □

Now we can establish that the τ-index is given by the "usual" formula:

LEMMA 12. *If $A \in \mathcal{F}_\tau(\mathcal{M})$, then*

$$\operatorname{ind}_\tau A = \dim_\tau \operatorname{Ker} A - \dim_\tau \operatorname{Ker} A^*.$$

PROOF. If $A = US$ is the polar decomposition of A then $U, S \in \mathcal{F}_\tau(\mathcal{M})$ and $\operatorname{ind}_\tau S = 0$ due to Lemmas 7 and 11. Hence, using Lemma 9, we obtain

$$\operatorname{ind}_\tau A = \operatorname{ind}_\tau U = \dim_\tau \operatorname{Ker} U - \dim_\tau \operatorname{Ker} U^* = \dim_\tau \operatorname{Ker} A - \dim_\tau \operatorname{Ker} A^*,$$

as required. □

Finally we can sum up everything in the following

THEOREM 1. 1) *Let $\mathcal{M}$ be a von Neumann algebra with a trace τ. For an operator $A \in \mathcal{M}$ the following conditions are equivalent:*

(i) *$A \in \mathcal{F}_\tau(\mathcal{M})$ (A is τ-Fredholm) i.e. there exists $B \in \mathcal{M}$ such that $BA - I$, $AB - I \in \mathcal{S}_1(\mathcal{M})$.*

(ii) $\dim_\tau \operatorname{Ker} A < \infty$ *and there exists a closed subspace $L \subset \mathcal{H}$, $L \eta \mathcal{M}$ such that $L \subset \operatorname{Im} A$ and* $\operatorname{codim}_\tau L < \infty$.

(iii) *If* $A = US$ *is the polar decomposition of* A *then* $\dim_\tau \operatorname{Ker} U < \infty$, $\dim_\tau \operatorname{Ker} U^* < \infty$ *and* $\tau(E(-\epsilon, \epsilon)) < \infty$ *for some* $\epsilon > 0$, *where* $E(\Delta)$ *is the spectral projection of* S *corresponding to a Borel set* $\Delta \subset \mathbb{R}$.

2) *If* $A \in \mathcal{F}_\tau(\mathcal{M})$ *then choosing* $B \in \mathcal{M}$ *such that* $I - BA$, $I - AB \in \mathcal{S}_1(\mathcal{M})$ *we obtain that the* τ*-index*

$$\operatorname{ind}_\tau A = \tau(I - BA) - \tau(I - AB)$$

does not depend on the choice of B *and is locally constant with respect to the norm topology on* $\mathcal{F}_\tau(\mathcal{M})$, *which is an open set in* $\mathcal{M}$ *with respect to the norm topology.*

3) $\operatorname{ind}_\tau(A_1A_2) = \operatorname{ind}_\tau A_1 + \operatorname{ind}_\tau A_2$ *for any* $A_1, A_2 \in \mathcal{F}_\tau(\mathcal{M})$.

4) $\operatorname{ind}_\tau A = \dim_\tau \operatorname{Ker} A - \dim_\tau \operatorname{Ker} A^*$, $A \in \mathcal{F}_\tau(\mathcal{M})$.

REMARK 2. In fact $A \in \mathcal{F}_\tau(\mathcal{M})$ implies that $\operatorname{Im} A$ is almost closed in the following sense: for every $\epsilon > 0$ there exists $L\eta\mathcal{M}$, $L \subset \operatorname{Im} A$ such that $\dim_\tau(\overline{\operatorname{Im} A} \ominus L) < \epsilon$. To prove this take the polar decomposition $A = US$ and the result reduces to the case of $S = S^* \in \mathcal{F}_\tau(\mathcal{M})$, where it becomes obvious since $\lim_{\epsilon\downarrow 0} \tau(E((0,\epsilon))) = 0$ due to the normality of the trace τ. (Here the normality should be applied to the family $E((0,\epsilon_0)) - E((0,\epsilon))$ where $\epsilon_0 > 0$ is chosen so that $\tau(E((0,\epsilon_0))) < \infty$.)

REMARK 3. We will mainly apply the results above to a particular case which is the most important for us: $\mathcal{M} = \mathcal{M}_\Gamma$ and $\tau = \operatorname{Tr}_\Gamma$ on a Γ-manifold.

EXERCISE 1. Give an example which shows that the conditions

$$\dim_\tau \operatorname{Ker} A < \infty, \qquad \dim_\tau \operatorname{Ker} A^* < \infty$$

are not sufficient for the operator A to be relatively Fredholm and that the difference $\dim_\tau \operatorname{Ker} A - \dim_\tau \operatorname{Ker} A^*$ is not necessarily locally constant outside $\mathcal{F}_\tau(\mathcal{M})$.

Γ-FREDHOLM OPERATORS AND Γ-INDEX

The results above provide an index theory for the operators $A \in \mathcal{M}$ where $\mathcal{M}$ is a von Neumann algebra with a trace τ in a fixed Hilbert space $\mathcal{H}$. In particular, we can consider an arbitrary Γ module $\mathcal{H}$ with the von Neumann algebra $\mathcal{M} = \mathcal{M}_\Gamma$ (which consists of all operators which commute with the action of Γ in $\mathcal{H}$) and with the trace Tr_Γ (see 2.14). However we need to consider elliptic operators from sections of one vector bundle to sections of another vector bundle, and also from one Sobolev space to another Sobolev space. So we see that it is convenient to extend our results to operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ where $\mathcal{H}_1, \mathcal{H}_2$ are two (possibly different) Γ-modules. In this case we will say that A is a Γ-*operator* or a Γ-*morphism* or a *morphism of Hilbert* Γ-*modules* if A is a bounded linear operator which commutes with the action of Γ in $\mathcal{H}_1$ and $\mathcal{H}_2$ (or, more precisely, intertwines these actions). This extension presents no difficulties, but we will formulate precise definitions. The results will be formulated in the form of exercises which can be done by routine repetition of the arguments given above for the case of the τ-Fredholm operators, τ-trace and τ-index.

DEFINITION 3. A Γ-operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is called Γ-*Fredholm* if there exists a Γ-operator $B : \mathcal{H}_2 \to \mathcal{H}_1$ such that the operators

$$I - BA : \mathcal{H}_1 \to \mathcal{H}_1 \quad \text{and} \quad I - AB : \mathcal{H}_2 \to \mathcal{H}_2$$

are in the Γ-trace class. Then the Γ-*index* of A is defined as

$$\operatorname{ind}_\Gamma A = \operatorname{Tr}_\Gamma(I - BA) - \operatorname{Tr}_\Gamma(I - AB).$$

Clearly in this case B is also Γ-Fredholm and $\operatorname{ind}_\Gamma B = -\operatorname{ind}_\Gamma A$.

We will denote the set of all Γ-Fredholm operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ by $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$.

A priori the Γ-index takes values in $\mathbb{C}$. However we will see later that in fact its values are always in $\mathbb{R}$.

The following definition provides an extension of the Definition in 2.23.

DEFINITION 4. 1) We will say that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a Γ-*Hilbert-Schmidt operator* if it is a Γ-morphism of Hilbert Γ-modules and $\operatorname{Tr}_\Gamma(A^*A) < \infty$. The set of all Γ-Hilbert-Schmidt operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ is denoted $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2; \operatorname{Tr}_\Gamma)$.

2) We will say that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a Γ-*trace class operator* if A is Γ-morphism of Hilbert Γ-modules and $\operatorname{Tr}_\Gamma |A| < \infty$. Here $|A| = \sqrt{A^*A}$ as in the generalized polar decomposition (see 2.25). The set of all Γ-trace class operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ is denoted $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2; \operatorname{Tr}_\Gamma)$.

LEMMA 13. (a) *$\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2; \operatorname{Tr}_\Gamma)$ and $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2; \operatorname{Tr}_\Gamma)$ are right modules over the ring of all Γ-operators in $\mathcal{H}_1$ and left modules over the ring of all Γ-operators in $\mathcal{H}_2$ (in particular they are both linear subspaces in the space of all Γ-operators $\mathcal{H}_1 \to \mathcal{H}_2$).*

(b) *If $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_1$ are Γ-operators such that one of them is in in the Γ-trace class or both are Γ-Hilbert-Schmidt, then both AB and BA are in the Γ-trace class and*

$$\operatorname{Tr}_\Gamma(AB) = \operatorname{Tr}_\Gamma(BA).$$

EXERCISE 2. Prove this Lemma using arguments from 2.18 and 2.19.

EXERCISE 3. Prove that the Γ-index of a Γ-Fredholm operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is well defined i.e. it does not depend on the choice of B in Definition 3.

Hint. Prove that $B - B_1 \in \mathcal{S}_1(\mathcal{H}_2, \mathcal{H}_1; \Gamma)$ for two different choices B, B_1 of the operator B in Definition 3.

EXERCISE 4. Prove that if $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ and $C \in \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ then $\operatorname{ind}_\Gamma(A + C) = \operatorname{ind}_\Gamma A$.

EXERCISE 5. Prove that if $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ and $B \in \mathcal{F}(\mathcal{H}_2, \mathcal{H}_3; \Gamma)$, then $BA \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_3; \Gamma)$ and

$$\operatorname{ind}_\Gamma(BA) = \operatorname{ind}_\Gamma A + \operatorname{ind}_\Gamma B.$$

DEFINITION 5. Denote by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ the norm closure of $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ The operators from $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ will be called Γ-*compact.*

EXERCISE 6. Prove that a Γ-morphism $A : \mathcal{H}_1 \to \mathcal{H}_2$ is Γ-Fredholm if and only if there exists a Γ-morphism $B : \mathcal{H}_2 \to \mathcal{H}_1$ such that both $I - BA$ and $I - AB$ are Γ-compact.

EXERCISE 7. Prove the following stability properties of the Γ-index:

(i) The set $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ is open in the set of all Γ-operators in the norm topology. The Γ-index

$$\operatorname{ind}_\Gamma : \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma) \longrightarrow \mathbb{C}$$

is locally constant in this open subset.

(ii) If $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ and $C \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$, then

$$\operatorname{ind}_\Gamma(A + C) = \operatorname{ind}_\Gamma A.$$

EXERCISE 8. Prove that for any Γ-Fredholm operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ the adjoint operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is also Γ-Fredholm and

$$\operatorname{ind}_\Gamma A^* = -\overline{\operatorname{ind}_\Gamma A},$$

where the bar over $\operatorname{ind}_\Gamma A$ means complex conjugation.

EXERCISE 9. If $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$, then

$$\dim_\Gamma \operatorname{Ker} A < \infty \quad \text{and} \quad \dim_\Gamma \operatorname{Ker} A^* < \infty.$$

EXERCISE 10. (i) Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a Γ-morphism of Hilbert Γ-modules, $A = US$ its polar decomposition. Then $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ if and only if $U \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ and $S \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_1; \Gamma)$.

(ii) For a partial isometry $U : \mathcal{H}_1 \to \mathcal{H}_2$ which is a Γ-operator, $U \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ if and only if $\dim_\Gamma \operatorname{Ker} U < \infty$ and $\dim_\Gamma \operatorname{Ker} U^* < \infty$. In this case

$$\operatorname{ind}_\Gamma U = \dim_\Gamma \operatorname{Ker} U - \dim_\Gamma \operatorname{Ker} U^*.$$

EXERCISE 11. For a Γ-operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ we have $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ if and only if the following two conditions are fulfilled:

(i) $\dim_\Gamma \operatorname{Ker} A < \infty$;

(ii) there exists a closed Γ-invariant subspace $L \subset \mathcal{H}_2$ such that $L \subset \operatorname{Im} A$ and $\operatorname{codim}_\Gamma L < \infty$. (Here $\operatorname{codim}_\Gamma L = \dim_\Gamma(\mathcal{H}_2 \ominus L)$.)

Let $S = S^*$ be a Γ-operator in a Hilbert Γ-module $\mathcal{H}$. Lemma 11 can be applied to the algebra $\mathcal{M}_\Gamma$ in $\mathcal{H}$, and we see that the inclusion $S \in \mathcal{F}(\mathcal{H}, \mathcal{H}; \Gamma)$ is equivalent to the existence of $\epsilon > 0$ such that $\operatorname{Tr}_\Gamma E((-\epsilon, \epsilon)) < \infty$. Besides in this case $\operatorname{ind}_\Gamma S = 0$.

EXERCISE 12. If $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$, then

$$\operatorname{ind}_\Gamma A = \dim_\Gamma \operatorname{Ker} A - \dim_\Gamma \operatorname{Ker} A^*.$$

Now we will summarize the main results in the form of the following

THEOREM 2. 1) *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert Γ-modules. For a Γ-operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ the following conditions are equivalent:*

(i) *$A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ (A is Γ-Fredholm) i.e. there exists a Γ-operator $B : \mathcal{H}_2 \to \mathcal{H}_1$ such that $BA - I$, $AB - I$ are in Γ-trace class.*

(ii) *$\dim_\Gamma \operatorname{Ker} A < \infty$ and there exists a closed Γ-invariant subspace $L \subset \mathcal{H}_2$, such that $L \subset \operatorname{Im} A$ and $\operatorname{codim}_\Gamma L < \infty$.*

(iii) *If $A = US$ is the polar decomposition of A then $\dim_\Gamma \operatorname{Ker} U < \infty$, $\dim_\Gamma \operatorname{Ker} U^* < \infty$ and $\operatorname{Tr}_\Gamma(E(-\epsilon, \epsilon)) < \infty$ for some $\epsilon > 0$.*

2) *If $A \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$ then choosing a Γ-operator $B : \mathcal{H}_2 \to \mathcal{H}_1$ such that $I - BA$, $I - AB$ are in Γ-trace class, we obtain that the Γ-index*

$$\operatorname{ind}_\Gamma A = \operatorname{Tr}_\Gamma(I - BA) - \operatorname{Tr}_\Gamma(I - AB)$$

does not depend on the choice of B and is locally constant with respect to the norm topology on $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2; \Gamma)$, which is an open set in the set of all Γ-operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ with respect to the norm topology.

3) *$\operatorname{ind}_\Gamma(A_1 A_2) = \operatorname{ind}_\Gamma A_1 + \operatorname{ind}_\Gamma A_2$ for any Γ-Fredholm operators $A_1 : \mathcal{H}_2 \to \mathcal{H}_3$, $A_2 : \mathcal{H}_1 \to \mathcal{H}_2$.*

4) $\operatorname{ind}_\Gamma A = \dim_\Gamma \operatorname{Ker} A - \dim_\Gamma \operatorname{Ker} A^*$ *for any* Γ*-Fredholm operator* $A : \mathcal{H}_1 \to \mathcal{H}_2$.

EXERCISE 13. Prove that for any Γ-Fredholm operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ its image $\operatorname{Im} A$ is almost closed in the following sense: for every $\epsilon > 0$ there exists a closed Γ-invariant subspace $L \subset \mathcal{H}_2$, such that $L \subset \operatorname{Im} A$ and $\dim_\Gamma(\overline{\operatorname{Im} A} \ominus L) < \epsilon$.

3.11. Γ-Fredholm properties and Γ-index of elliptic operators

In this section M will be a Γ-manifold with a quotient $X = M/\Gamma$ which is a compact closed manifold. We will start by discussing Γ-Fredholm properties of elliptic pseudo-differential operators on M.

We will assume that a Γ-invariant smooth positive measure $\tilde{\mu}$ (given locally by a smooth positive density) is fixed on M. Then it is actually a lifting of a smooth positive measure μ on X.

It was explained in Remark 1 in 3.9 that we can not work with arbitrary Γ-invariant pseudo-differential operators on M because Γ-invariant operators with smooth kernels can be even unbounded in L^2. There we restricted ourselves to properly supported Γ-invariant pseudo-differential operators. In particular, we required the Γ-invariant operators with smooth kernels to be properly supported. But this occurs to be too restrictive for some purposes, as will be clear in this section. By this reason we will extend the class of properly supported Γ-invariant operators to a wider class. In this class smoothing operators will be the operators which not only have smooth kernels but act appropriately in Sobolev spaces. Even though an explicit description of the corresponding kernels is more difficult, the operators themselves naturally appear in the calculus of operators (e.g. taking inverse operators) and they are easy to handle.

DEFINITION 1. Let us consider a Γ-invariant linear operator

$$A : C_c^\infty(M) \to \mathcal{D}'(M),$$

where $\mathcal{D}'(M)$ is the space of all distributions on M. We will say that A is *S-smoothing* if for any $s, t \in \mathbb{R}$ this operator extends to a continuous linear map of Sobolev spaces

$$A_{t,s} : H^s(M) \to H^t(M).$$

(Here we consider $H^t(M)$ as a subspace in $\mathcal{D}'(M)$, using the measure $\tilde{\mu}$.) The set of all S-smoothing operators will be denoted $S\Psi_\Gamma^{-\infty}(M)$.

REMARK 1. (i) We can equivalently say that A extends to a continuous linear map $A : H^{-\infty}(M) \to H^\infty(M)$ where

$$H^{-\infty}(M) = \bigcup_{s\in\mathbb{R}} H^s(M), \quad H^\infty(M) = \bigcap_{s\in\mathbb{R}} H^s(M).$$

Here $H^\infty(M)$ is considered with the Frechet topology given by the seminorms $\|\cdot\|_s$, $s \in \mathbb{R}$, where $\|\cdot\|_s$ is the norm in $H^s(M)$, and $H^{-\infty}(M)$ is considered with the inductive topology (or direct limit topology), so that the continuity of the map above is equivalent to saying that the restriction of this map to $H^s(M)$ is a continuous map $H^s(M) \to H^\infty(M)$ for any $s \in \mathbb{R}$.

(ii) The desired continuity property of the operator $A_{t,s}$ is automatically satisfied due to the Closed Graph Theorem if we know that $A_{t,s}$ maps $H^s(M)$ to $H^t(M)$ and this map is continuous if considered as a map from $H^s(M)$ to $\mathcal{D}'(M)$.

LEMMA 1. (i) *If A is a properly supported Γ-invariant operator with a Schwartz kernel $K_A \in C^\infty(M \times M)$, then $A \in S\Psi_\Gamma^{-\infty}(M)$.*
(ii) *If $A \in S\Psi_\Gamma^{-\infty}(M)$, then $K_A \in C^\infty(M \times M)$.*

PROOF. (i) follows from Theorem 1 in 3.9.

Let us prove (ii). To this end note first that A maps any Sobolev space $H^s(M)$ to $C^\infty(M)$ because $H^\infty(M) \subset C^\infty(M)$ due to the Sobolev embedding theorem (see 3.3). Now K_A can be expressed through A by the formula

$$K_A(x, y) = (A\delta_y)(x) = (A\delta_y, \delta_x),$$

where δ_y means the Dirac delta-function supported at y and taken with respect to the measure $\tilde{\mu}$, so that formally

$$\int_M \delta_y(z)\phi(z)d\tilde{\mu}(z) = \phi(y), \quad \phi \in C_c^\infty(M);$$

$(\cdot, \cdot)$ is the duality between functions and distributions which extends the inner product in $L^2(M)$.

It is easy to see that $y \mapsto \delta_y$ is a C^k-function of y with values in $H^s(M)$ if $s < -n/2 - k$. Hence the function $y \mapsto A\delta_y$ will be a C^k-function of y with values in $H^t(M)$ for any fixed $t \in \mathbb{R}$ if $A \in S\Psi_\Gamma^{-\infty}(M)$. By the Sobolev embedding theorem we conclude that $K_A \in C^k(M \times M)$ which implies the desired result because k is arbitrary. $\square$

LEMMA 2. *If $C \in S\Psi_\Gamma^{-\infty}(M)$, then C is a Γ-trace class operator.*

PROOF. Let us choose $m > n$ and an operator $A \in \Psi^m_{\mathrm{cl};\Gamma}(M)$ which is elliptic of order m and properly supported. Let $B \in \Psi^{-m}_{\mathrm{cl};\Gamma}(M)$ be its properly supported parametrix, so in particular $BA = I - T$ where $T \in \Psi_\Gamma^{-\infty}(M)$ is also properly supported. Then we can write $C = B(AC) + TC$. Now note that B and T are in the Γ-trace class due to Theorem 4 in 3.9. Since AC is bounded in $L^2(M)$ as a composition of bounded operators

$$C : L^2(M) \to H^m(M) \quad \text{and} \quad A : H^m(M) \to L^2(M),$$

we see that $B(AC)$ is also in the Γ-trace class. Similarly TC is in the Γ-trace class because C is bounded in $L^2(M)$. It follows that C is in the Γ-trace class. $\square$

Now we will introduce a useful class of Γ-invariant pseudo-differential operators, which will extend and replace the class of properly supported operators.

DEFINITION 2. A Γ-invariant pseudo-differential operator operator $A \in \Psi_\Gamma^m(M)$ is called *S-regular* if it can be presented in the form $A = A_1 + R$, where $A_1 \in \Psi_\Gamma^m(M)$ is properly supported and $R \in \Psi_\Gamma^{-\infty}(M)$. Denote by $S\Psi_\Gamma^m(M)$ the class of all such operators A. The class of all operators of the form above with a properly supported classical $A_1 \in \Psi^m_{\mathrm{cl};\Gamma}(M)$ will be denoted $S\Psi^m_{\mathrm{cl};\Gamma}(M)$.

We will naturally denote

$$S\Psi_\Gamma^\infty(M) = \bigcup_{m \in \mathbb{R}} S\Psi_\Gamma^m(M), \quad S\Psi^\infty_{\mathrm{cl};\Gamma}(M) = \bigcup_{m \in \mathbb{R}} S\Psi^m_{\mathrm{cl};\Gamma}(M).$$

Clearly

$$S\Psi_\Gamma^{-\infty}(M) = \bigcap_{m \in \mathbb{R}} S\Psi_\Gamma^m(M) = \bigcap_{m \in \mathbb{R}} S\Psi^m_{\mathrm{cl};\Gamma}(M).$$

Obviously we can also define the corresponding classes of operators in sections of vector Γ-bundles over M. Choosing a Γ-invariant hermitian metric on such a bundle E, we can define Sobolev spaces of sections of E. If E, F are two such bundles over M, then the notations $S\Psi^m_\Gamma(M; E, F)$, $S\Psi^m_\Gamma(M; E)$ etc. have a natural meaning.

The following Corollary usefully extends Theorem 4 from 3.9 to the case of operators which are not necessarily properly supported.

COROLLARY 1. *Suppose that C is a bounded Γ-invariant operator in $L^2(M)$ which satisfies one of the following conditions:*

1) *$C \in S\Psi^\ell_\Gamma(M)$, $\ell < -n$;*
2) *There exists an operator $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M)$ which is elliptic of order $m > n$ such that AC is bounded in $L^2(M)$;*
3) *$C = C_1 + R$, where C_1 is properly supported with the Schwartz kernel $K_{C_1} \in C^{n+2}(M \times M)$, and $R \in \Psi^{-\infty}_\Gamma(M)$;*
4) *$C = C_1 + R$, where $R \in \Psi^{-\infty}_\Gamma(M)$, C_1 is properly supported and $K_{C_1} \in H^s(U) \otimes L^2(U)$ with $s > n/2$ on any coordinate neighborhood U with a compact closure, such that the local coordinates can be extended to a neighborhood of $\bar{U}$;*
5) *$\operatorname{Im} C \subset H^{n+\epsilon}(M)$ with $\epsilon > 0$.*

Then C is a Γ-trace class operator.

PROOF. Under the conditions 1), 3), 4) the result immediately follows from Lemma 2 above and Theorem 4 in 3.9.

Assuming 2) is satisfied, we can write $A = A_1 + R$, where $A_1 \in \Psi^m_{\mathrm{cl};\Gamma}(M)$ and $R \in \Psi^{-\infty}_\Gamma(M)$. Then RC is obviously bounded in $L^2(M)$, hence $A_1 C$ is bounded too. Hence the result in this case follows from Theorem 4 from 3.9, which should be applied with A replaced by A_1 in the condition 2) there.

If the condition 5) is fulfilled, then the arguments from the proof of Theorem 4 in 3.9 can be applied without any changes. □

Now we will give a necessary and sufficient condition (though not very explicit) for the operator A to be in $S\Psi^{-\infty}_\Gamma(M)$; it will be formulated in terms of the Schwartz kernel K_A. We already know that the following conditions are necessary:

(a) $K_A \in C^\infty(M \times M)$;
(b) $K_A(gx, gy) = K_A(x, y)$ for all $x, y \in M$, $g \in \Gamma$.

PROPOSITION 1. *For an operator $A : C^\infty_c(M) \to \mathcal{D}'(M)$ to be in $S\Psi^{-\infty}_\Gamma(M)$ it is necessary and sufficient that its Schwartz kernel K_A satisfies the following condition:*

(c) *The function $x, y \mapsto B_1(x, D_x)B_2(y, D_y)K_A(x, y)$ is in $L^2\left(\frac{M \times M}{\Gamma}\right)$ for arbitrary Γ-invariant differential operators B_1, B_2 on M.*

PROOF. 1) We already proved that (a) and (b) are necessary (see Lemma 1 in this section and Lemma in 2.22). Therefore in proving that (c) is necessary we can assume that (a) and (b) are fulfilled.

So let us assume that $A \in S\Psi^{-\infty}_\Gamma(M)$. Then $BAC \in S\Psi^{-\infty}_\Gamma(M)$ for arbitrary Γ-invariant differential operators $B = B(x, D_x)$, $C = C(x, D_x)$ on M. In particular, BAC is bounded in $L^2(M)$ for any such B, C. It easily follows that A is a Γ-Hilbert-Schmidt operator and even a Γ-trace class operator due to Corollary 1 above. In particular, we have $K_A \in L^2\left(\frac{M \times M}{\Gamma}\right)$. But the same arguments apply to BAC with arbitrary B, C, so BAC is a Γ-Hilbert-Schmidt operator too. Now it remains to note that the Schwartz kernel K_{BAC} can be obtained from K_A by the formula

$$K_{BAC}(x,y) = B(x, D_x)C^t(y, D_y)K_A(x,y),$$

where C^t is the transposed differential operator to C i.e. it is defined by the relation

$$\langle Cu, v\rangle = \langle u, C^t v\rangle, \quad u, v \in C_c^\infty(M),$$

where

$$\langle u, v\rangle = \int_M u(x)v(x)d\tilde{\mu}(x) = (u, \bar{v}),$$

the bar over v means the complex conjugation.

2) Vice versa, assume that the condition (c) holds. Note first that (c) implies (b) because we can take $B_1 = B_2 = I$. It also implies (a) due to the Sobolev embedding theorem. So in proving that (c) is sufficient, we can assume that (a) and (b) hold too.

We have also seen in the first part of this proof that (c) is equivalent to saying that the operator BAC is Γ-Hilbert-Schmidt for any Γ-invariant differential operators B, C. In particular, BAC is bounded in $L^2(M)$ for any such B, C. But this immediately implies that $A \in S\Psi_\Gamma^{-\infty}(M)$. □

Remark 2. The condition (c) provides some information about the behavior of $K_A(x,y)$ at infinity (when $d(x,y) \to \infty$). In particular, it follows from this condition that $K_A(\cdot, x), K_A(x, \cdot) \in H^\infty(M)$ for every $x \in M$ uniformly in x.

The boundedness Theorem 1 from 3.9 extends to S-regular operators as follows:

Proposition 2. *If $A \in S\Psi_\Gamma^m(M; E, F)$, then A extends to a bounded linear map*

$$A : H^s(M, E) \longrightarrow H^{s-m}(M, F)$$

for any $s \in \mathbb{R}$.

Proof. In the decomposition $A = A_1 + R$ with A_1 properly supported and $R \in S\Psi_\Gamma^{-\infty}(M; E, F)$ both A_1 and R have the desired extension due to Theorem 1 in 3.9 for A_1 and Definition 1 above for R. □

Proposition 3 (Uniform Elliptic Regularity). *Let $A \in S\Psi_\Gamma^m(M; E, F)$ be elliptic of order m, $u \in H^{-\infty}(M, E)$ and $Au = f \in H^s(M, F)$ with some $s \in \mathbb{R}$. Then $u \in H^{s+m}(M, E)$.*

Proof. Let us again decompose $A = A_1 + R$ where A_1 is properly supported and $R \in S\Psi_\Gamma^{-\infty}(M; E, F)$. Then A_1 is elliptic of order m and

$$A_1 u = f - Ru \in H^s(M, F).$$

Therefore we can use Uniform Elliptic Regularity for properly supported operators (Theorem 2 in 3.9) to conclude that $u \in H^{s+m}(M, E)$. □

It is clear that the set $S\Psi_\Gamma^\infty(M; E)$ of S-regular Γ-invariant operators in sections of a vector Γ-bundle E is an algebra, and the set $S\Psi_\Gamma^{-\infty}(M; E)$ of S-smoothing Γ-invariant operators in sections of E is a two-sided ideal in this algebra.

Now we can formulate the crucial fact about the S-regular operators.

THEOREM 1. *If $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ is elliptic (of order m) and there exists $s \in \mathbb{R}$ such that the operator*

$$A : H^s(M, E) \longrightarrow H^{s-m}(M, F)$$

is bijective, then $A^{-1} \in S\Psi^{-m}_{\mathrm{cl};\Gamma}(M; F, E)$.

PROOF. 1) For simplicity of notations we will assume that both E, F are trivial with the one-dimensional fiber $\mathbb{C}$. So in particular $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M)$.

2) Note that $A^{-1} : H^{s-m} \to H^s(M)$ is bounded due to the Inverse Operator Theorem.

Let A^+ denote the formally adjoint operator to A, i.e.

$$(Au, v) = (u, A^+v), \quad u, v \in C_c^\infty(M),$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(M)$. Then $A^+ \in S\Psi^m_{\mathrm{cl};\Gamma}(M)$ because taking the adjoint operator preserves both the class of properly supported Γ-invariant classical pseudo-differential operators and the class $S\Psi^{-\infty}_\Gamma M$. The later is true because if we know that $A : H^s(M) \to H^t(M)$ is bounded, then A^+ defines a bounded linear operator $A^+ : H^{-t}(M) \to H^{-s}(M)$ and vice versa.

It follows from the requirement in the theorem that $\operatorname{Ker} A = \{0\}$ in $H^{-\infty}(M)$ because in fact $\operatorname{Ker} A \subset H^\infty(M)$ due to the Uniform Elliptic Regularity (Proposition 3). Similarly $\operatorname{Ker} A^+ = \{0\}$ in $H^{-\infty}(M)$.

The same elliptic regularity implies that $A : H^t(M) \to H^{t-m}(M)$ is surjective (hence, bijective) for any $t \geq s$. Indeed, for any $f \in H^{t-m}(M) \subset H^{s-m}(M)$ we can find $u \in H^s(M)$ such that $Au = f$. But then $u \in H^t(M)$ due to the elliptic regularity. It follows from the Inverse Operator Theorem that the operator $A^{-1} : H^{t-m} \to H^t(M)$ is bounded for all such t.

Now taking the adjoint operator (with respect to the L^2 inner product) we find that $A^+ : H^{-t+m} \to H^{-t}$ is bijective and the inverse operator bounded if $t \geq s$. Changing notations we conclude that $A^+ : H^t(M) \to H^{t-m}(M)$ is a topological isomorphism for all $t \leq -s+m$. But then this holds *for all* $t \in \mathbb{R}$ due to the elliptic regularity and the Closed Graph Theorem. Taking the adjoint operator we see that the same holds for A as well.

3) We proved that $A : H^t(M) \to H^{t-m}(M)$ is a topological isomorphism for any $t \in \mathbb{R}$. Now let us present $A = A_1 + R$ where $A_1 \in \Psi^m_{\mathrm{cl};\Gamma}(M)$ is properly supported, $R \in S\Psi^{-\infty}_\Gamma(M)$, and take a parametrix $B \in \Psi^{-m}_{\mathrm{cl};\Gamma}(M)$ of A_1 such that B is properly supported and

$$A_1 B = I - T_1, \quad T_1 \in \Psi^{-\infty}_\Gamma(M).$$

Note that T_1 is properly supported too (hence $T_1 \in S\Psi^{-\infty}_\Gamma(M)$).

It follows that

$$AB = I - T, \quad T \in S\Psi^{-\infty}_\Gamma(M),$$

hence

$$A^{-1} = B + A^{-1}T,$$

and it remains to note that $A^{-1}T \in S\Psi^{-\infty}_\Gamma(M)$ because T maps $H^s(M)$ to $H^t(M)$ for amy $s, t \in \mathbb{R}$ and A^{-1} maps $H^t(M)$ to $H^{t+m}(M)$ also for any $t \in \mathbb{R}$ as was established above. ☐

COROLLARY 2. *Let us assume that* $A \in \Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ *is elliptic (of order* m*) and properly supported (in particular* A *can be any elliptic* Γ*-invariant differential operator). If there exists* $s \in \mathbb{R}$ *such that the operator*

$$A : H^s(M, E) \longrightarrow H^{s-m}(M, F)$$

is bijective, then $A^{-1} \in S\Psi^{-m}_{\mathrm{cl};\Gamma}(M; F, E)$.

REMARK 3. Note that it is not true that A^{-1} is properly supported even for an elliptic Γ-invariant differential operator A. For example if we take the operator $(I - \Delta)$ in $\mathbb{R}^3$ with the standard metric (it fits into the picture e.g. with $\Gamma = \mathbb{Z}^3$ acting by translations), then the inverse $(I - \Delta)^{-1}$ has the Schwartz kernel

$$K(x, y) = (4\pi|x - y|)^{-1} e^{-|x-y|},$$

which is positive everywhere off the diagonal. The same holds in $\mathbb{R}^n$ for arbitrary n due to the maximum principle or due to explicit formulas for the fundamental solution of $I - \Delta$ (see e.g. [**Tay2**], Ch.3, Sections 5,6).

EXERCISE 1. Assume that $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M)$ is elliptic of order m, and $\operatorname{Ker} A = \operatorname{Ker} A^+ = \{0\}$ in $H^\infty(M)$.

1) Prove that the following 41 conditions are equivalent:

(a) $A(H^{-\infty}(M)) \supset H^\infty(M)$;
(b) $A(H^\infty(M)) = H^\infty(M)$;
(c) $A(H^{-\infty}(M)) = H^{-\infty}(M)$;
(d) $\exists s \in \mathbb{R}$, $A(H^s(M)) \supset H^\infty(M)$;
(e) $\exists t \in \mathbb{R}$, $A(H^{-\infty}(M)) \supset H^t(M)$;
(f) $\exists s, t \in \mathbb{R}$, $A(H^s(M)) \supset H^t(M)$;
(g) $\forall s \in \mathbb{R}$, $A(H^s(M)) \supset H^\infty(M)$;
(h) $\forall t \in \mathbb{R}$, $A(H^{-\infty}(M)) \supset H^t(M)$;
(i) $\forall s \in \mathbb{R}, \exists t \in \mathbb{R}$, $A(H^s(M)) \supset H^t(M)$;
(j) $\forall t \in \mathbb{R}, \exists s \in \mathbb{R}$, $A(H^s(M)) \supset H^t(M)$;
(a_1) $A^{-1} : H^\infty(M) \to H^{-\infty}(M)$ is continuous;
(b_1) $A^{-1} : H^\infty(M) \to H^\infty(M)$ is continuous;
(c_1) $A^{-1} : H^{-\infty}(M) \to H^{-\infty}(M)$ is continuous;
(d_1) $\exists s \in \mathbb{R}$, $A^{-1} : H^\infty(M) \to H^s(M)$ is continuous;
(e_1) $\exists t \in \mathbb{R}$, $A^{-1} : H^t(M) \to H^{-\infty}(M)$ is continuous;
(f_1) $\exists s, t \in \mathbb{R}$, $A^{-1} : H^t(M) \to H^s(M)$ is continuous;
(g_1) $\forall s \in \mathbb{R}$, $A^{-1} : H^\infty(M) \to H^s(M)$ is continuous;
(h_1) $\forall t \in \mathbb{R}$, $A^{-1} : H^t(M) \to H^{-\infty}(M)$ is continuous;
(i_1) $\forall s \in \mathbb{R}, \exists t \in \mathbb{R}$, $A^{-1} : H^t(M) \to H^s(M)$ is continuous;
(j_1) $\forall t \in \mathbb{R}, \exists s \in \mathbb{R}$, $A^{-1} : H^t(M) \to H^s(M)$ is continuous;
(a^+), …,(j_1^+): the same 20 conditions as above with A replaced by A^+;
(k) $A^{-1} \in S\Psi^{-m}_{\mathrm{cl};\Gamma}(M)$.

2) If $m > 0$ then the conditions above are equivalent to the following condition:

(l) 0 is not in the spectrum of the self-adjoint operator which is the closure of A^+A in $L^2(M)$ (from the initial domain $C_c^\infty(M)$ or $H^\infty(M)$).

Note that though many implications between statements in this exercise are trivial by purely logical reasons, there is no equivalence which can be established without analytic arguments.

BIBLIOGRAPHICAL REMARK. A discussion on a topic similar to this exercise is contained in [**Sh2**] (see e.g. Proposition 2.2 where 80 equivalent conditions of invertibility of an almost periodic pseudo-differential operator in $\mathbb{R}^n$ are given).

Let again E be a vector Γ-bundle with a Γ-invariant hermitian metric on M.

LEMMA 3. *For any $s \in \mathbb{R}$ there exists a formally self-adjoint pseudo-differential operator $I_s \in S\Psi^s_{\mathrm{cl};\Gamma}(M;E)$ such that I_s is elliptic (of order s) and*

$$I_s : H^s(M,E) \longrightarrow L^2(M,E)$$

is a topological isomorphism of Hilbert Γ-modules.

If $s \geq 0$, then for any $\epsilon > 0$ this operator can be chosen ϵ-local.

PROOF. Assume first that $s \geq 0$. Let us start from an arbitrary $\epsilon/2$-local elliptic operator $\Lambda_{s/2} \in \Psi^{s/2}_{\mathrm{cl};\Gamma}(M;E)$ of order $s/2$ in $C_c^\infty(M,E)$, and then take

$$I_s = (\Lambda_{s/2})^* \Lambda_{s/2} + I,$$

where I is the identity operator and the adjoint operator is taken with respect to the inner product in $L^2(M,E)$. Then I_s is an elliptic ϵ-local Γ-invariant pseudo-differential operator of order s in $C_c^\infty(M,E)$. Let $\bar{I}_s$ be the closure of I_s in $L^2(M,E)$. Then $\bar{I}_s$ has the domain $D(\bar{I}_s) = H^s(M,E)$ and with this domain it is a self-adjoint operator in $L^2(M,E)$ due to Theorem 3 in 3.9.

We obviously have $||u||^2 \leq (I_s u, u) \leq ||I_s u||||u||$, hence

$$||u|| \leq ||I_s u||, \quad u \in C_c^\infty(M,E).$$

Taking closure, we see that the same inequality holds for $\bar{I}_s$ and for all $u \in H^s(M,E)$. Therefore the operator $\bar{I}_s$ is invertible in $L^2(M,E)$ and the inverse $(\bar{I}_s)^{-1}$ is bounded. Hence the domain of the inverse (which is the image of $\bar{I}_s$) is closed. Since $(\bar{I}_s)^* = \bar{I}_s$ and $\operatorname{Ker} \bar{I}_s = \{0\}$, we see that $\operatorname{Im} \bar{I}_s = L^2(M,E)$. This proves Lemma for $s \geq 0$.

If $s < 0$, we can take $I_s = I_{-s}^{-1}$ where I_{-s} was constructed above. Then we will have $I_s \in S\Psi^s_{\mathrm{cl};\Gamma}(M)$ due to Theorem 1. □

COROLLARY 3. *In the notations of Lemma 2 the space $H^s(M,E)$ is a free Hilbert Γ-module for any $s \in \mathbb{R}$.*

PROPOSITION 4. *Let $A \in S\Psi^m_\Gamma(M,E)$ be an elliptic operator of order $m > 0$, and $A = A^+$ i.e. A is formally self-adjoint. Then A is essentially self-adjoint, and for its spectral projection $E(\Delta)$, corresponding to a bounded Borel set $\Delta \subset \mathbb{R}$, we have*

$$E(\Delta) \in S\Psi^{-\infty}_\Gamma(M,E).$$

In particular, $E(\Delta)$ is in the Γ-trace class.

PROOF. 1) For simplicity of notations we will assume that E is trivial and has one-dimensional fiber $\mathbb{C}$. Proof of the essential self-adjointness almost coincides with the proof of Theorem 3 in 3.9. The closure $\bar{A}$ of A from the original domain $C_c^\infty(M)$ has the domain $D(\bar{A}) = H^m(M)$. Indeed, $D(\bar{A}) \supset H^m(M)$ because $C_c^\infty(M)$ is dense in $H^m(M)$ and the operator $A : H^m(M) \to L^2(M)$ is continuous. On the other hand $u \in D(A^*)$ implies $u \in H^m(M)$ due to the uniform elliptic regularity (see Proposition 3 above). So we obtain

$$H^m(M) \subset D(\bar{A}) \subset D(A^*) \subset H^m(M),$$

which implies both that $\bar{A}$ is self-adjoint (or, equivalently, that A is essentially self-adjoint) and $D(\bar{A}) = H^m(M)$.

2) Since $\Delta \subset \mathbb{R}$ is bounded, for any integer $N \geq 0$ the operator

$$A^N E(\Delta) : L^2(M) \longrightarrow H^{-Nm}(M)$$

is in fact bounded in $L^2(M)$-norm, hence maps $L^2(M)$ to $L^2(M)$. Therefore it is a bounded self-adjoint operator in $L^2(M)$ because $E(\Delta)$ commutes with A^N. So for any $u \in L^2(M)$ we have $A^N E(\Delta)u \in L^2(M)$, hence $E(\Delta)u \in H^{Nm}(M)$ due to the elliptic regularity.

We have seen that in fact $E(\Delta)$ maps $L^2(M)$ to $H^{Nm}(M)$. This map is continuous by the Closed Graph Theorem. But since $E(\Delta)$ is self-adjoint in $L^2(M)$, it extends by duality to a bounded linear map $E(\Delta) : H^{-Nm}(M) \to L^2(M)$. Now using the identity $E(\Delta) = (E(\Delta))^2$ we obtain that $E(\Delta)$ extends to a bounded linear map

$$E(\Delta) : H^{-Nm}(M) \longrightarrow H^{Nm}(M).$$

Recalling that N is arbitrary and $m > 0$, we immediately obtain the desired inclusion $E(\Delta) \in S\Psi_\Gamma^{-\infty}(M)$. □

COROLLARY 4. *In the conditions of Proposition 4 assume that A is semi-bounded below. Then $E_\lambda \in S\Psi_\Gamma^{-\infty}(M)$ for all $\lambda \in \mathbb{R}$. (Here $E_\lambda = E((-\infty, \lambda])$.)*

COROLLARY 5. *Let $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ be elliptic of order $m > 0$, and $P_{\operatorname{Ker} A}$ the orthogonal projection onto* Ker A *in $L^2(M, E)$. Then $P_{\operatorname{Ker} A} \in S\Psi_\Gamma^{-\infty}(M; E)$.*

In particular, the Schwartz kernel of $P_{\operatorname{Ker} A}$ is in C^∞ on $M \times M$.

PROOF. Let us consider the operator $B = A^+ A \in S\Psi^{2m}_{\mathrm{cl};\Gamma}(M; E)$. We claim that $\operatorname{Ker} A = \operatorname{Ker} B$ in $L^2(M)$. Indeed, both these null-spaces are subspaces in $H^\infty(M, E)$, and

$$(A^+ A u, u) = (Au, Au), \quad u \in H^\infty(M, E),$$

because $H^\infty(M, E)$ is in the domain of the adjoint operator to A^+. (In fact this domain is $H^m(M, E)$ due to Proposition 3.) It follows that $\operatorname{Ker} B \subset \operatorname{Ker} A$ and the opposite inclusion is obvious.

Now note that B satisfies the conditions of Proposition 4 which gives the desired result if we apply it to the spectral projection $E(\{0\}) = P_{\operatorname{Ker} B}$. □

REMARK 4. There is another way to prove that the Schwartz kernel $p = p(x, y)$ of $P = P_{\operatorname{Ker} A}$ is in C^∞ on $M \times M$: to establish that p satisfies an elliptic equation on $M \times M$. We will sketch the corresponding arguments now.

For simplicity consider first the case when A is properly supported, the bundles E, F are trivial and have one-dimensional fiber $\mathbb{C}$. Then the Schwartz kernel p is a distribution on $M \times M$. The obvious relation $AP = 0$ implies that

$$A_x p(x, y) = 0 \quad \text{or} \quad (A \otimes I)p = 0.$$

(Here A_x means the operator A applied with respect to x.) Therefore

$$A_x^* A_x p(x, y) = 0 \quad \text{or} \quad (A^* A \otimes I)p = 0.$$

Now let us take into account that $P^* = P$ which formally means in terms of kernels that $\overline{p(x, y)} = p(y, x)$, where the bar means complex conjugation. Therefore

$$A_y \overline{p(x, y)} = 0.$$

Now let us denote the operation of complex conjugation on functions by h (it is an antilinear operation and $h^2 = \mathrm{Id}$, or $h^{-1} = h$). Denote

$$\tilde{A} = h^{-1}Ah : C_c^\infty(M) \longrightarrow C_c^\infty(M).$$

If A is a differential operator which locally has the form $A = \sum a_\alpha(x)\partial_x^\alpha$, then $\tilde{A} = \sum \overline{a_\alpha(x)}\partial_x^\alpha$, i.e. $\tilde{A}$ is obtained from A by taking complex conjugate coefficients. Generally $\tilde{A}$ is a linear properly supported elliptic pseudo-differential operator of order m and

$$\tilde{A}_y p(x,y) = 0 \quad \text{or} \quad (I \otimes \tilde{A})p = 0.$$

Therefore,

$$\tilde{A}_y^* \tilde{A}_y p(x,y) = 0 \quad \text{or} \quad (I \otimes \tilde{A}^*\tilde{A})p = 0,$$

and

$$(A_x^* A_x + \tilde{A}_y^* \tilde{A}_y)p(x,y) = 0 \quad \text{or} \quad (A^*A \otimes I + I \otimes \tilde{A}^*\tilde{A})p = 0.$$

It remains to notice that

$$L = A_x^* A_x + \tilde{A}_y^* \tilde{A}_y = A^*A \otimes I + I \otimes \tilde{A}^*\tilde{A}$$

is a properly supported elliptic pseudo-differential operator of order $2m$ on $M \times M$.

To be able to do the same for arbitrary vector bundles E, F (with hermitian metrics on fibers) we need an appropriate algebraic formalism. Let us assume as above that A is properly supported. Note first that formally $p(x,y)$ is a linear map (of fibers)

$$p(x,y) : E_y \longrightarrow E_x$$

for any $x, y \in M$. More precisely, a priori we can only say that p is a distributional section of the vector bundle $E \boxtimes E'$ over $M \times M$, whose fiber over (x,y) is

$$(E \boxtimes E')_{(x,y)} = E_x \otimes E'_y = \mathrm{Hom}_{\mathbb{C}}(E_y, E_x),$$

where $E'_y = \mathrm{Hom}_{\mathbb{C}}(E_y, \mathbb{C})$, $E' = \mathrm{Hom}(E, M \times \mathbb{C})$ is the dual bundle to E.

Since $AP_{\mathrm{Ker}\,A} = 0$, we obtain as above

$$(A \otimes I)p(x,y) = A_x p(x,y) = 0$$

in the sense of distributions.

Now note that the self-adjointness of $P_{\mathrm{Ker}\,A}$ implies $p(x,y) = (p(y,x))^*$ where the star means taking hermitian adjoint operator, i.e.

$$(p(y,x))^* : E_y \longrightarrow E_x$$

is hermitian adjoint to $p(y,x) : E_x \longrightarrow E_y$. Hence we can define an elliptic pseudo-differential operator

$$\tilde{A} : C^\infty(M, E') \to C^\infty(M, F')$$

by the formula $\tilde{A} = h_F A h_E^{-1}$, where $h_E : E \to E'$ denotes the antilinear isometric isomorphism induced by the hermitian metric on E, and h_F is a similar operator in F. Then

$$(I \otimes \tilde{A})p(x,y) = \tilde{A}_y p(x,y) = 0.$$

It follows that p is a solution of an elliptic equation

$$(A_x^* A_x + \tilde{A}_y^* \tilde{A}_y)p(x,y) = 0,$$

hence $p \in C^\infty$.

If we do not assume that A is properly supported, then we first have to establish that A_x and $\tilde{A}_y$ can be applied to p. The simplest way to do this is to use the arguments from the second part of the proof of Proposition 4 above to see that P

is S-smoothing. Then it follows e.g. that $y \mapsto p(\cdot, y)$ is a C^∞-function of y with values in the Sobolev space H^s (of sections of an appropriate vector bundle) for an arbitrary $s \in \mathbb{R}$. In particular we obtain that $A_x p(x,y)$ is well defined, and the same equation for p can be easily established.

EXERCISE 2. In the notations of the previous remark prove that if $\{\varphi_n | n = 1, 2, \dots\}$ is an orthonormal basis in $\operatorname{Ker} A$, $\bar{\varphi}_n = h\varphi_n$, then

$$p(x,y) = \sum_{n=1}^{\infty} \varphi_n(x) \otimes \bar{\varphi}_n(y),$$

where the series converges in C^∞, i.e. uniformly on compact sets of $M \times M$, together with all its derivatives.

Γ-Fredholm properties of elliptic operators

The following theorem lists the most important Γ-Fredholm properties of Γ-invariant elliptic S-regular operators.

THEOREM 2. *Let $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M; E, F)$ be an elliptic operator of order $m \in \mathbb{R}$, E, F vector Γ-bundles over M with Γ-invariant hermitian metrics on them. Then the following is true.*

(i) *For any $s \in \mathbb{R}$ the operator $A_s : H^s(M,E) \to H^{s-m}(M,F)$ is Γ-Fredholm.*
(ii) *The Γ-index of A_s does not depend of s and we will denote it $\operatorname{ind}_\Gamma A$. It is given by the formula*

$$\operatorname{ind}_\Gamma A = \dim_\Gamma \operatorname{Ker} A - \dim_\Gamma \operatorname{Ker} A^+,$$

where $A^+ \in S\Psi^m_{\mathrm{cl};\Gamma}(M; F, E)$ is formally adjoint to A in L^2.
(iii) *$\operatorname{ind}_\Gamma A$ depends only on the principal symbol of A and does not change under homotopy of this symbol (in the uniform topology on the symbols).*

PROOF. (i) Let $B \in S\Psi^{-m}_{\mathrm{cl};\Gamma}(M; F, E)$ be a parametrix of A, so that

$$BA = I - T_1, \qquad AB = I - T_2,$$

where $T_1 \in S\Psi^{-\infty}_\Gamma(M; E)$, $T_2 \in S\Psi^{-\infty}_\Gamma(M; F)$. Let us show that T_1, T_2 are in the Γ-trace class in $H^s(M,E)$ and $H^{s-m}(M,F)$ respectively. To this end we can choose an operator $I_s \in S\Psi^s_{\mathrm{cl};\Gamma}(M; E)$ which is elliptic of order s and provides a topological isomorphism of Hilbert Γ-modules $I_s : H^s(M,E) \to L^2(M,E)$ (see Lemma 3). Then $I_s^{-1} \in S\Psi^{-s}_{\mathrm{cl};\Gamma}(M; E)$ by Theorem 1. The operator $\tilde{T}_1 = I_s T_1 I_s^{-1}$ is still in $S\Psi^{-\infty}_\Gamma(M; E)$, hence in the Γ-trace class in $L^2(M,E)$. Therefore the same holds for T_1 in $H^s(M,E)$.

Similarly we conclude that T_2 is in the Γ-trace class in $H^{s-m}(M,F)$. It follows that A_s is Γ-Fredholm.

(ii) It follows from Theorem 2 in 3.10 that

$$\operatorname{ind}_\Gamma A_s = \dim_\Gamma \operatorname{Ker} A_s - \dim_\Gamma \operatorname{Ker} A_s^*,$$

where $A_s^* : H^{s-m}(M,F) \to H^s(M,E)$ is the Hilbert space adjoint operator to A_s, i.e. the operator defined from the relation

$$(A_s u, v)_{s-m} = (u, A_s^* v), \qquad u \in H^s(M,E),\ v \in H^{s-m}(M,F),$$

where $(\cdot,\cdot)_s$, $(\cdot,\cdot)_{s-m}$ denote the Hilbert inner products in $H^s(M,E)$, $H^{s-m}(M,F)$ respectively.

Clearly $\operatorname{Ker} A_s$ is independent of s since $\operatorname{Ker} A_s = \operatorname{Ker} A \subset H^\infty(M, E)$ due to the uniform elliptic regularity (Proposition 3). Therefore $\dim_\Gamma \operatorname{Ker} A_s = \dim_\Gamma \operatorname{Ker} A$ does not depend on s. So to prove (ii) it remains to establish that

$$\dim_\Gamma \operatorname{Ker} A_s^* = \dim_\Gamma \operatorname{Ker} A^+.$$

Let us consider the range $\operatorname{Im} A_s$ in $H^{s-m}(M, F)$. Then $\operatorname{Ker} A_s^*$ is its orthogonal complement with respect to the Hilbert inner product $(\cdot,\cdot)_{s-m}$. It can be also called the annihilator of $\operatorname{Im} A_s$ with respect to the non-degenerate sesquilinear duality between two copies of $H^{s-m}(M, F)$ induced by the inner product. Here the non-degeneracy is an obvious corollary from the following much stronger fact: the map $v \mapsto (\cdot, v)_{s-m}$ is an anti-linear unitary isomorphism

$$U_1 : H^{s-m}(M, F) \longrightarrow \left(H^{s-m}(M, F)\right)^*,$$

where $\left(H^{s-m}(M, F)\right)^*$ is the Hilbert space of all linear maps $\tilde{v} : H^{s-m}(M, F) \to \mathbb{C}$.

On the other hand $\operatorname{Ker} A^+$ is the annihilator of $\operatorname{Im} A_s$ in $H^{-(s-m)}(M, F)$ with respect to another non-degenerate sesquilinear duality between $H^{s-m}(M, F)$ and $H^{-(s-m)}(M, F)$ which extends the $L^2(M, F)$ inner product $(\cdot,\cdot) = (\cdot,\cdot)_0$. Moreover, the map $v \mapsto (\cdot, v)$ is an anti-linear unitary isomorphism

$$U_2 : H^{-(s-m)}(M, F) \longrightarrow \left(H^{s-m}(M, F)\right)^*.$$

Now let us define

$$U = U_2^{-1} U_1 : H^{s-m}(M, F) \longrightarrow H^{-(s-m)}(M, F),$$

or, equivalently, by the following commutative diagram

$$\begin{tikzcd}
& H^{s-m}(M,F) \arrow[dl, "U_1"'] \arrow[dd, "U"] \\
\left(H^{s-m}(M,F)\right)^* & \\
& H^{-(s-m)}(M,F) \arrow[ul, "U_2"]
\end{tikzcd}$$

Clearly, U is linear. Since all the inner products are Γ-invariant, the operators U_1, U_2, U commute with the actions of Γ. Therefore U is an isomorphism of Hilbert Γ-modules. Let $(\operatorname{Im} A)^\circ$ be the annihilator of $\operatorname{Im} A_s$ in $\left(H^{s-m}(M, F)\right)^*$. Then by the definitions of U_1, U_2 we have

$$U_1(\operatorname{Ker} A_s^*) = U_2(\operatorname{Ker} A^+) = (\operatorname{Im} A)^\circ.$$

Therefore, $U(\operatorname{Ker} A_s^*) = \operatorname{Ker} A^+$, hence $\operatorname{Ker} A_s^*$ and $\operatorname{Ker} A_s^*$ have equal Γ-dimensions due to the results in 2.16.

(iii) Note that Theorem 2 in 3.10 implies that $\operatorname{ind}_\Gamma A$ does not change under any homotopy $A(t)$, $t \in [0,1]$, in the class $S\Psi^m_{\mathrm{cl};\Gamma}(M, E, F)$, such that all operators $A(t)$ are elliptic of order m and the homotopy is continuous with respect to the operator norm $\|A_m\| = \|A : H^m(M, E) \to L^2(M, F)\|$. Analyzing the proof of the boundedness results (see Proposition 2 in this section and Theorem 1 in 3.9) we see that this is the case for the homotopy $A(t) = A + tB$ where $B \in S\Psi^m_{\mathrm{cl};\Gamma}(M, E, F)$, as soon as $A+tB$ remains elliptic in the course of the homotopy. In particular, this works for $B \in S\Psi^{m-1}_{\mathrm{cl};\Gamma}(M,E,F)$ and any $t \in \mathbb{R}$. (In fact, $B \in S\Psi^{m-\epsilon}_{\Gamma}(M,E,F)$ with any $\epsilon > 0$ would do as well, though then $A+tB$ will not generally be classical.) It follows that $\operatorname{ind}_\Gamma A$ depends on the principal symbol of A only. A more general uniformly continuous homotopy of principal symbols can be replaced by a locally linear one, yielding the same result. □

Atiyah Index Theorem

Let E be a vector Γ-bundle with a Γ-invariant hermitian metric on a Γ-manifold M, $X = M/\Gamma$ is a compact closed manifold. Then in fact $E = \pi^*\underline{E}$, where $\underline{E}$ is a vector bundle on X, $\pi : M \to X$ is the canonical projection. (Recall that π^*E is the vector bundle on M whose fibers over all points $x \in \pi^{-1}(\underline{x})$, $\underline{x} \in X$, are identified with the fiber $\underline{E}_{\underline{x}}$ of the bundle $\underline{E}$ over $\underline{x}$.) The Γ-invariant hermitian metric on E can be also considered as a lifting of an hermitian metric on $\underline{E}$.

The space $L^2(M,E)$ of L^2-sections of E and the Sobolev spaces $H^s(M,E)$, $s \in \mathbb{R}$, are well defined. They are Hilbert Γ-modules (see Corollary 2 in 3.9).

Let

$$A : C_c^\infty(M,E) \to C^\infty(M,F)$$

be an elliptic S-regular classical Γ-invariant pseudo-differential operator in sections of vector Γ-bundles E, F, i.e. $A \in S\Psi^m_{\mathrm{cl};\Gamma}(M;E,F)$ and A is elliptic of order m. Then obviously the principal symbol of A can be obtained as a lifting to M of a principal symbol of an elliptic classical pseudo-differential operator

$$\underline{A} : C^\infty(X,\underline{E}) \to C^\infty(X,\underline{F}),$$

where $\underline{E}, \underline{F}$ are vector bundles on X such that $\pi^*\underline{E} = E$ and $\pi^*\underline{F} = F$.

Theorem 3 (M. Atiyah [**A**]). $\operatorname{ind}_\Gamma A = \operatorname{ind} \underline{A}$.

Proof. Let us recall that the indices $\operatorname{ind}_\Gamma A, \operatorname{ind}\underline{A}$ depend only on the principal symbols of $A, \underline{A}$ respectively (see Theorem 2 above in this section). Therefore choosing an arbitrary $\epsilon > 0$, we can replace $\underline{A}$ by an ϵ-local classical pseudo-differential operator with the same principal symbol (see 3.9). We will also denote such an operator by $\underline{A}$.

Let $\underline{B}$ be an ϵ-local parametrix of the operator $\underline{A}$, where $\epsilon > 0$ is sufficiently small. So

$$\underline{B} \cdot \underline{A} = I - \underline{T}_1, \quad \underline{A} \cdot \underline{B} = I - \underline{T}_2,$$

where $\underline{T}_1, \underline{T}_2$ are 2ϵ-local operators with C^∞ Schwartz kernels. Now we can lift all the operators here to M, thus obtaining

$$B \cdot A = I - T_1, \quad A \cdot B = I - T_2,$$

where B, T_1, T_2 are Γ-invariant 2ϵ-local operators, and T_1, T_2 have C^∞ kernels. In particular, T_1, T_2 are in the Γ-trace class.

Now by definition of lifting and Γ-trace (see 2.22) we get

$$\operatorname{Tr}_\Gamma T_j = \operatorname{Tr} \underline{T}_j, \quad j = 1, 2.$$

Therefore, using the Atiyah-Bott formula and its Γ-version (see 3.7 and 3.10), we obtain

$$\operatorname{ind}_\Gamma A = \operatorname{Tr}_\Gamma T_1 - \operatorname{Tr}_\Gamma T_2 = \operatorname{Tr} \underline{T}_1 - \operatorname{Tr} \underline{T}_2 = \operatorname{ind} \underline{A},$$

as required. □

Note that $\operatorname{ind}\underline{A}$ can be calculated by the Atiyah-Singer index formula.

Let us consider a particular case of the Atiyah Theorem when Γ is finite, hence M is compact. Then obviously for Γ-invariant operators C on M with C^∞-kernels we have $\operatorname{Tr} C = |\Gamma| \operatorname{Tr}_\Gamma C$, hence $\operatorname{ind} A = |\Gamma| \operatorname{ind}\underline{A}$, where $|\Gamma|$ is the cardinality of Γ.

How can the Atiyah Theorem be applied when Γ is infinite? The simplest applications can be given through the following

Corollary 6. *If* $\operatorname{ind}_\Gamma A > 0$ *then* $\dim_\Gamma \operatorname{Ker} A > 0$ *and there exist non-trivial* L^2*-solutions of* $Au = 0$*. If* Γ *is infinite then these solutions form a closed* Γ*-invariant infinite-dimensional subspace in* $L^2(M, E)$ *which is actually included in* $H^\infty(M, E)$*, hence in* $C^\infty(M, E)$*.*

The proof is obvious if we remark that $\dim_\Gamma \operatorname{Ker} A \geq \operatorname{ind}_\Gamma A$ and that for an infinite Γ and for a Γ-invariant projection P in $L^2(M, E)$,

$$\{\operatorname{Tr}_\Gamma P > 0\} \Longrightarrow \{\operatorname{Tr} P = \infty\}.$$

Example. Let X be a compact Riemannian surface of genus $g \geq 2$,

$$\underline{A} = \bar{\partial} : C^\infty(X) \to \Lambda^{0,1}(X), \quad \underline{A}^t = \bar{\partial} : \Lambda^{1,0}(X) \to \Lambda^{1,1}(X) = \Lambda^2(X).$$

Then it can be easily proved that $\operatorname{ind}\underline{A} = -\operatorname{ind}\underline{A}^t = 1-g$, hence $\operatorname{ind}\underline{A}^t = g-1 > 0$. Let M be the universal covering space for X which is the upper half-plane. Then $\operatorname{ind}_\Gamma A^t = g - 1 > 0$, which gives the existence of non-trivial holomorphic L^2-forms of type $1, 0$ on M (and, moreover, the existence of an infinite-dimensional space of such forms). This of course can be seen without the index theorem but the index theorem gives the same result if we add 0-order term to $\underline{A}$ (and then the result becomes not obvious). More generally, this idea allows to construct the discrete series of representations of semi-simple Lie groups (see [**AS**]).