CHAPTER 4

L^2-invariants in topology

4.1. L^2-cohomology of coverings and L^2 Betti numbers

Let M be a Riemannian manifold. The Riemannian metric on M induces a positive Riemannian density on M and also inner products in all tensor bundles, in particular, in $\Lambda^p T^*M$ (p^{th} exterior power of the cotangent bundle). This induces a Hermitian inner product in the complexifications $\Lambda^p T^*M \otimes \mathbb{C}$ and in the space of sections of these bundles with compact support $\Lambda^p_c(M) = C^\infty_c(M, \Lambda^p T^*M \otimes \mathbb{C})$. We shall denote the inner product by $(\cdot,\cdot)$ and the completion of $\Lambda^p_c(M)$ with respect to the corresponding norm by $L^2\Lambda^p(M)$. So $L^2\Lambda^p(M)$ is the Hilbert space of square-integrable complex-valued exterior p-forms on M.

As usual $\Lambda^p(M)$ will denote the space of all C^∞ complex-valued p-forms on M, i.e. $\Lambda^p(M) = C^\infty(M, \Lambda^p T^*M \otimes \mathbb{C})$.

Now let us introduce the exterior differential $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$, which is a first order differential operator. It defines two unbounded linear operators $d_{\min}$ and $d_{\max}$, both acting from $L^2\Lambda^p(M)$ to $L^2\Lambda^{p+1}(M)$. Here $d_{\min}$ is the closure of d from the domain $\Lambda^p_c(M)$, and $d_{\max}$ has the domain

$$D(d_{\max}) = \{\omega \mid \omega \in L^2\Lambda^p(M), d\omega \in L^2\Lambda^{p+1}(M)\},$$

where d is applied in the sense of distributions. Let δ denote the formally adjoint differential operator to d i.e. $\delta : \Lambda^{p+1}(M) \to \Lambda^p(M)$ and

$$(d\omega, \alpha) = (\omega, \delta\alpha), \quad \omega \in \Lambda^p_c(M),\ \alpha = \Lambda^{p+1}_c(M).$$

Then $d_{\max} = \delta^*$ where the adjoint operator is understood as the adjoint in the Hilbert space operator sense. Note that d^* is densely defined, hence the closure $d_{\min}$ is really well defined.

Let us also denote $d_p = d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ to show explicitly where d acts; $\delta_p = \delta : \Lambda^{p+1}(M) \to \Lambda^p(M)$. The notations $d_{p,\min}$, $d_{p,\max}$, $\delta_{p,\min}$, $\delta_{p,\max}$ have an obvious meaning. Now define the Laplacian $\Delta = \delta d + d\delta : \Lambda^p(M) \to \Lambda^p(M)$, or more precisely $\Delta_p = \Delta = \delta_p d_p + d_{p-1}\delta_{p-1}$.

Exercise 1. Prove that Δ is an elliptic operator of order 2 and its principal symbol at $(x,\xi) \in T^*M$ is $|\xi|^2 I$ where I is the identity map of $\Lambda^p T^*_x M \otimes \mathbb{C}$ at any point $x \in M$, $|\xi|^2$ is the square of the length of the cotangent vector (x,ξ) (the length is defined by the given Riemannian metric).

Definition 1. The space of *harmonic forms* of degree p on M is

$$\mathcal{H}^p(M) = \operatorname{Ker} d_{p,\max} \cap \operatorname{Ker} \delta_{p-1,\max} = \{\omega \mid \omega \in L^2\Lambda^p(M), d\omega = 0, \delta\omega = 0\},$$

where d, δ are applied in the sense of distributions.

Lemma 1. *$\mathcal{H}^p(M) \subset \Lambda^p(M)$ (i.e. elements of $\mathcal{H}^p(M)$ are in fact C^∞-forms on M), and $\mathcal{H}^p(M)$ is a closed subspace in $L^2\Lambda^p(M)$.*

PROOF. The subspace $\mathcal{H}^p(M)$ is closed in $L^2\Lambda^p(M)$ because both operators $d_{p,\max}$, $\delta_{p-1,\max}$ are closed. Also $\mathcal{H}^p(M) \subset \operatorname{Ker}\Delta$ i.e.

$$\mathcal{H}^p(M) \subset \{\omega \mid \omega \in L^2\Lambda^p(M), \Delta\omega = 0\}.$$

Since Δ is elliptic we obtain $\mathcal{H}^p(M) \subset \Lambda^p(M)$. □

PROPOSITION 1 (Hodge-Kodaira decomposition). *There is an orthogonal decomposition*

$$L^2\Lambda^p(M) = \overline{d\Lambda_c^{p-1}(M)} \oplus \mathcal{H}^p(M) \oplus \overline{\delta\Lambda_c^{p+1}(M)}$$

where the bars mean closures in $L^2\Lambda^p(M)$. Besides

$$\operatorname{Ker} d_{p,\max} = \overline{d\Lambda_c^{p-1}(M)} \oplus \mathcal{H}^p(M),$$
$$\operatorname{Ker} \delta_{p-1,\max} = \mathcal{H}^p(M) \oplus \overline{\delta\Lambda_c^{p+1}(M)}.$$

PROOF. The equality $d^2 = 0$ (or $d_{p+1}d_p = 0$) obviously implies that the subspaces $d\Lambda_c^{p-1}(M)$ and $\delta\Lambda_c^{p+1}(M)$ are orthogonal, so their closures are also orthogonal. Now

$$(d\Lambda_c^{p-1}(M) \oplus \delta\Lambda_c^{p+1}(M))^\perp = (d\Lambda_c^{p-1}(M))^\perp \cap (\delta\Lambda_c^{p+1}(M))^\perp$$

consists exactly of all $\omega \in L^2\Lambda^p(M)$ such that $\delta\omega = 0$ and $d\omega = 0$ in the sense of distributions. Hence this orthogonal complement coincides with $\mathcal{H}^p(M)$ which proves the first decomposition.

Note that obviously $\overline{d\Lambda_c^{p-1}(M)} \subset \operatorname{Ker} d_{p,\max}$, $\mathcal{H}^p(M) \subset \operatorname{Ker} d_{p,\max}$. To prove the equality for $\operatorname{Ker} d_{p,\max}$ we have to check that $\operatorname{Ker} d_{p,\max} \cap \overline{\delta\Lambda_c^{p+1}(M)} = \{0\}$. But in fact $\operatorname{Ker} d_{p,\max} \perp \delta\Lambda_c^{p+1}(M)$, hence $\operatorname{Ker} d_{p,\max} \perp \overline{\delta\Lambda_c^{p+1}(M)}$, and the desired equality follows. The last equality is proved similarly. □

Lemma 1 and Proposition 1 hold on any Riemannian manifold (we never used any special feature of the manifold). The next Lemma however will not be true in such generality.

Let us assume that M is a Riemannian Γ-manifold. This means that M is a Γ-manifold with a Γ-invariant Riemannian metric. The corresponding positive Γ-invariant density and inner products on tensor bundles will also be Γ-invariant.

PROPOSITION 2. *Suppose that M is a Riemannian Γ-manifold with a compact closed $X = M/\Gamma$. Then $\Delta_{p,\max} = \Delta_{p,\min}$ (i.e. Δ_p is essentially self-adjoint), and*
(i) $\mathcal{H}^p(M) = \operatorname{Ker}\Delta_p$, where $\Delta_p = \Delta_{p,\max} = \Delta_{p,\min}$;
(ii) $d_{p,\min} = d_{p,\max}$, $\delta_{p,\min} = \delta_{p,\max}$.

Note first that the equality $\Delta_{p,\max} = \Delta_{p,\min}$ follows from a more general essential self-adjointness result (Theorem 3 in 3.9). It can actually be proved for any complete Riemannian manifold (see [**Ch**]). Now we need the following

LEMMA 2. *Suppose that A, A^+ are two linear operators in a Hilbert space $\mathcal{H}$ which are formally adjoint to each other i.e.*

$$(Au, v) = (u, A^+v), \quad u \in D(A), v \in D(A^+).$$

Assume that the operator A^+A is essentially self-adjoint. Then the closures $\bar{A}, \bar{A^+}$ are mutually adjoint as the operators in a Hilbert space i.e. $\bar{A} = (A^+)^*, \bar{A^+} = A^*$.

REMARK. This statement is in a way inverse to Lemma 2 in 3.8. The proof is taken from [**Sh1**], though it was probably known for a long time before.

PROOF OF LEMMA 2. We will use the following fact: a closed symmetric densely defined operator S is self-adjoint if and only if $\operatorname{Ker}(S^* + iI) = \operatorname{Ker}(S^* - iI) = 0$, i.e. if $\pm i$ are not eigenvalues of S^* (see e.g. [**ReSi**], Theorem VIII.3 and its Corollary). Now consider the matrix operator $M = \begin{pmatrix} 0 & A \\ A^+ & 0 \end{pmatrix}$ in $\mathcal{H} \oplus \mathcal{H}$. Obviously $\bar{M} = \begin{pmatrix} 0 & \bar{A} \\ \overline{A^+} & 0 \end{pmatrix}$ and we have to prove that $(\bar{M})^* = \bar{M}$. Clearly we can assume that A, A^+ are closed, so M is also closed and we need to establish that $M^* = M$. To do this, calculate e.g. $\operatorname{Ker}(M^* - iI)$. Note that $M^* = \begin{pmatrix} 0 & (A^+)^* \\ A^* & 0 \end{pmatrix}$. The equality $(M^* - iI)(u \oplus v) = 0$ means that

$$(A^+)^* v = iu \qquad \text{and} \qquad A^* u = iv.$$

It follows that $A^*(A^+)^* v = -v$. But $A^*(A^+)^* \subset (A^+ A)^*$ (see Exercise 4 in 3.8) and $(A^+ A)^* = \overline{A^+ A}$ due to our assumption. The quadratic form of the operator $\overline{A^+ A}$ is obviously non-negative. Therefore the same is true for $A^*(A^+)^*$, hence it can not have an eigenvalue -1. It follows that $u = v = 0$ which ends the proof. □

We shall also need the following modification of Lemma 2:

LEMMA 3. *Suppose that $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $A^+ : \mathcal{H}_2 \to \mathcal{H}_1$ are two linear (unbounded) operators in Hilbert spaces, such that*

$$(Au, v)_{\mathcal{H}_2} = (u, A^+ v)_{\mathcal{H}_1}, \quad u \in D(A), v \in D(A^+).$$

Suppose that the operator $A^+ A$ is essentially self-adjoint. Then the closures $\bar{A}, \bar{A^+}$ are mutually adjoint as operators in Hilbert spaces, i.e. $\bar{A} = (A^+)^*, \overline{A^+} = A^*$.

The proof is similar to the proof of Lemma 2.

PROOF OF PROPOSITION 2. Let us discuss (ii) first. Let us consider Δ_p in the Hodge-Kodaira decomposition given in Proposition 1. Clearly Δ_p has the form $d\delta, 0, \delta d$ on the spaces $d\Lambda_c^{p-1}(M), \mathcal{H}^p(M), \delta\Lambda_c^{p+1}(M)$ respectively. Therefore the closure $\bar{\Delta}_p$ also splits into a direct sum of operators $\overline{d\delta}, 0, \overline{\delta d}$. Also clearly the adjoint operator Δ_p^* splits as a direct sum of the operators $(d\delta)^*, 0, (\delta d)^*$. Since, as noted above, $\Delta_p^* = \bar{\Delta}_p$, we see that $\overline{d\delta} = (d\delta)^*, \overline{\delta d} = (\delta d)^*$ which means that the operators $d\delta, \delta d$ are essentially self-adjoint. Using Lemma 3, we obtain that $d^* = \bar{\delta}, \delta^* = \bar{d}$ in corresponding Hilbert spaces, hence in $L^2\Lambda^p(M)$. This is a reformulation of (ii).

Now (i) is an obvious corollary of (ii) which implies that

$$\Delta_{\max} = \Delta_{\min} = d^*_{\max} d_{\max} + \delta^*_{\max} \delta_{\max},$$

hence $\operatorname{Ker} \Delta_{\max} = \operatorname{Ker} d_{\max} \cap \operatorname{Ker} \delta_{\max}$. □

REMARK. Proposition 2 holds for any complete Riemannian manifold M.

DEFINITION 2. The *reduced L^2-cohomology* of degree p of the Riemannian manifold M is the Hilbert space

$$\bar{H}^p_{(2)}(M) = \operatorname{Ker} d_{p,\max} / \overline{d\Lambda_c^{p-1}(M)}.$$

The Hodge-Kodaira decomposition obviously implies that there exists a natural linear topological isomorphism

$$\mathcal{H}^p(M) \cong \bar{H}^p_{(2)}(M)$$
$$\omega \mapsto \omega \bmod \overline{d\Lambda_c^{p-1}(M)}.$$

Now if M is a Γ-manifold with a Γ-invariant Riemannian metric, then all spaces $L^2\Lambda^p(M)$ become free Hilbert Γ-modules, hence $\operatorname{Ker} d_{p,\max}$, $\overline{d\Lambda_c^{p-1}(M)}$, $\mathcal{H}^p(M)$ and $\bar{H}^p_{(2)}(M)$ become (projective) Hilbert Γ-modules.

Definition 3 (M. Atiyah [**A**]). The Γ-dimensions of the Hilbert Γ-modules $\bar{H}^p_{(2)}(M)$ (or $\mathcal{H}^p(M)$) are called *L^2-Betti numbers*, or *von Neumann Betti numbers*, or *Γ-Betti numbers*, and are denoted by $\bar{b}_p(M)$ or $\bar{b}_p(M,\Gamma)$, i.e.

$$\bar{b}_p(M) = \bar{b}_p(M,\Gamma) = \dim_\Gamma \bar{H}^p_{(2)}(M) = \dim_\Gamma \mathcal{H}^p(M).$$

These numbers were introduced by M. Atiyah [**A**].

So a priori $\bar{b}_p(M) \in [0,\infty]$.

Lemma 4. *If $X = M/\Gamma$ is a compact closed manifold, then $\bar{b}_p(M) < \infty$ for all $p = 0, 1, \dots, n$, where $n = \dim M$.*

Proof. The orthogonal projection on $\mathcal{H}^p(M)$ is a Γ-smoothing operator, hence it is in Γ-trace class (see Corollary 5 in 3.11). □

Let $\chi(X) = \sum_{p=0}^n (-1)^p b_p(X)$ be the usual Euler characteristic of X (here $b_p(X) = \dim_{\mathbb{C}} H^p(X,\mathbb{C})$ are the usual Betti numbers.)

Theorem 1. *In the notations above*

$$\sum_{p=0}^n (-1)^p \bar{b}_p(M) = \chi(X),$$

i.e. the L^2 Euler characteristic of M (or von Neumann Euler characteristic of M) coincides with the usual Euler characteristic of X.

Proof. Consider the differential operator

$$A = d + \delta : \bigoplus_p \Lambda^{2p}(M) \to \bigoplus_p \Lambda^{2p+1}(M).$$

Obviously

$$A^* = d + \delta : \bigoplus_p \Lambda^{2p+1}(M) \to \bigoplus_p \Lambda^{2p}(M).$$

Since $d^2 = \delta^2 = 0$, we have $(d+\delta)^2 = d\delta + \delta d = \Delta$, or, more precisely,

$$A^*A = \bigoplus_p \Delta_{2p}, \quad AA^* = \bigoplus_p \Delta_{2p+1}.$$

It follows that A is elliptic and

$$\operatorname{Ker} A = \bigoplus_p \mathcal{H}^{2p}(M), \qquad \operatorname{Ker} A^* = \bigoplus_p \mathcal{H}^{2p+1}(M).$$

Hence

$$\operatorname{ind}_\Gamma A = \sum_{p=0}^n (-1)^p \dim_\Gamma \mathcal{H}^p(M) = \sum_{p=0}^n (-1)^p \bar{b}_p(M).$$

But A is obviously the lifting of the same operator $d+\delta$ on X, which has the usual index $\sum_{p=0}^{n}(-1)^p b_p(X) = \chi(X)$ since we may repeat the reasoning on X. Now the result immediately follows from the Atiyah Γ-index theorem (Theorem 3 in 3.11). □

Now let us give a simplest example that shows that the L^2-Betti numbers might differ from the usual Betti numbers.

LEMMA 5. *Let M has no compact connected components. Then $\bar{b}_0(M) = 0$.*

PROOF. $\mathcal{H}^0(M)$ consists of all locally constant L^2 functions on M. So the imposed restriction implies that $\mathcal{H}^0(M) = \{0\}$, hence $\bar{b}_0(M) = 0$. □

Actually more can be said if we use the following analogue of the Poincaré duality:

LEMMA 6. *Suppose that M is orientable. Then $\bar{H}^p_{(2)}(M)$ and $\bar{H}^{n-p}_{(2)}(M)$ are isomorphic as Γ-modules. In particular $\bar{b}_p(M) = \bar{b}_{n-p}(M)$ for every $p = 0, 1, \ldots, n$.*

PROOF. We shall use the Hodge duality operator $* : \Lambda^p(M) \to \Lambda^{n-p}(M)$ (which is in fact induced by an isomorphism of vector bundles $\Lambda^p T^*M \to \Lambda^{n-p}T^*M$) defined by the given Γ-invariant Riemannian metric. It commutes with the action of Γ and with the Laplacian Δ. It follows that it induces an isomorphism of Γ-modules $* : \mathcal{H}^p(M) \to \mathcal{H}^{n-p}(M)$ which proves the desired result. □

COROLLARY 1. *If M is orientable and has no compact connected components then $\bar{b}_0(M) = \bar{b}_n(M) = 0$.*

REMARK. Note that if M is simply connected then M is orientable (e.g. this is true if M is the universal covering of X, hence $\Gamma = \pi_1(X)$).

EXAMPLE 1. Let $M = \mathbb{H}^2$ be the hyperbolic plane, Γ a discrete group of isometries such that the action of Γ is free and $X = M/\Gamma$ is a compact surface of genus g. The L^2-Betti numbers depend on Γ also (and not only on M), so we shall denote them $\bar{b}_p(M;\Gamma)$. Since $\bar{b}_0(M;\Gamma) = \bar{b}_2(M;\Gamma) = 0$ we have

$$-\bar{b}_1(M;\Gamma) = \chi(X) = 2 - 2g \quad \text{if } X \text{ is orientable,}$$

and

$$-\bar{b}_1(M;\Gamma) = \chi(X) = 1 - g \quad \text{if } X \text{ is not orientable,}$$

so

$$\bar{b}_1(M;\Gamma) = 2g - 2 \quad \text{if } X \text{ is orientable,}$$

and

$$\bar{b}_1(M;\Gamma) = g - 1 \quad \text{if } X \text{ is not orientable.}$$

Now note that the space $\mathcal{H}^1(M)$ of all harmonic square-integrable 1-forms on M does not depend on Γ, so the Schwartz kernel of the projection P_1 on $\mathcal{H}^1(M)$ does not depend on Γ as well. Denote this kernel $K_1 = K_1(x,y)$. Now use the fact that $\mathbb{H}^2$ is a homogeneous space i.e. the isometry group acts transitively. Since all isometries commute with the Laplacian, the space $\mathcal{H}^1(M)$ is isometry invariant. It follows that $\operatorname{Tr} K_1(x,x) = c_1 = \text{constant}$. Hence

$$\bar{b}_1(M;\Gamma) = \int_F \operatorname{Tr} K_1(x,x)\, d\mu(x) = c_1 \operatorname{vol} F = \begin{cases} 2g-2 & \text{if } X \text{ is orientable.} \\ \\ g-1 & \text{if } X \text{ is not orientable.} \end{cases}$$

where F is a fundamental domain of the action of Γ on $\mathbb{H}^2$, $d\mu(x)$ is the Riemannian density on $\mathbb{H}^2$, $\operatorname{vol} F$ is the Riemannian volume of F. It can be proved by a straightforward calculation that in fact $c_1 = (2\pi)^{-1}$.

It follows that the possible volumes of fundamental domains are quantized according to the Euler characteristic of $X = M/\Gamma$

$$\operatorname{vol} F = \begin{cases} 2\pi(2g-2) & \text{if } X \text{ is orientable,} \\ \\ 2\pi(g-1) & \text{if } X \text{ is not orientable.} \end{cases}$$

The same arguments can be applied to any homogeneous Riemannian manifold (in particular to any symmetric space).

EXAMPLE 2. It can be proved that if $\mathbb{H}^m$ denotes the m-dimensional hyperbolic space, then

$$\bar{b}_p(\mathbb{H}^{2n+1}) = 0 \quad \text{for all } p = 0, 1, \ldots, 2n+1;$$
$$\bar{b}_p(\mathbb{H}^{2n}) = 0 \quad \text{for all } p \neq n;$$
$$\bar{b}_n(\mathbb{H}^{2n}; \Gamma) = c_n \operatorname{vol}(\mathbb{H}^{2n}/\Gamma),$$

where c_n is a constant that depends only on n (and not on Γ).

4.2. L²-cohomology of simplicial and CW Γ-complexes

Let M be a Γ-manifold with a compact closed quotient $X = M/\Gamma$. Then X can be represented as a finite simplicial complex K. We can then lift all simplices to M which makes M a simplicial (generally infinite) Γ-complex $\hat{K}$. This means that Γ acts freely on the sets of simplices of every dimension and this action commutes with the boundary operator ∂ on chains.

Let us forget about manifolds for a while and consider a simplicial Γ-complex $\hat{K}$ with a finite quotient complex $K = \hat{K}/\Gamma$. By definition, an L^2-chain of $\hat{K}$ is a formal linear combination $c = \sum_\sigma c_\sigma \sigma$, where σ runs over oriented simplices in $\hat{K}$, $c_\sigma \in \mathbb{C}$ and $\sum_\sigma |c_\sigma|^2 < \infty$. Obviously, the space of all L^2-chains is a Hilbert space which we will denote $C^{(2)}(\hat{K})$. It is obviously a Hilbert Γ-module. Denote by $C_p^{(2)}(\hat{K})$ its subspace which consists of L^2 linear combinations of the simplices of dimension p. It is also a Hilbert Γ-module. The boundary map extends to a bounded linear map $\partial : C_p^{(2)}(\hat{K}) \to C_{p-1}^{(2)}(\hat{K})$, such that $\partial^2 = 0$. Obviously ∂ is a morphism of Hilbert Γ-modules.

Let us denote the dual space $(C_p^{(2)}(\hat{K})'$ by $C^p_{(2)}(\hat{K})$. So $C^p_{(2)}(\hat{K})$ is also a Hilbert space that can be identified with $C_p^{(2)}(\hat{K})$. We have a morphism of Hilbert Γ-modules

$$d : C^p_{(2)}(\hat{K}) \to C^{p+1}_{(2)}(\hat{K})$$

which is adjoint to $\partial : C_{p+1}^{(2)}(\hat{K}) \to C_p^{(2)}(\hat{K})$. Obviously $d^2 = 0$.

Now we can consider reduced simplicial L^2-homology and L^2-cohomology $\bar{H}_p(\hat{K})$ and $\bar{H}^p(\hat{K})$ which are reduced homology and reduced cohomology of the Hilbert complexes $(C_p^{(2)}(\hat{K}), \partial)$ and $(C^p_{(2)}(\hat{K}), d)$ respectively. Then $\bar{H}_p(\hat{K}), \bar{H}^p(\hat{K})$ are

Hilbert Γ-modules. It is easy to check that they are dual (hence isomorphic) Hilbert Γ-modules, so the simplicial L^2-Betti numbers

$$\bar{b}_p(\hat{K}) = \bar{b}_p(\hat{K}, \Gamma) = \dim_\Gamma \bar{H}_p(\hat{K}) = \dim_\Gamma \bar{H}^p(\hat{K})$$

are well defined.

THEOREM 1 (J. Dodziuk [**Do**]). 1) *Let $\hat{K}$ be a simplicial Γ-complex with a finite quotient complex $K = \hat{K}/\Gamma$. Then the isomorphism classes of the Hilbert Γ-modules $\bar{H}^p(\hat{K})$ are homotopy invariants of the pair $(|\hat{K}|, \Gamma)$, where $|\hat{K}|$ is the topological Γ-space defined by the complex $\hat{K}$. In particular, if $\Gamma = \pi_1(K)$ (and $\hat{K}$ is the universal covering of K), then the isomorphism classes of the Hilbert Γ-modules $\bar{H}^p(\hat{K})$ are homotopy invariants of $|K|$.*

2) *Let M be a Γ-manifold with a compact closed quotient $X = M/\Gamma$, X is represented as $X = |K|$, where K is a finite simplicial complex, and $\hat{K}$ is the simplicial Γ-complex obtained by lifting simplices of K to M (so in particular $M = |\hat{K}|$). Then there is an isomorphism of Hilbert Γ-modules $\bar{H}^p(M) \cong \bar{H}^p(\hat{K})$, i.e. the reduced de Rham L^2-cohomologies are isomorphic to reduced simplicial L^2-cohomologies as Hilbert Γ-modules for all $p = 0, 1, \ldots, n$, where $n = \dim M$.*

COROLLARY 1. 1) *The L^2-Betti numbers $\bar{b}_p(\hat{K}) = \bar{b}_p(\hat{K}, \Gamma)$ are homotopy invariants of the pair $(|\hat{K}|, \Gamma)$.*

2) *Let $M, \hat{K}$ are as in the second part of Theorem. then*

$$\bar{b}_p(M) = \bar{b}_p(\hat{K}), \quad p = 0, 1, \ldots, n.$$

In particular, $\bar{b}_p(M)$ are homotopy invariants of the pair (M, Γ).

COROLLARY 2. 1) *If $|\hat{K}|$ is simply connected then $\bar{b}_p(\hat{K})$ are homotopy invariants of $|K|$, where $K = \hat{K}/\Gamma$, $\Gamma = \pi_1(|K|)$.*

2) *If M is simply connected then $\bar{b}_p(M)$ are homotopy invariants of the quotient space $X = M/\Gamma$, $\Gamma = \pi_1(X)$, $p = 0, 1, \ldots n$.*

REMARK. Let Y_1, Y_2 be topological Γ-spaces (i.e. Γ is a discrete group acting freely on Y_j, $j = 1, 2$). Then we say that the pairs $(Y_1, \Gamma), (Y_2, \Gamma)$ are *homotopy equivalent* if there exist Γ-maps $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ such that both compositions $f \circ g$ and $g \circ f$ are homotopic to the identity maps in the class of Γ-maps. So we say that something is *a homotopy invariant* of a pair (Y, Γ) if it coincides for $(Y_1, \Gamma), (Y_2, \Gamma)$ provided $(Y_1, \Gamma), (Y_2, \Gamma)$ are homotopy equivalent.

We shall not prove the Theorem now (see [**Do**] for a detailed proof). It will follow from a more general result which will be established later in this chapter. We just note for the reader who is well versed in algebraic topology that the first part can be proved by reproducing the routine proof which is described in textbooks on Algebraic Topology in case of trivial Γ, and the second part can be obtained by reproducing the classical Whitney proof of the de Rham isomorphism provided we use Sobolev forms instead of L^2-forms to be able to integrate forms over simplices of all dimensions.

Now an important step is to replace simplicial complexes by arbitrary CW-complexes with Γ-invariant cell structures. This can be done and the reduced L^2-(co)homologies defined by cells and incidence coefficients, depend on the underlying Γ-space only. This can be done because the same homological algebra technique

can be used in the Γ-invariant situation. In fact it is even convenient to consider the following more general situation.

Let X be a finite CW-complex, $\Gamma_0 \subset \pi_1(X)$ a normal subgroup of the fundamental group. Let M be the corresponding covering CW-complex, so that $\Gamma = \pi_1(X)/\Gamma_0$ acts freely on M with $X = M/\Gamma$. Let a unitary representation $\rho : \Gamma \to U(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ be given (here $U(\mathcal{H})$ denotes the set of all unitary operators in $\mathcal{H}$). Then we can form a flat Hilbert vector bundle with the fiber $\mathcal{H}$ (a Hilbert bundle with a flat unitary connection) over X as $E = M \underset{\Gamma}{\times} \mathcal{H}$, where $M \underset{\Gamma}{\times} \mathcal{H} = (M \times \mathcal{H})/\Gamma$ if we define the action of Γ on $M \times \mathcal{H}$ by

$$g\{y, \xi\} = \{gy, \rho(g)\xi\}.$$

We have a natural projection $\tilde{\pi} : E \to X$ given by $\tilde{\pi}(\Gamma\{y, \xi\}) = \pi(y)$, where $\pi : M \to X$ is the canonical projection. If $x_0 \in X$, U is a small connected neighborhood of x_0 in X and $y_0 \in M$ is chosen, $y_0 \in \pi^{-1}(x_0)$, then we have a trivialization of $\tilde{\pi}^{-1}(U)$ which is obtained if we choose a small connected neighbourhood $\tilde{U}$ of y_0 such that $\pi(\tilde{U}) = U$, and then identify $\Gamma\{y, \xi\}$ with $\{x, \xi\}$ provided $y \in \tilde{U}$ and $\pi(y) = x$. These trivializations provide a flat connection on E with the monodromy given by the composition $\pi_1(X) \to \Gamma \xrightarrow{\rho} U(\mathcal{H})$, where the first map is the canonical projection.

Now the sheaf of locally constant (or locally parallel with respect to the given flat connection) sections of E is well defined. Let us denote this sheaf by $\underline{E}$. Denote by $\bar{H}^p(X, \underline{E})$ its reduced cohomologies which are well defined as Čech cohomologies of appropriate coverings, or with the help of the cell structure and reduced cohomologies of the CW-complex X with coefficients in the local system defined by the sheaf $\underline{E}$ (in both cases, the spaces of L^2 chains form Hilbert spaces and the differentials are bounded linear operators).

Now let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$, $\mathcal{M}'$ its commutant. Let $U(\mathcal{M})$, $U(\mathcal{M}')$ denote the sets of unitary operators in $\mathcal{M}, \mathcal{M}'$ respectively. Suppose that in fact ρ is a representation $\rho : \Gamma \to U(\mathcal{M}) \subset U(\mathcal{H})$. Then $\mathcal{M}'$ will act in all fibers, sections and cohomologies described above. Actually spaces of chains and cochains will have the form $\mathcal{H}^N = \mathcal{H} \otimes \mathbb{C}^N$ and (co)homology spaces $\bar{H}^p(X, \underline{E})$ can be identified with some closed subspaces in $\mathcal{H}^N$ affiliated with the von Neumann algebra $\mathcal{M}_N = \mathcal{M} \otimes \mathrm{End}(\mathbb{C}^N)$. Now if we have a finite trace τ on $\mathcal{M}$, then we can take the trace $\tau_N = \tau \otimes \mathrm{Tr}$ on $\mathcal{M}_N$ (here Tr is the usual matrix trace on $\mathrm{End}(\mathbb{C}^N)$). We shall always suppose that τ is normalized by the condition $\tau(I) = 1$. We can use the dimension function $\dim_\tau$ associated with τ and τ_N to define real-valued von Neumann Betti numbers

$$\bar{b}_p^\rho = \dim_\tau \bar{H}^p(X, \underline{E}).$$

They do not actually depend on the representation of X as a CW-complex. If X is a compact manifold, then $\bar{H}^p(X, \underline{E})$ can also be defined by an appropriate de Rham complex of L^2-forms with coefficients in E.

Example 1. Assume that $X = S^1 = \mathbb{R}/\mathbb{Z}$, and S^1 is represented as a CW-complex with 2 cells : a 0-dimensional cell $\sigma_0 = 0 \bmod \mathbb{Z}$ and a 1-dimensional cell $\sigma_1 = (0, 1) \bmod \mathbb{Z}$. Suppose that a unitary representation $\rho : \mathbb{Z} = \pi_1(S) \to U(\mathcal{M})$ is given. It is defined of course by a single unitary operator $V = \rho(1) \in \mathcal{M}$. The bundle E is then of course trivial on $(0, 1)$ and the trivialization extends to $[0, 1]$

after being lifted to the universal covering $\mathbb{R}$. Then the complex that defines the corresponding homology looks as follows:

$$0 \to \mathcal{H} \xrightarrow{\partial} \mathcal{H} \to 0$$

where $\partial = 1 - V$. So the image of ∂ can be not closed if 1 is not an isolated point in the spectrum of V. Hence we have

$$\bar{H}^0(S^1, \underline{E}) = \operatorname{Ker}(1-V), \quad \bar{H}^1(S^1, \underline{E}) = \operatorname{Ker}(1-V^*) = \operatorname{Ker}(1-V).$$

So if we have a normalized finite trace τ on $\mathcal{M}$, then

$$\bar{b}_0^\rho = \bar{b}_1^\rho = \dim_\tau \operatorname{Ker}(1-V).$$

In particular, we see that $\bar{b}_0^\rho = \bar{b}_1^\rho$ can be any real number in $[0,1]$.

Now let X be a compact Riemannian manifold, $\mathcal{H} = L^2\Gamma$ and $\mathcal{M} = \mathcal{M}_\Gamma$ i.e. $\mathcal{M}$ is generated by right translations. Then it is easy to see that if $\rho : \Gamma \to U(\mathcal{M})$ is the natural representation, $\rho(g) = R_g$, then there is a natural isomorphism $L^2(M) = L^2(X, E)$. More precisely, let U_0 be a small connected neighbourhood of $x_0 \in X$, $y_1, y_2 \in \pi^{-1}(x_0)$. Let U_1, U_2 be small connected neighbourhoods of y_1, y_2, such that $\pi(U_j) = U_0$, $j = 1, 2$. Then we have $\Gamma U_1 = \Gamma U_2 = \pi^{-1}(U_0)$ and we have two identifications of $\pi^{-1}(U_0)$ with $U_0 \times \Gamma$: $gy \leftrightarrow \{x, g\}$ if $y \in U_j$, $\pi(y) = x$, where $j = 1, 2$. Now suppose that $y_2 = g_0 y_1$. If $\{x, g_1\}, \{x, g_2\}$ correspond to the same point in M under two chosen identifications of $\pi^{-1}(U_0)$ with $U_0 \times \Gamma$, then we should have $g_1 y_1 = g_2 y_2 = g_2 g_0 y_1$, hence $g_1 = g_2 g_0$. Using these identifications to obtain trivializations of E we see that the transition function between two copies of $L^2(U_0) \otimes L^2\Gamma$, corresponding to the two given trivializations is just $I \otimes R_{g_0}$. This exactly gives the desired result. So in this case all reduced cohomologies will be projective Hilbert Γ-modules. Hence the corresponding Betti Γ-numbers are well defined. Of course they coincide with the Betti Γ-numbers defined with the help of the de Rham L^2-complex.

4.3. Morse inequalities

We will start by recalling the classical Morse inequalities. Let X be a compact closed C^∞-manifold, $f : X \to \mathbb{R}$ a C^∞-function on X. A point $x_0 \in X$ is called a *critical point* of f if $df(x_0) = 0$. A critical point x_0 is called *non-degenerate* if in some local coordinates near x_0 the matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right]$ is non-degenerate (i.e. $\det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right] \neq 0$). This is a symmetric matrix and the number of its negative eigenvalues is called *index of* f at the critical point x_0. All non-degenerate critical points are isolated (in the set of all critical points). A function f is called *Morse function* if all its critical points are non-degenerate. It follows that f has only a finite number of critical points. Denote the number of critical points of index p by $m_p = m_p(f)$, $p = 0, 1, \ldots, n$, where $n = \dim X$. Let $b_p = b_p(X)$ be the Betti numbers of X.

THEOREM 1 (The Morse inequalities). *For any Morse function $f : X \to \mathbb{R}$*

$$\sum_{p=0}^{k} (-1)^{k-p} m_p(f) \geq \sum_{p=0}^{k} (-1)^{k-p} b_p(X), \quad k = 0, 1, \ldots, n.$$

This inequality becomes an equality when $k = n = \dim X$.

COROLLARY 1. $m_p(f) \geq b_p(X)$, $p = 0, 1, \ldots, n$.

This follows if we just sum two previous inequalities with $k = p$ and $k = p - 1$.

In this chapter we will briefly discuss a topological proof of Theorem 1, which can be found e.g. in [**Mi**]. Another (analytic) proof was suggested by E. Witten [**Wi**] (see also Ch. 11 in [**CFKS**] for more details). It is based on semi-classical asymptotics of spectra of some Schrödinger-type operators. This proof and its generalization to a more general case ([**Sh7**]) will be discussed in the next chapter.

Let us recall the main construction of the Morse theory that is used to prove the Morse inequalities and their generalizations that will be given later. We refer to [**Mi**] for more details.

For every $t \in \mathbb{R}$ consider $X_t = \{x \mid x \in X,\ f(x) \leq t\} \subset X$. If t is a regular value of f (i.e. t does not coincide with the value of f at a critical point) then X_t is a compact manifold with boundary. Let $x_1, \ldots, x_N$ be the set of all critical points of f. Slightly perturbing f we may assume that $f(x_i) \neq f(x_j)$ if $i \neq j$, i.e. that all critical values are different. Denote $a_i = f(x_i)$ and assume that $a_1 < a_2 < \cdots < a_N$. Then $X_t = \emptyset$ if $t < a_1$, X_t is diffeomorphic to a ball if $a_1 < t < a_2$, because a_1 must be the absolute minimum of f and x_1 must be its non-degenerate minimum. Generally if $t', t'' \in (a_i, a_{i+1})$, then moving along the gradient flow of f we can estabish that $X_{t'}$ is diffeomorphic to $X_{t''}$. Let us look what happens if t increases and crosses a_i. It can be proved that homotopically a cell is glued at the moment of crossing, and the dimension of the cell is equal to the index of the critical point x_i. So raising t from $a_1 - \epsilon$ to $a_N + \epsilon$ with $\epsilon > 0$, we see that the following Theorem is true:

THEOREM 2. *X is homotopy equivalent to a CW-complex which has exactly $m_p(f)$ cells of dimension p.*

Now we will show how this theorem can be used to deduce the Morse inequalities. We can use the CW-complex from Theorem 2 to calculate the cohomology of X. This gives us a complex of finite-dimensional vector spaces

$$0 \to \mathbb{C}^{m_0} \xrightarrow{d_0} \mathbb{C}^{m_1} \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} \mathbb{C}^{m_n} \to 0$$

with the cohomologies $H^p(X) = H^p(X, \mathbb{C})$. Now break this complex at $(k+1)$th place and form a shorter complex

$$0 \to \mathbb{C}^{m_0} \xrightarrow{d_0} \mathbb{C}^{m_1} \xrightarrow{d_1} \ldots \xrightarrow{d_{k-1}} \mathbb{C}^{m_k} \xrightarrow{d_k} \operatorname{Im} d_k \to 0$$

which has cohomologies $H^0(X), H^1(X), \ldots, H^k(X)$. Hence by a standard algebraic property of the Euler characteristic (see Lemma in 2.27) we obtain

$$\sum_{p=0}^{k} (-1)^p m_p + (-1)^{k+1} \dim_{\mathbb{C}} \operatorname{Im} d_k = \sum_{p=0}^{k} (-1)^p b_p(X),$$

hence

$$\sum_{p=0}^{k} (-1)^{k-p} m_p = \sum_{p=0}^{k} (-1)^{k-p} b_p(X) + \dim_{\mathbb{C}} \operatorname{Im} d_k \geq \sum_{p=0}^{k} (-1)^{k-p} b_p(X),$$

with equality in the case when $k = n = \dim X$, as required.

The following theorem generalizes Theorem 2.

THEOREM 3 ([**NS1**]). *Let X be a finite CW-complex that can be represented as $X = M/\Gamma$, where Γ acts freely on a CW-complex M. Let X have m_p cells of dimension p, $p = 0, 1, \dots, n$. Suppose that a representation $\rho : \Gamma \to U(\mathcal{M})$ is given where $\mathcal{M}$ is a von Neumann algebra with a finite trace τ normalized by the condition $\tau(I) = 1$. Then*

$$\sum_{p=0}^{k} (-1)^{k-p} m_p \geq \sum_{p=0}^{k} (-1)^{k-p} \bar{b}_p^{\rho}, \quad k = 0, \dots, n.$$

with equality in the case when $k = n$.

PROOF. If $\mathcal{M}$ acts in $\mathcal{H}$, then we have the complex

$$0 \to \mathcal{H}^{m_0} \xrightarrow{d_0} \mathcal{H}^{m_1} \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \mathcal{H}^{m_n} \to 0$$

where d_j is a bounded linear operator that commutes with the action of $\mathcal{M}'$ (by the operators of the form $a' \otimes I$ on $\mathcal{H}^k = \mathcal{H} \otimes \mathbb{C}^k$) and with the reduced cohomologies equal to $\bar{H}^p(X, \underline{E})$, where $\underline{E}$ is the local system (the sheaf of flat sections of the flat vector bundle E) constructed with the help of the representation ρ. Note that the differentials d_p can be then represented as matrices with elements in $\mathcal{M}$, or

$$d_p \in \mathcal{M} \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{m_p}, \mathbb{C}^{m_{p+1}}).$$

Hence the Laplacians $\Delta_p = d_p^* d_p + d_{p-1} d_{p-1}^*$ are also matrices with elements from $\mathcal{M}$, more precisely,

$$\Delta_p \in \mathcal{M} \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{m_p}, \mathbb{C}^{m_p}) = \mathcal{M} \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{m_p}).$$

It follows that the spaces $\operatorname{Ker} \Delta_p = \bar{H}^p(X, \underline{E})$ are affiliated with the von Neumann algebra $\mathcal{M}_{m_p} = \mathcal{M} \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{m_p})$. The von Neumann dimensions of the cohomologies are $\bar{b}_p^{\rho} = \dim_{\tau} \bar{H}^p(X, \underline{E})$ (see 4.2).

Let us again cut the complex at $(k+1)$th place, obtaining the complex

$$0 \to \mathcal{H}^{m_0} \xrightarrow{d_1} \mathcal{H}^{m_1} \to \dots \xrightarrow{d_{k-1}} \mathcal{H}^{m_k} \xrightarrow{d_k} \overline{\operatorname{Im} d_k} \to 0$$

Now we can use the von Neumann version of the algebraic Lemma about the Euler characteristic (cf. Lemma in 2.27) that gives us the desired inequalities exactly as in the classical case. □

COROLLARY 2. *In the notation above, one has*

$$\sum_{p=0}^{k} (-1)^{k-p} m_p \geq \sum_{p=0}^{k} (-1)^{k-p} \bar{b}_p(M, \Gamma), \quad k = 0, 1, \dots, n,$$

with equality when $k = n = \dim X$.

COROLLARY 3. *If X is a compact closed C^∞-manifold, $X = M/\Gamma$, where Γ is a discrete group acting freely on Γ, $f : X \to \mathbb{R}$ be a Morse C^∞-function and $\rho : \Gamma \to U(\mathcal{M})$ be a unitary representation of Γ to a von Neumann algebra $\mathcal{M}$ with a normalized trace τ, then*

$$\sum_{p=0}^{k} (-1)^{k-p} m_p \geq \sum_{p=0}^{k} (-1)^{k-p} \bar{b}_p^{\rho}, \quad k = 0, 1, \dots, n,$$

with equality when $k = n = \dim X$. *Here* $n = \dim X$ *and* m_p *is the number of critical points of the index* p *for* f. *In particular*

$$\sum_{p=0}^{k}(-1)^{k-p}m_p \geq \sum_{p=0}^{k}(-1)^{k-p}\bar{b}_p(M,\Gamma),$$

with equality when $k = n$.

REMARK 1. On a simply-connected manifold X, the Morse inequalities can not be improved in the sense that there exists a Morse function $f : X \to \mathbb{R}$ such that $m_p(f)$ is exactly equal to the maximum of $b_p(X,\mathbb{F})$ over all finite fields $\mathbb{F}$. As we shall see now it is not the case for non-simply connected manifolds.

Let Γ be a finitely presented group, i.e. a finitely generated group with a finite number of relations between generators. A.A. Markov [**Mar**] proved that any such group is the fundamental group of a compact closed connected manifold X of dimension 4. Let us take Γ infinite with $\Gamma = [\Gamma,\Gamma]$, where $[\Gamma,\Gamma]$ is the commutant of Γ which is the normal subgroup generated by all the elements of the form $aba^{-1}b^{-1}$, $a, b \in \Gamma$. Since $H_1(X,\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$, we then have $b_1(X) = b_3(X) = 0$. Since $b_0(X) = b_4(X) = 1$, we have for the Euler characteristic

$$\chi(X) = 2 + b_2(X) = \bar{b}_2(M,\Gamma) - 2\bar{b_1}(M,\Gamma)$$

because $\bar{b}_0(M,\Gamma) = \bar{b}_4(M,\Gamma) = 0$ and $\bar{b}_3(M,\Gamma) = \bar{b_1}(M,\Gamma)$ (here M is the universal covering of X). Hence for any Morse function $f : X \to \mathbb{R}$ we will have

$$m_2(f) - m_1(f) + m_0(f) \geq \bar{b}_2(M,\Gamma) - \bar{b}_1(M,\Gamma) \tag{4.1}$$

$$= b_2(X) + 2 + \bar{b}_1(M,\Gamma) \geq b_2(X) + 2. \tag{4.2}$$

Even in the case $\bar{b}_1(M,\Gamma) = 0$ this inequality is better than the classical one by 1.

EXERCISE 1. Prove that the same reasoning works for an arbitrary representation $\rho : \Gamma \to U(\mathcal{M})$ where $\mathcal{M}$ is a von Neumann algebra with a normalized trace on it. Namely,

$$m_2(f) - m_1(f) + m_0(f) \geq b_2(X) + 2 + \bar{b}_1^{\rho}.$$

Hint. You have to check that $\bar{b}_0^{\rho} = \bar{b}_4^{\rho} = 0$.

REMARK 2. The same is true not only on 4-dimensional closed manifolds but also on any closed manifold Y that has 3-skeleton which is homotopy equivalent to the 3-skeleton of a closed 4-dimensional manifold X, because all the Betti numbers (usual and von Neumann) in dimensions ≤ 2 depend on the 3-skeleton only.

REMARK 3. The Morse inequalities can also work in the opposite direction. Namely, we can always construct a CW-complex $\hat{X}$ which is homotopy equivalent to our manifold X and such that it has exactly 1 cell of dimension 0, m_1 cells of dimension 1 and m_2 cells of dimension 2 where m_1 is the number of generators of $\Gamma = \pi_1(X)$ and m_2 is the number of relations between them. Since for any representation $\rho : \Gamma \to U(\mathcal{M})$ we have then

$$m_2 - m_1 + 1 \geq \bar{b}_2^{\rho} - \bar{b}_1^{\rho},$$

we obtain

$$\bar{b}_1^{\rho} \geq m_1 - m_2 - 1.$$

In particular this implies that if $m_1 > m_2 + 1$ (this means that Γ is nearly free) then there is an infinite-dimensional space of L^2 harmonic 1-forms on the universal cover of X.

4.4. Near-cohomology

We have seen that the Betti numbers (either usual or von Neumann ones) are dimensions of the spaces of harmonic (L^2-)forms. It occurs that quantitative characteristics of the behaviour of the spectra of Laplacians on covering spaces near 0 provide other topological invariants of the base which is a non-simply connected manifold with an infinite fundamental group. These invariants can also be defined in terms of the behaviour of heat-kernels at infinity. But their motivation comes from their expression in terms of near-cohomology, which will be our starting point.

The original idea of homology due to A. Poincaré looks as follows:

$$H_*(M) = \frac{\{\text{closed submanifolds in } M\}}{\{\text{boundaries of submanifolds in } M\}} = \\ = \frac{\{c \mid \partial c = 0\}}{\{c \mid \text{there exists } a, \text{ such that } \partial a = c\}} ,$$

where ∂ means taking the boundary. The chain c in the numerator can be imagined as a closed submanifold with a "black hole" inside, so that this "black hole" does not belong to M and so does not allow c to be a boundary of what lies inside it.

The dimension grading can be made, so $H_k(M)$ is obtained if we take only chains or closed submanifolds c such that $\dim c = k$.

Similarly cohomology can be defined as

$$H^*(M) = \frac{\{\omega \mid d\omega = 0\}}{\{\omega \mid \text{there exist } \alpha, \text{ such that } d\alpha = \omega\}}$$

where d is in a way dual to ∂. For example d can be thought of as the de Rham exterior differential, so ω, α are differential forms on M. The grading is then made similarly.

Now let us first proceed to define near-(co)homology heuristically (the rigorous treatment will be given later). Let us try to define near-homology as

$$NH_*(M) = \frac{\{\text{submanifolds with a small boundary in } M\}}{\{\text{submanifolds in } M \text{ which are close to closed submanifolds in } M\}} = \\ \frac{\{c \mid \partial c \text{ is small}\}}{\{c \mid \operatorname{dist}(c, \operatorname{Ker}\partial) \text{ is small}\}} .$$

The chain c in the numerator can be imagined as a bottle with a very small neck, but an infinitely thin "black hole" wire may go through the neck and not permit to close the bottle by a small lid. The same bottle without the wire (but still with the "black hole" inside) should be considered negligible because it can be closed with a small lid.

It might seem more natural to replace $\operatorname{Ker}\partial$ by $\operatorname{Im}\partial$ in the denominator so as to make it closer to the definition of homology.

This is also possible but will lead to an object that includes homology itself as a direct summand; we prefer to extract a pure addition to (co)homology instead of considering a bigger object.

Similarly, heuristically near-cohomologies are defined as

$$NH^*(M) = \frac{\{\omega \mid d\omega \text{ is small}\}}{\{\omega \mid \operatorname{dist}(\omega, \operatorname{Ker} d) \text{ is small}\}}$$

The grading on all near-(co)homologies is of course also available.

However the formulas for the near-(co)homologies do not make sense because neither numerator nor denominator are vector spaces. The idea for making it rigorous is to use norms and compare e.g. how small $\|d\omega\|$ is with respect to $\|\omega\|_{\operatorname{mod} \operatorname{Ker} d} = \operatorname{dist}(\omega, \operatorname{Ker} d)$. More exactly, we will consider the asymptotic of sets

$$\{\omega \mid \omega \in E/\operatorname{Ker} d, \|d\omega\| \leq \lambda \|\omega\|_{\operatorname{mod} \operatorname{Ker} d}\}, \quad \lambda \downarrow 0,$$

where E is a normed space where all ω belong.

Now we shall pass to a rigorous treatment. First we have to modify the definition of the Hilbert complex by allowing unbounded differentials.

DEFINITION 1. A *Hilbert complex* is a sequence

$$E : 0 \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \to E_k \xrightarrow{d_k} E_{k+1} \to \ldots \xrightarrow{d_{N-1}} E_N \to 0$$

where E_k is a Hilbert space, the differential $d_k : E_k \to E_{k+1}$ is closed densely defined linear operators (with the domain $D(d_k)$) and $d_{k+1} \cdot d_k = 0$ on $D(d_k)$ or, equivalently, $\operatorname{Im} d_k \subset \operatorname{Ker} d_{k+1}$. Note that $\operatorname{Ker} d_k$ is always a closed linear subspace in E_k.

Let E' be another Hilbert complex of the same length N (if the length differ then we can always formally extend the shorter complex by adding zero spaces at the end; so for the sake of simplicity we shall always suppose that all complexes have the same length N). The corresponding spaces and differentials will be denoted E'_k and d'_k. A *morphism* $f : E \to E'$ of the Hilbert complexes is a collection of *bounded* linear operators $f_k : E_k \to E'_k$ such that $f_{k+1} d_k \subset d'_k f_k$, which means that $f_{k+1} d_k = d'_k f_k$ on $D(d_k)$. In particular we require that $f_k(D(d_k)) \subset D(d'_k)$.

If $f : E \to E'$ and $g : E' \to E''$ are two morphisms of Hilbert complexes then their composition $g \circ f : E \to E''$ is a morphism defined as the collection of compositions $g_k \circ f_k$, $k = 0, 1, \ldots, N$.

DEFINITION 2. Let $f, g : E \to E'$ be two morphisms of the same Hilbert complexes. A *homotopy* (between f and g) is a collection T of *bounded* linear operators $T_k : E_k \to E'_{k-1}$ such that

$$f_k - g_k = T_{k+1} d_k + d'_{k-1} T_k \text{ on } D(d_k) \quad k = 0, 1, \ldots, N$$

(or, in other words, $f_k - g_k - T_{k+1} d_k \subset d'_{k-1} T_k$, $k = 0, 1, \ldots, N$). In particular this means that $T_k(D(d_k)) \subset D(d'_{k-1})$. If there exists a homotopy between morphisms f and g, then f and g are called *homotopic* and we denote it as $f \sim g$.

EXERCISE 1. Check that being homotopic is really an equivalence relation between morphisms.

Hilbert complexes E, E' are called *homotopy equivalent* if there exist morphisms $f : E \to E'$ and $g : E' \to E$ such that $g \circ f \sim \operatorname{Id}_E$, $f \circ g \sim \operatorname{Id}_{E'}$ where Id_E and $\operatorname{Id}_{E'}$ are identity morphisms of the corresponding Hilbert complexes. We shall denote the homotopy equivalence between E and E' as $E \sim E'$.

EXERCISE 2. Check that cohomologies $H^k(E) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}$ and reduced cohomologies $\bar{H}^k(E) = \operatorname{Ker} d_k / \overline{\operatorname{Im} d_{k-1}}$ are homotopy functors in the category of Hilbert complexes with morphisms and homotopy as before. This means that any morphism $f : E \to E'$ induces maps

$$f_k^* : H^k(E) \to H^k(E') \quad \text{and} \quad \bar{f}_k^* : \bar{H}^k(E) \to \bar{H}^k(E'),$$

so that $(\mathrm{Id}_E)_k^*$ and $(\overline{\mathrm{Id}_E})_k^*$ are identity morphisms,

$$(f \circ g)_k^* = f_k^* \circ g_k^* \quad \text{and} \quad (\overline{f \circ g})_k^* = \bar{f}_k^* \circ \bar{g}_k^*$$

for two morphisms $f : E \to E'$ and $g : E'' \to E$, and $f_k^* = h_k^*$, $\bar{f}_k^* = \bar{h}_k^*$ for two homotopic morphisms $f, h : E \to E'$. In particular if $E \sim E'$ (i.e. E, E' are homotopy equivalent as Hilbert complexes), then

$$H^k(E) = H^k(E') \quad \text{and} \quad \bar{H}^k(E) = \bar{H}^k(E').$$

Now let us introduce for a given Hilbert complex E the following special families of quadric cones, depending on the degree k and on a positive parameter λ :

$$B_\lambda^{(k)} = \{\omega \mid \omega \in E_k / \operatorname{Ker} d_k, \|d_k\omega\| \le \lambda \|\omega\|_{\operatorname{mod} \operatorname{Ker} d}\},$$

where $\|\omega\|_{\operatorname{mod} \operatorname{Ker} d} = \operatorname{dist}(\omega, \operatorname{Ker} d_k)$ is the norm of ω in the quotient space $E_k / \operatorname{Ker} d_k$, $\|d_k\omega\|$ means the norm of $d_k\omega$ in E_{k+1}. It is understood that here we should only take cosets in $E_k / \operatorname{Ker} d_k$ defined by the elements $\omega \in D(d_k)$ to make $d_k\omega$ well defined. So $B_\lambda^{(k)}$ becomes a conic set in the Hilbert space $E_k / \operatorname{Ker} d_k$ (which can also be identified with $(\operatorname{Ker} d_k)^\perp$, the orthogonal complement of $\operatorname{Ker} d_k$ in E_k). It can be proved that $B_\lambda^{(k)}$ is always closed (this is obvious if d_k is bounded) but this fact is not important for us and we omit the proof.

Let us consider a morphism of Hilbert complexes $f : E \to E'$ defined by a collection of bounded linear operators $f_k : E_k \to E'_k$, $k = 0, \dots, N$. Then $f_k(\operatorname{Ker} d_k) \subset \operatorname{Ker} d'_k$ so f_k naturally defines a bounded linear operator

$$\hat{f}_k : E_k / \operatorname{Ker} d_k \to E'_k / \operatorname{Ker} d'_k$$

THEOREM 1 ([**GS2**]). *Let Hilbert complexes E and E' be homotopy equivalent and $f : E \to E'$, $g : E' \to E$ be morphisms such that $f \circ g \sim \mathrm{Id}_{E'}$, $g \circ f \sim \mathrm{Id}_E$. Let $B_\lambda^{(k)}$, ${}'B_\lambda^{(k)}$ be the special families of cones associated with the complexes E, E' respectively. Then there exist $\lambda_0 > 0, C > 0$ such that for every $\lambda \in (0, \lambda_0)$*

(i) $\hat{f}_k(B_\lambda^{(k)}) \subset {}'B_{C\lambda}^{(k)}$, $\|\omega\| \le C\|\hat{f}_k\omega\|$ *if* $\omega \in B_\lambda^{(k)}$;
(ii) $\hat{g}_k({}'B_\lambda^{(k)}) \subset B_{C\lambda}^{(k)}$, $\|\omega'\| \le C\|\hat{g}_k\omega'\|$ *if* $\omega' \in {}'B_\lambda^{(k)}$.

PROOF. We shall prove (i), then (ii) will follow by symmetry. Let us consider $\omega \in B_\lambda^{(k)}$ and let $\omega_1 \in (\operatorname{Ker} d_k)^\perp$ represent ω i.e. $\omega_1 \bmod \operatorname{Ker} d_k = \omega$. Then $\|d_k\omega_1\| \le \lambda\|\omega_1\|$. It follows that $f_k\omega_1 \in D(d'_k), d'_k f_k \omega_1 = f_{k+1} d_k \omega_1$ and

$$\|d'_k(f_k\omega_1)\| = \|f_{k+1}(d_k\omega_1)\| \le \|f_{k+1}\|\|d_k\omega_1\| \le \lambda\|f_{k+1}\|\|\omega_1\|.$$

Now we should estimate $\|\omega_1\|$ by $C_1\|f_k\omega_1\|_{\operatorname{mod} \operatorname{Ker} d'}$ provided $\lambda \in (0, \lambda_0)$ with a small $\lambda_0 > 0$ and with a constant C_1 which does not depend on ω_1 or λ. Then (i) will follow.

Let us split $f_k\omega_1$ into the sum $f_k\omega_1 = \omega'_1 + \omega'_2$ with $\omega'_2 \in \operatorname{Ker} d'_k$, $\omega'_1 \perp \operatorname{Ker} d'_k$. Hence $\|f_k\omega_1\|_{\operatorname{mod} \operatorname{Ker} d'} = \|\omega'_1\|$.

Now let us use a homotopy T between $g \circ f$ and Id_E. In particular we have

$$g_k \circ f_k = \mathrm{Id}_E + T_{k+1} d_k + d_{k-1} T_k \text{ on } D(d_k),$$

hence

$$\omega_1 = g_k(f_k\omega_1) - T_{k+1}d_k\omega_1 - d_{k-1}T_k\omega_1 = g_k(\omega_1' + \omega_2') - T_{k+1}d_k\omega_1 - d_{k-1}T_k\omega_1.$$

Clearly $g_k\omega_2' \in \operatorname{Ker} d_k$, hence

$$\omega_1 \equiv g_k\omega_1' - T_{k+1}d_k\omega_1 \bmod \operatorname{Ker} d_k.$$

It follows that

$$\begin{aligned}\|\omega_1\| &\le \|g_k\omega_1' - T_{k+1}d_k\omega_1\| \le \|g_k\|\|\omega_1'\| + \|T_{k+1}\|\|d_k\omega_1\| \le \\ &\le \|g_k\|\|\omega_1'\| + \lambda\|T_{k+1}\|\|\omega_1\|,\end{aligned}$$

hence

$$\|\omega_1\| \le (1 - \lambda\|T_{k+1}\|)^{-1}\|g_k\|\|\omega_1'\| \le 2\|g_k\|\|\omega_1'\|$$

if $\lambda \in (0, \lambda_0)$ where $\lambda_0 = (2\|T_{k+1}\|)^{-1}$. This gives the required estimate that proves (i), hence Theorem. □

Now we can introduce the notion of near-cohomology of Hilbert complexes. First let us give a general definition.

Definition 3. A *special family of quadric cones* in a Hilbert space E is a family of closed subsets $B_\lambda \subset E$ defined for all $\lambda > 0$ as follows

$$B_\lambda = \{\omega \mid \omega \in D(A),\ \|A\omega\| \le \lambda\|\omega\|\},$$

where $A : E \to E_1$ is a closed densely defined linear operator (E_1 is another Hilbert space) with the domain $D(A)$, the norms $\|A\omega\|$ and $\|\omega\|$ are taken in E_1 and E respectively.

Two such families $B_\lambda \subset E$ and $'B_\lambda \subset E'$ are called *equivalent* if there exist two bounded linear operators $f : E \to E'$, $g : E' \to E$ and positive constants $\lambda_0 > 0$, $C > 0$ such that for every $\lambda \in (0, \lambda_0)$

(i) $f(B_\lambda) \subset {}'B_{C\lambda}$, $\|\omega\| \le C\|f\omega\|$ if $\omega \in B_\lambda$;
(ii) $g({}'B_\lambda) \subset B_{C\lambda}$, $\|\omega'\| \le C\|g\omega'\|$ if $\omega' \in {}'B_\lambda$.

So in fact up to the equivalence only the germ of the family B_λ near 0 is important.

Definition 4. Let E be a Hilbert complex. Its *near-cohomology* $NH^k(E)$ of degree k is the equivalence class of the special family of quadric cones $B_\lambda^{(k)}$.

Theorem 1 means that the near-cohomology is a homotopy invariant of the Hilbert complex if the homotopy equivalence is defined as a chain homotopy equivalence with bounded morphisms and homotopy operators.

Actually the equivalence relation between the families of cones makes the near-cohomology independent on the choice of the space where all the cones belong. For instance if we have a special family $B_\lambda \subset E$, then for every Hilbert space E_1 this family is equivalent to the family $B_\lambda \oplus 0 \subset E \oplus E_1$. Indeed, the canonical inclusion $f : E \to E \oplus E_1$, $f(x) = x \oplus 0$, and projection $g : E \oplus E_1 \to E$, $g(x \oplus y) = x$, provide the desired equivalence.

The general definition might be useless if we do not introduce additional structures. One of the useful possibilities is the Γ-structure i.e. the structure of the Hilbert Γ-modules on all spaces E_k, such that all differentials commute with the action of Γ. More exactly if L_g denotes the (unitary) operator which gives the action of $g \in \Gamma$ in all spaces E_k then we should have $L_g d_k = d_k L_g$ for all k and all $g \in \Gamma$ (this means in particular that $L_g(D(d_k)) = D(d_k)$). If we have such a

Γ-structure on E then we shall say that E is a *Hilbert Γ-complex*. Now the definitions of morphisms, homotopy and homotopy equivalence have to be modified in the obvious way: we have to require that the morphism and homotopy operators commute with the action of Γ. If this is the case, then we shall call the corresponding objects *Γ-morphisms* (or *morphisms of Hilbert Γ-complexes*), *Γ-homotopy* etc. In a Hilbert Γ-complex we obviously have $L_g(B_\lambda^{(k)}) = B_\lambda^{(k)}$, so the family of cones $B_\lambda^{(k)}$ is Γ-invariant. The equivalence relation should be also understood in the restricted sense, so that only the use of Γ-morphisms is allowed. We shall denote the near-cohomology obtained as the Γ-equivalence classes of $B_\lambda^{(k)}$, by $NH^k(E,\Gamma)$.

DEFINITION 5. Let M be a Γ-manifold (possibly with a boundary) with a compact quotient $X = M/\Gamma$. Then the *near-cohomologies* of the pair (M,Γ) are defined as

$$NH^k(M,\Gamma) = NH^k(L^2\Lambda^\bullet(M),\Gamma).$$

We shall also denote them as $NH^k_\Gamma(X)$ if we want to consider them as invariants of X (then Γ should be quotient of $\pi_1(X)$ by a normal subgroup). Also we shall write $NH^k(X)$ instead of $NH^k_\Gamma(X)$ if $\Gamma = \pi_1(X)$ (and hence M is the universal covering of X).

THEOREM 2. *The near-cohomologies $NH^k(M,\Gamma)$ are homotopy invariants of the pair (M,Γ) in the homotopy category of Γ-manifolds (possibly with boundary) and Γ-maps. In particular $NH^k_\Gamma(X)$ are homotopy invariants of X.*

We will sketch a proof of this theorem. But we will start with some preparatory material.

The theorem will be proved if we check that (geometric) homotopy equivalence of (M,Γ) and (M',Γ) implies that the Hilbert Γ-complexes $L^2\Lambda^\bullet(M)$ and $L^2\Lambda^\bullet(M')$ are homotopy equivalent. Let us recall the classical construction that gives similar result for complexes of C^∞-forms. Denote $I = [0,1]$ and let $f_0, f_1 : M_1 \to M_2$ be two homotopic C^∞-maps of manifolds with boundaries. Let us denote the corresponding smooth homotopy by F, i.e. $F : M_1 \times I \to M_2$ is a C^∞-map such that $F(x,j) = f_j(x),\ j = 0,1$.

LEMMA 1 (Chain homotopy formula). *For every $\omega \in \Lambda^k(M_2)$*

$$f_1^*\omega - f_0^*\omega = (dT + Td)\omega,$$

where $f_j^* : \Lambda^k(M_2) \to \Lambda^k(M_1)$ are maps on forms induced by the maps f_j, and the homotopy operator $T : \Lambda^k(M_2) \to \Lambda^{k-1}(M_1)$ is defined by the formula

$$T\omega = \int_0^1 i_t^* i_{\partial/\partial t}(F^*\omega)\, dt.$$

Here $i_{\partial/\partial t}(F^*\omega)$ is the result of substitution of the vector field $\partial/\partial t$ to the form $F^*\omega$ at the first place (or inner multiplication of $F^*\omega$ by $\partial/\partial t$), i_t^* is induced by the map $i_t : M_1 \to M_1 \times I$, $i_t(x) = (x,t)$, so the integrand is a $(k-1)$-form on M_1 depending on $t \in I$.

PROOF. Calculating $dT\omega$, we find

$$dT\omega = \int_0^1 di_t^* i_{\partial/\partial t}(F^*\omega)\, dt = \int_0^1 i_t^* di_{\partial/\partial t}(F^*\omega)\, dt.$$

Similarly,

$$Td\omega = \int_0^1 i_t^* i_{\partial/\partial t}(F^* d\omega)\, dt = \int_0^1 i_t^* i_{\partial/\partial t} d(F^*\omega)\, dt.$$

Let us add these two formulas and use the Cartan formula

$$\mathcal{L}_Z = di_Z + i_Z d,$$

where Z is a vector field on a manifold, $\mathcal{L}_Z$ is the corresponding Lie derivative on the exterior forms. We obtain then

$$(dT + Td)\omega = \int_0^1 i_t^* \mathcal{L}_{\partial/\partial t}(F^*\omega)\, dt.$$

By the definition of the Lie derivative, taking $\mathcal{L}_{\partial/\partial t}(F^*\omega)$ means that we need to replace the coefficients of the form $F^*\omega$ by their partial derivatives with respect to t (in the product coordinates). Then j_t^* removes the terms with dt, and the integration in t exactly gives $f_1^*\omega - f_0^*\omega$ because $i_j^* F^*\omega = f_j^*$, $j = 0, 1$. □

It follows that $f_0^*\omega = f_1^*\omega \bmod \mathrm{Im}\, d$ for closed forms ω (i.e. forms ω such that $d\omega = 0$). Hence f_0^*, f_1^* define the same maps on the de Rham cohomologies. It follows also that the de Rham complexes of C^∞-forms $\Lambda^\bullet(M_1)$ and $\Lambda^\bullet(M_2)$ are chain homotopy equivalent if M_1 and M_2 are homotopy equivalent in the geometric sense.

Now if we want to apply this argument to the de Rham L^2-complexes we meet with the following basic difficulty : even for a C^∞-map $f : M_1 \to M_2$ of two compact manifolds, f^* can not be generally extended to a bounded linear map $f^* : L^2\Lambda^k(M_2) \to L^2\Lambda^k(M_1)$ due to possible local degenerations of f. A simple example: take $M_1 = M_2 = [0, 1]$ and $f(x) = x^2$. Then $\varphi(x) = x^{-1/3}$ is in L^2 but $(f^*\varphi)(x) = x^{-2/3}$ is not in L^2. So we can not use all C^∞-maps, but have to restrict ourselves to maps that do give bounded linear maps on L^2-forms. A suitable class of such maps is formed by *submersions* $f : M_1 \to M_2$ i.e. C^∞-maps such that $df_x : T_x M_1 \to T_{f(x)} M_2$ is surjective for every $x \in M_1$.

LEMMA 2. *Let M_1, M_2 be compact manifolds (possibly with boundaries), $f : M_1 \to M_2$ be a submersion. Then the induced map f^* extends to a bounded linear map*

$$f^* : L^2\Lambda^k(M_2) \to L^2\Lambda^k(M_1)$$

for every k.

The same statement holds for Γ-manifolds M_1, M_2 and a Γ-submersion f if we use Γ-invariant metrics on M_1, M_2 to define the spaces $L^2\Lambda^k(M_j)$, $j = 1, 2$.

PROOF. For compact M_1, M_2 the statement obviously reduces to a local one, i.e. it is sufficient to prove that for any $x_0 \in M_1$ there exists a neighborhood U of x_0 such that the statement holds if we replace M_1 by U. But locally any submersion is reduced (by a diffeomorphism) to the canonial projection map $\pi_{n,k} : I^n \to I^k$ with $n \geq k$, $I = [0, 1]$, and the statement is obvious for $\pi_{n,k}$.

The proof for Γ-manifolds and Γ-submersions is easily reduced to the compact case by use of a Γ-invariant partition of unity. □

EXERCISE 3. Let M_1, M_2 be compact C^∞-manifolds (possibly with boundary), and assume that $f : M_1 \to M_2$ is a C^∞-map. Then $f^* : C^\infty(M_2) \to C^\infty(M_1)$ extends to a bounded linear map $f^* : L^2(M_2) \to L^2(M_1)$ if and only if f is a

submersion. (As we know from Lemma 2, in this case, f^* extends also to a bounded linear map $f^* : L^2\Lambda^k(M_2) \to L^2\Lambda^k(M_1)$ for every k.)

Now the idea is to replace all maps by submersions up to a homotopy. But this is not always possible. For example, if $\dim M_1 < \dim M_2$ then there is no submersions $M_1 \to M_2$. So two steps are necessary. The first one is given by the following

LEMMA 3. *Let M be a Γ-manifold (possibly with boundary) with a compact quotient $X = M/\Gamma$. Let $B = B^N$ be the unit ball in $\mathbb{R}^N$ (with the center at 0). Then the Hilbert Γ-complexes $L^2\Lambda^\bullet(M)$ and $L^2\Lambda^\bullet(M \times B)$ are Γ-homotopy equivalent.*

PROOF. 1) Let $p : M \times B \to M$ denote the canonical projection. It induces bounded linear maps $p^* : L^2\Lambda^k(M) \to L^2\Lambda^k(M \times B)$ which define a morphism of complexes of Hilbert Γ-modules. Let us also define a map

$$J : L^2\Lambda^\bullet(M \times B) \to L^2\Lambda^\bullet(M)$$

by the formula

$$J\omega = \frac{1}{\operatorname{vol} B} \int_B (i_t^*\omega)\, dt$$

where $i_t : M \to M \times I$, $i_t(x) = (x,t)$, $t \in I$. This formula defines J on smooth forms, but it is easy to prove J is well defined and bounded in the spaces of L^2 forms. It is also a morphism of complexes of Hilbert Γ-modules. Indeed, the identity $dJ = Jd$ is easily checked on C^∞ forms with compact supports. By taking limits from such forms we see that it holds also for $\omega \in D(d_{\max})$ i.e. for $\omega \in L^2$ such that $d\omega \in L^2$. (This is in fact a local statement with respect to $x \in M$ and we can use appropriate mollifiers in local coordinates.)

2) Now we will start checking that the Γ-morphisms p^* and J give the desired homotopy equivalence. Obviously $J^*p^* = \mathrm{Id}$ on $L^2\Lambda^\bullet(M)$. So we have to construct a homotopy operator

$$T : L^2\Lambda^\bullet(M \times B) \longrightarrow L^2\Lambda^\bullet(M),$$

such that T is bounded and $\mathrm{Id} - p^*J^* = dT + Td$ on $D(d)$.

The idea which allows to construct T is as follows. The maps

$$p : M \times B \to M \quad \text{and} \quad i_t : M \to M \times B$$

provide a (Γ-)homotopy equivalence of $M \times B$ and M. So the induced operators

$$p^* : \Lambda^\bullet(M) \to \Lambda^\bullet(M \times B) \quad \text{and} \quad i_t^* : \Lambda^\bullet(M \times B) \to \Lambda^\bullet(M)$$

are chain homotopy equivalent. However the operator i_t^* is not bounded in the corresponding L^2 spaces. This is the reason why we need to replace it by J which is the average of i_t^* with respect to $t \in B$. The chain homotopy operator

$$T_t : \Lambda^\bullet(M \times B) \longrightarrow \Lambda^\bullet(M \times B),$$

such that $\mathrm{Id} - p^*i_t^* = dT_t + T_t d$, can be constructed as in Lemma 1, but generally it will not be bounded either. Note however that if

$$T = \frac{1}{\operatorname{vol} B} \int_B T_t dt$$

makes sense, then we formally have $\mathrm{Id} - p^*J = dT + Td$, and T has a chance to be bounded because of the integration. As we will see, this works if T_t is induced by

the simplest canonical geometric homotopy between the maps Id and $j_t = i_t p$ in $M \times B$. This canonical homotopy will be done by homotheties, which can be used because B is convex.

3) So let us take

$$F_t : M \times B \times I \to M \times B,$$
$$F_t(x, s; \tau) = (x, t + \tau(s - t)) = (x, (1 - \tau)t + \tau s),$$

where $x \in M$, $t, s \in B$, $\tau \in I = [0, 1]$, so τ is the homotopy parameter. So if x, t, τ are fixed, then $s \mapsto F_t(x, s; \tau)$ is a homothety of B with the center at t. Clearly $F_t(x, s, 0) = (x, t)$, $F_t(x, s, 1) = (x, s)$, so F_t is indeed a homotopy between j_t and Id on $M \times B$.

From the general chain homotopy formula (Lemma 1) we get

$$\mathrm{Id} - j_t^* = dT_t + T_t d$$

with T_t given by the formula

$$T_t \omega = \int_0^1 i_\tau^* i_{\partial/\partial\tau} F_t^* \omega d\tau, \quad \omega \in \Lambda^\bullet(M \times B).$$

Denote by T the average of T_t over $t \in B$ as above. Then the argument above shows that T indeed gives a chain homotopy between Jp^* and Id on $\Lambda^\bullet(M \times B)$.

4) Now we have to prove that T is bounded as a linear operator

$$T : L^2\Lambda^\bullet(M \times B) \to L^2\Lambda^\bullet(M \times B).$$

Explicitly writing $T\omega$ for $\omega \in \Lambda^\bullet(M \times B)$, we obtain

$$T\omega = \frac{1}{\mathrm{vol}\, B} \int_B dt \int_0^1 d\tau\, i_\tau^* i_{\partial/\partial\tau} F_t^* \omega = \frac{1}{\mathrm{vol}\, B} \int_{B \times I} i_\tau^* i_{\partial/\partial\tau} F_t^* \omega dt d\tau.$$

It is clear that T does not touch the variable $x \in M$, i.e. it has the form $T = I \otimes A$ with

$$A\alpha = \frac{1}{\mathrm{vol}\, B} \int_{B \times I} i_\tau^* i_{\partial/\partial\tau} G_t^* \alpha dt d\tau,$$

where $\alpha \in \Lambda^\bullet(B)$, and

$$G_t : B \times I \to B, \quad (s, \tau) \mapsto (1 - \tau)t + \tau s,$$

is just F_t with a fixed $x \in M$, so that $F_t = \mathrm{Id}_M \otimes G_t$. Therefore it is sufficient to prove that A is bounded in $L^2\Lambda^\bullet(B)$.

Since the bundle T^*B has a natural trivialization over B, we can identify each fiber of $\Lambda^\bullet T^*B$ with a standard fiber and write A in the form

$$A\alpha(s) = \int_0^1 d\tau \int_B dt\, L\alpha((1 - \tau)t + \tau s),$$

where $L = i_{\partial/\partial\tau}$ is a vector bundle homomorphism.

Now let us split the integral in τ above into the sum of integrals over $[0, 1/2]$ and $[1/2, 1]$, and take the corresponding splitiing $A = A_0 + A_1$ of A.

If $\tau \in [1/2, 1]$, then the operators

$$\alpha(\cdot) \mapsto L\alpha((1 - \tau)t + \tau\cdot)$$

are bounded in $L^2\Lambda^\bullet(B)$ uniforlmly in t, τ. Therefore A_1 is bounded in $L^2\Lambda^\bullet(B)$.

If $\tau \in [0, 1/2]$ then the operators

$$\alpha(\cdot) \mapsto \int_B L\alpha((1 - \tau)t + \tau\cdot)dt$$

are bounded in $L^2\Lambda^\bullet(B)$ uniformly in τ by the Cauchy-Schwarz inequality. Therefore A_0 is also bounded in $L^2\Lambda^\bullet(B)$.

5) We have proved that T is bounded in $L^2\Lambda^\bullet(M\times B)$. The identity $\mathrm{Id}-p^*J^* = dT+Td$ was established on smooth forms but it is also true on $D(d)$ which is easily proved by a mollifiers approximation. □

LEMMA 4. *For every compact C^∞-manifold X (possibly with a boundary) there exists an integer $N>0$ and a submersion $s=s_{x_0}: B^N\to X$ with $s(0)=x_0\in \mathrm{Int}\, X$, such that s smoothly depends on x_0.*

SKETCH OF THE PROOF. We can imbed X into $\mathbb{R}^N$ and take the orthogonal projection of a small ball $B(x_0,r(x_0))$ in $\mathbb{R}^N$ centered at x_0 with the radius $r(x_0)$ to a neigbourhood of x_0 in X, composing it with an affine map of B^N onto $B(x_0,r(x_0))$ ($r(x_0)$ and the affine map can be chosen smoothly depending on $x_0\in \mathrm{Int}\, X$). This gives the desired submersion. □

LEMMA 5. *Let M_1, M_2 be two Γ-manifolds (with boundaries) with compact quotients $X_j=M_j/\Gamma$, $j=1,2$, and $f: M_1\to \mathrm{Int}\, M_2$ be a smooth Γ-map. Then there is an integer $N>0$ and a smooth Γ-map $F: M_1\times B^N\to \mathrm{Int}\, M_2$ such that F is a submersion and $f=F\circ i$, where $i: M_1\to M_1\times B^N$ is the natural inclusion, i.e. $i(x)=(x,0)$, $x\in M_1$.*

SKETCH OF THE PROOF. Let us use notations $\pi_j: M_j\to X_j$ for the natural projections, $j=1,2$. Due to Lemma 2 we can choose a submersion $s: X_1\times B^N\to X_2$ such that $s(x,0)=\pi_2 f(\tilde{x})$, $x\in X$, $\tilde{x}\in M_1, \pi_1(\tilde{x})=x$. Then there exists a unique smooth map $F: M_1\times B^N\to M_2$ such that $F\circ i=f$ and $\pi_2\circ F = s\circ(\pi_1\times \mathrm{Id})$ i.e. the diagram

$$\begin{array}{ccccc} M_1 & \xrightarrow{\ i\ } & M_1\times B^N & \xrightarrow{\pi_1\times \mathrm{Id}} & X_1\times B^N \\ & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle s} \\ & & M_2 & \xrightarrow{\ \pi_2\ } & X_2 \end{array}$$

is commutative. It is easy to check that F satisifies all the conditions. □

SKETCH OF THE PROOF OF THEOREM 2. So a smooth Γ-map $f: M_1\to M_2$ can be replaced by a Γ-submersion $F: M_1\times B^N\to M_2$ which does not homotopically differ from f. The same can be done with Γ-homotopies. Using Lemma 1 then we can produce a homotopy equivalence of the Hilbert Γ-complexes $L^2\Lambda^\bullet(M_1)$ and $L^2\Lambda^\bullet(M_2)$ from a homotopy Γ-equivalence of M_1 and M_2 which will prove the Theorem. □

REMARK. Using a representation of a Γ-manifold M (with boundary) as a CW Γ-complex (with the quotient $X=M/\Gamma$ which is a finite CW-complex) we can consider the combinatorial Γ-complex of L^2-cochains $L^2C^\bullet(M)$. Then using the Sobolev forms (instead of L^2-forms) and duality arguments based on the Whitney integration technique one can prove that the Hilbert Γ-complexes $L^2\Lambda^\bullet(M)$ and $L^2C^\bullet(M)$ are homotopically equivalent (A. Efremov [**Ef**]).

4.5. Variational principle for the spectrum distribution function

The classical variational principle (Courant) for eigenvalues of self-adjoint operators plays an important role in Mathematical Physics. We shall recall it in a more convenient form of the Glazman Lemma, for simplicity restricting ourselves

to the finite dimensional case, and then switching to a much more general situation of the von Neumann algebra with a trace. The resulting variational principle will be important when we shall pass from near-cohomologies to simpler invariants.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ be a linear self-adjoint operator, $\dim \mathcal{H} = N$ and $\lambda_1, \dots, \lambda_N$ are all eigenvalues of A (repeated according to their multiplicities). Define the distribution function of the eigenvalues of A by

$$N(\lambda) = N(\lambda; A) = \mathrm{Card}\{j \mid \lambda_j \leq \lambda\}$$

(the number of eigenvalues which are less or equal than λ). Then the Glazman Lemma in this situation looks as follows:

$$N(\lambda) = \sup\{\dim L \mid L \text{ is a linear subspace in } \mathcal{H},\ A(x,x) \leq \lambda(x,x) \text{ for every } x \in L\}$$

Exercise 1. (i) Prove this formula.

(ii) Prove the same formula for a self-adjoint operator with a discrete spectrum in a (infinite-dimensional) separable Hilbert space, i.e. such a self-adjoint operator A that it has an orthonormal basis of eigenvectors $\{\psi_j \mid j = 1, 2, \dots\}$ with eigenvalues $\{\lambda_j \mid j = 1, 2, \dots\}$ such that $\lambda_j \to +\infty$ as $j \to \infty$.

Now we shall pass to the generalization to the von Neumann algebra situation.

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a von Neumann algebra in $\mathcal{H}$, τ a (faithful, normal, semifinite) trace on $\mathcal{M}$, $\dim_\tau$ the corresponding dimension function. Now let A be a self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}$; in particular if $A = \int \lambda \, dE_\lambda$ is the spectral decomposition of A, then $E_\lambda \in \mathcal{M}$.

Definition 1. $N_{\mathcal{M}}(\lambda) = N_{\mathcal{M}}(\lambda; A) = \tau(E_\lambda)$.

Obviously $N_{\mathcal{M}}$ is a non-decreasing function on $\mathbb{R}$ with values in $[0, \infty]$.

Exercise 2. Prove that if $N_{\mathcal{M}}(\lambda_0) < \infty$ for some $\lambda_0 \in \mathbb{R}$ then the following is true:

(i) $N_{\mathcal{M}}(\lambda + 0) = N_{\mathcal{M}}(\lambda)$ if $\lambda < \lambda_0$ (i.e. $N_{\mathcal{M}}$ is right continuous on $(-\infty, \lambda_0)$);
(ii) $N_{\mathcal{M}}(-\infty) = 0$ i.e. $\lim_{\lambda \to -\infty} N_{\mathcal{M}}(\lambda) = 0$
(iii) the spectrum of $\sigma(A)$ on $(-\infty, \lambda_0)$ coincides with the set of all points of growth of $N_{\mathcal{M}}$ i.e.

$$\sigma(A) \cap (-\infty, \lambda_0) = \{\lambda \mid \lambda \in (-\infty, \lambda_0), N_{\mathcal{M}}(\lambda + \epsilon) - N_{\mathcal{M}}(\lambda - \epsilon) > 0 \text{ for every } \epsilon > 0\}$$

Theorem 1 (Variational Principle or Glazman Lemma).

$$N_{\mathcal{M}}(\lambda) = \sup\{\dim_\tau L \mid L \eta \mathcal{M}, L \subset D(A), (Ax, x) \leq \lambda(x,x) \textit{ for every } x \in L\}$$
$$= \sup\{\tau(P) \mid P \in \mathrm{Proj}(\mathcal{M}), P\mathcal{H} \subset D(A), P(A - \lambda I)P \leq 0\}.$$

Proof. 1) Consider first $E_{\lambda,\mu} = E_\lambda - E_\mu$, $\mu < \lambda$. Then $E_{\lambda,\mu} \in \mathrm{Proj}(\mathcal{M})$, $E_{\lambda,\mu}\mathcal{H} \subset D(A)$ and $E_{\lambda,\mu} \nearrow E_\lambda$ as $\mu \to -\infty$. Hence

$$N_{\mathcal{M}}(\lambda) = \tau(E_\lambda) = \lim_{\mu \to -\infty} \tau(E_{\lambda,\mu}).$$

Now obviously $E_{\lambda,\mu}(A - \lambda I)E_{\lambda,\mu} \leq 0$, hence

$$N_{\mathcal{M}}(\lambda) \leq \sup\{\tau(P) \mid P \in \mathrm{Proj}(\mathcal{M}), P\mathcal{H} \subset D(A),\ P(A - \lambda I)P \leq 0\}$$

2) To prove the inverse inequality we have to check that if $P \in \mathrm{Proj}(\mathcal{M})$, $P\mathcal{H} \subset D(A)$ and $P(A - \lambda I)P \leq 0$ then $\tau(P) \leq N_{\mathcal{M}}(\lambda) = \tau(E_\lambda)$. First note that $P\mathcal{H} \cap$

$(I - E_\lambda)\mathcal{H} = \{0\}$ because if $x \in P\mathcal{H}$ then $x \in D(A)$ and $(Ax, x) \le \lambda(x, x)$, but if $x \in D(A)$, $x \neq 0$ and $x \in (I - E_\lambda)\mathcal{H}$ then $(Ax, x) > \lambda(x, x)$. Now consider the operator $B = E_\lambda P : P\mathcal{H} \to E_\lambda \mathcal{H}$. Obviously $\operatorname{Ker} B = \{0\}$ so B defines an almost isomorphism of $P\mathcal{H}$ to a subspace in $E_\lambda \mathcal{H}$. Hence $\dim_\tau(P\mathcal{H}) \le \dim_\tau(E_\lambda \mathcal{H})$ or $\tau(P) \le \tau(E_\lambda) = N_{\mathcal{M}}(\lambda)$ as required. □

Corollary 1. *Let A_1, A_2 be two self-adjoint operators affiliated with $\mathcal{M}$, $D(A_2) \subset D(A_1)$ and $A_1 \le A_2$ (i.e. $(A_1, x, x) \le (A_2, x, x)$ for every $x \in D(A_2)$). Then $N_{\mathcal{M}}(\lambda; A_1) \ge N_{\mathcal{M}}(\lambda; A_2)$ for every $\lambda \in \mathbb{R}$.*

Proof. If $L\eta\mathcal{M}$, $L \subset D(A_2)$ and $(A_2 x, x) \le \lambda(x, x)$ for every $x \in L$, then the same is true if we replace A_2 by A_1. Hence we have a wider choice of subspaces L for A_1 than for A_2 and the desired inequality follows from the variational principle. □

The variational principle has a lot of applications in Spectral Theory but we shall not write about them now. Instead we shall use it as a heuristic hint to introduce real valued characteristics arising from near-cohomology.

Let

$$E : 0 \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \to E_k \xrightarrow{d_k} E_{k+1} \to \ldots \xrightarrow{d_{N-1}} E_N \to 0$$

be a Hilbert Γ-complex. Recall the notation

$$B_\lambda^{(k)} = \{\omega \mid \omega \in E_k / \operatorname{Ker} d_k,\ \|d_k \omega\| \le \lambda \|\omega\|_{\operatorname{mod} \operatorname{Ker} d}\}.$$

Definition 2. Let $F_k(\lambda) = \sup\{\dim_\Gamma L \mid L \subset B_\lambda^{(k)}\}$. It is understood of course that L should be closed and Γ-invariant (otherwise $\dim_\Gamma L$ does not make sense). Note that $E_k / \operatorname{Ker} d_k$ is a projective Hilbert Γ-module since $\operatorname{Ker} d_k$ is Γ-invariant. So $F_k(\lambda)$ is in fact the spectrum distribution function of $d_k^* d_k$ on $E_k / \operatorname{Ker} d_k$.

Theorem 2. *Let E, E' be homotopy equivalent Hilbert Γ-complexes, F_k, F_k' corresponding functions. Then there exist $C > 0, \lambda_0 > 0$ such that for every $k = 0, 1, \ldots, N$*

$$F_k(C^{-1}\lambda) \le F_k'(\lambda) \le F_k(C\lambda), \quad \lambda \in (0, \lambda_0).$$

Proof. Proof is obvious due to the homotopy invariance of the near-cohomologies. □

Corollary 2. *Let M, M' be two Γ-homotopy equivalent Γ-manifolds, possibly with boundaries, with compact quotients $X = M/\Gamma$ and $X' = M'/\Gamma$. Let F_k, F_k' be functions defined by Hilbert complexes $L^2\Lambda^\bullet(M)$, $L^2\Lambda^\bullet(M')$. Then there exist $C > 0, \lambda_0 > 0$ such that*

$$F_k(C^{-1}\lambda) \le F_k'(\lambda) \le F_k(C\lambda), \quad \lambda \in (0, \lambda_0).$$

Let us write $F_k \overset{d}{\sim} F_k'$ near 0 (or say that F_k' is *dilatationally equivalent* to F_k near 0) if these inequalities are fulfilled. It follows that the dilatation equivalence class of F_k is a homotopy invariants of the Γ-manifold M (or a homotopy invariant of $X = M/\Gamma$ if $\Gamma = \pi_1(X)$ and M is the universal covering of X).

Even simpler invariant can be produced if we compare $F_k(\lambda)$ with λ^β near 0. If $M^{-1}\lambda^{\beta_k} \le F_k(\lambda) \le M\lambda^{\beta_k}$ near 0 (with a positive constant M) then

$$\beta_k = \lim_{\lambda \downarrow 0} \frac{\log F_k(\lambda)}{\log \lambda}.$$

It is not clear whether the limit exists in general, so we can put e.g.

$$\beta_k = \liminf_{\lambda \downarrow 0} \frac{\log F_k(\lambda)}{\log \lambda}$$

If β_k' is defined in the same way by F_k' and $F_k' \overset{d}{\sim} F_k$ near 0, then obviously $\beta_k' = \beta_k$. So we have

COROLLARY 3. *β_k is a homotopy invariant of the Γ-manifold M (or of non-simply connected manifold $X = M/\Gamma$ if $\Gamma = \pi_1(X)$).*

Now let us return to the case when M has no boundary and X is a closed compact manifold. We shall briefly sketch a description of the connection of the spectra of the Laplacians on $L^2\Lambda^k(M)$ and the functions $F_k(\lambda)$. Let $\Delta_k = d\delta + \delta d$ denote the Laplacian on $\Lambda^k(M)$. This is an elliptic Γ-invariant operator (the Riemannian metric on M is supposed to be Γ-invariant) with a positive principal symbol. Hence its spectral projections $E_\lambda^{(k)}$ are in Γ-trace class (they have C^∞ Schwartz kernels because all operators of the form $\Delta_k^p E_\lambda^{(k)} \Delta_k^q$, $p, q \in \mathbb{Z}_+$, are bounded). Hence the following spectrum distribution function is finite for every $\lambda \in \mathbb{R}$:

$$N_k(\lambda) = \operatorname{Tr}_\Gamma E_{\lambda^2}^{(k)} \text{ if } \lambda \geq 0 \text{ and } 0 \text{ otherwise.}$$

Now due to the variational principle

$$N_k(\lambda) = \sup\{\dim_\Gamma L \mid L \subset D(\Delta_k),\ \|d\omega\|^2 + \|\delta\omega\|^2 \leq \lambda^2 \|\omega\|^2 \text{ if } \omega \in L\}.$$

Let us use the Hodge-Kodaira decomposition

$$L^2\Lambda^k(M) = \overline{d\Lambda_c^{k-1}(M)} \oplus \mathcal{H}^k(M) \oplus \overline{\delta\Lambda_c^{k+1}(M)}.$$

Since all three subspaces in this decomposition are Δ_k-invariant, the spectrum distribution function $N_k(\lambda)$ decomposes into the sum of the spectrum distribution functions of Δ_k on the subspaces:

$$N_k(\lambda) = G_k(\lambda) \oplus \bar{b}_k \oplus F_k(\lambda), \quad \lambda > 0.$$

Here $\bar{b}_k$ is the Γ-Betti number (and the spectrum distribution function of Δ_k on $\mathcal{H}^k(M)$ is equal $\bar{b}_k$ if $\lambda \geq 0$ and 0 otherwise), F_k is exactly the function that was introduced earlier, because $\Delta_k = \delta d$ on $\delta\Lambda_c^{k+1}(M)$, and

$$G_k(\lambda) = \sup\{\dim_\Gamma L \mid L \subset D(\Delta_k) \cap \overline{d\Lambda_c^{k-1}(M)},\ \|\delta w\| \leq \lambda \|w\| \text{ if } w \in L\}$$

or, in other words, $\lambda \mapsto G_k(\sqrt{\lambda})$ for positive λ is the spectrum distribution function of the self-adjoint operator $\bar{d}\bar{\delta}$ on the Hilbert Γ-module $\overline{d\Lambda_c^{k-1}(M)}$, i.e. the Γ-trace of its spectral projection.

Note that it follows from these arguments that $F_k(\lambda) < \infty$ for all $\lambda > 0$, since $F_k(\lambda) \leq N_k(\lambda)$. This finiteness result can be obtained for Γ-manifolds M with boundary also (provided the quotient $X = M/\Gamma$ is compact, see [LL]).

Now it occurs that in fact the function G_k contains nothing new because

$$G_k(\lambda) = F_{k-1}(\lambda), \quad \lambda \geq 0,$$

for all $k = 0, 1, \ldots, n$, where $n = \dim M$ and by definition $F_{-1}(\lambda) \equiv 0$.

The reason of this relation is as follows. Denote d_k, δ_{k-1} restrictions of d, δ on k-forms (so d_k and δ_k are formally adjoint to each other). Then $\Delta_{k+1} d_k = d_k \Delta_k$, hence formally $d_k(\Delta_k - \mu)^{-1} = (\Delta_{k+1} - \mu)^{-1} d_k$. Since the spectral projections are strong limits of the resolvent operators we obtain $E_{\lambda^2}^{(k+1)} d_k = d_k E_{\lambda^2}^{(k)}$ on $D(d_k)$.

This equality gives us non-trivial information only on the subspace $(\operatorname{Ker}\bar{d}_k)^\perp = \overline{\delta\Lambda_c^{k+1}(M)}$ and there $\bar{d}_k$ has no kernel. Hence it gives an operator similarity of $E^{(k)}_{\lambda^2} \mid \overline{\delta\Lambda_c^{k+1}(M)}$ and $E^{(k+1)}_{\lambda^2} \mid \overline{d\Lambda_c^k(M)}$. It is natural to expect then (and this can be proved rigorously by a use of the polar decomposition of $\bar{d}$) that the traces of the operators $E^{(k)}_{\lambda^2} \mid \overline{\delta\Lambda_c^{k+1}(M)}$ and $E^{(k+1)}_{\lambda^2} \mid \overline{d\Lambda_c^k(M)}$ coincide, which means that $F_k(\lambda = G_{k+1}(\lambda)$ as required.

Now we see that

$$N_k(\lambda) = \bar{b}_k + F_{k-1}(\lambda) + F_k(\lambda).$$

In particular it follows that the following Theorem is true:

THEOREM 3 (M.Gromov and M.Shubin). *Let M, M' be Γ-manifolds with compact closed quotients $X = M/\Gamma, X' = M'/\Gamma$, and let M, M' are Γ-homotopy equivalent. Let N_k, N'_k be the spectrum distribution functions of the Laplacians on k-forms on M, M'. Then $N'_k \overset{d}{\sim} N_k$ near 0.*

COROLLARY 4. *Let $\alpha_k = \liminf\limits_{\lambda\downarrow 0} \dfrac{\log(N_k(\lambda) - \bar{b}_k)}{\log\lambda}$. Then α_k is a homotopy invariant of (M,Γ) or of $X = M/\Gamma$ (if $\Gamma = \pi_1(X)$).*

In fact it is easy to prove from the decomposition of N_k that

$$\alpha_k = \min(\beta_{k-1}, \beta_k).$$

The numbers α_k can be also obtained from the large time behaviour of heat kernels. Namely, if $\theta_k(t) = \operatorname{Tr}_\Gamma e^{-t\Delta_k}$ then $\theta_k(t) = \int e^{-t\lambda^2} dN_k(\lambda)$, so θ_k is the Laplace transform of the measure dN_k up to a trivial change of variable. If N'_k, θ'_k are obtained similarly from M', and M' is Γ-homotopy equivalent to M, then $N'_k \overset{d}{\sim} N_k$ near 0, which implies that $\theta'_k \overset{d}{\sim} \theta_k$ near infinity i.e. there exist $t_0 > 0, C > 0$ such that

$$\theta_k(Ct) \le \theta'_k(t) \le \theta_k(C^{-1}t) \text{ if } t \ge t_0.$$

It is easy to prove that $\lim_{t\to+\infty} \theta_k(t) = \bar{b}_k$ and the numbers α_k can be expressed in terms of θ_k as follows:

$$\alpha_k = 2 \liminf_{t\to+\infty} \frac{-\log(\theta_k(t) - \bar{b}_k)}{\log t}.$$

Actually this gives the most convenient way of calculating α_k. Let us give two examples where such calculation can be done.

EXAMPLE 1. If $M = \mathbb{R}^n$ and Γ is a lattice in $\mathbb{R}^n$ (e.g. $\mathbb{Z}^n$) then Δ_k is the direct sum of the scalar Laplacians. Using the standard formulas for the heat kernels we obtain easily that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = n$. It follows that $\beta_0 = \beta_1 = \cdots = \beta_{n-1} = n$.

(Note that $\bar{b}_0 = \bar{b}_1 = \cdots = \bar{b}_n = 0$ in this case)

EXAMPLE 2. (a) Let $M = \mathbb{H}^{2n}$ (the $2n$-dimensional hyperbolic space), Γ a discrete group of its isometries. Then $\bar{b}_k = 0$ if $k \ne n$; $\bar{b}_n = (-1)^n\chi(M/\Gamma) > 0$. Also $\alpha_k = \beta_k = \infty$ for all k since the spectrum of Δ_k always has a gap near 0 (i.e. there exists $\epsilon > 0$ such that $\sigma(\Delta_k) \cap (0,\epsilon) = 0$).

(b) Let $M = \mathbb{H}^{2n+1}$. Then $\bar{b}_k = 0$ for all $k = 0, 1, \ldots, 2n+1$. But here the spectrum of Δ_k has no gap near 0 if $k = n, n+1$. So $\alpha_k = \infty$ if $k \ne n, n+1$. A

calculation of heat kernels shows that $\alpha_n = \alpha_{n+1} = 1$. It follows that $\beta_k = \infty$ if $k \neq n$ and $\beta_n = 1$.