Preface

Von Neumann algebras and von Neumann dimensions are among the most beautiful and useful inventions of mathematics of the 20th century.

Let us recall that the dimension of a finite-dimensional linear space in Linear Algebra can be $0, 1, 2, \ldots$. It is used to compare "size" of different subspaces, count the number of linearly independent solutions of systems of linear equations etc.

In 1929 J. von Neumann [**Ne1**] initiated a new direction in Functional Analysis, which was developed in his joint papers with F.J. Murray in the mid 1930th [**MN**], and in his book [**Ne2**]. One of the important features of the theory is that it is possible to measure some infinite-dimensional subspaces in a Hilbert space, assigning to them dimensions which are arbitrary real (and not necessarily integer) numbers. For example the von Neumann dimension of a subspace can be equal to $\sqrt{2}$, $\pi/4$ etc. Besides these dimensions have all properties of the usual dimension (which can only take values $0, 1, 2, \ldots$ and $+\infty$). The only difference is that a von Neumann dimension does not measure *all* (closed) subspaces, but only some special class of them: the ones which are affiliated with a special algebra of operators (i.e. such that the corresponding orthogonal projection belongs to the algebra). The most appropriate algebras of operators were the main object of study in the papers quoted above, and they are now called the *von Neumann algebras* or *W^*-algebras* (though in some terminology these two terms refer to slightly different objects).

Von Neumann dimensions and von Neumann algebras proved to be quite useful in Analysis (including Complex Analysis), Geometry, Topology and Statistical Mechanics, where appropriate infinite-dimensional spaces arise as spectral subspaces of periodic or almost-periodic differential operators (which usually have continuous spectrum), the spaces of harmonic forms on covering manifolds etc. Recently an extensive use of these ideas was made in the theory of L^2-invariants of manifolds and more general CW-complexes.

Let us discuss some applications more specifically.

Index theories of non-Fredholm operators

A generalized index of an operator A is usually defined by the formula

$$\text{"ind"}A = \text{"dim"}\operatorname{Ker} A - \text{"dim"}\operatorname{Ker} A^*,$$

where A^* is the adjoint operator to A, "dim" is a generalized dimension. Here $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ can be infinite-dimensional in the usual sense but finite-dimensional with respect to "dim". So "ind"A is a real number (and not necessarily an integer).

The papers by L. Coburn, R. Douglas, D. Schaeffer, I. Singer [**CDSS**] and L. Coburn, R. Moyer, I. Singer [**CMS**] presented first examples of such index theories for almost periodic Wiener-Hopf operators or pseudo-differential operators.

Another index theory due to M. Atiyah [**A**] treats Γ-invariant pseudo-differential operators on a regular Γ-covering M of a compact manifold. Here Γ is a discrete group which acts freely on M with a discrete quotient M/Γ. We have chosen this analytic context to develop several applications of von Neumann algebras techniques in detail. In particular, we will provide a detailed proof of the Atiyah index theorem. An important application of this theorem is a construction of the discrete series representations of semi-simple Lie groups which is due to M. Atiyah and W. Schmid [**AS**].

Spectral theory

The von Neumann dimensions can be used in spectral theory to define a spectrum distribution function of some self-adjoint operators with a continuous spectrum. Recall that the usual spectrum distribution function $N(\lambda) = N(\lambda; A)$ of a semi-bounded below self-adjoint operator A with a discrete spectrum is defined as the number of elgenvalues which are $\leq \lambda$ (multiplicities counted). If we denote by L_λ the linear span of all eigenvectors with eigenvalues $\leq \lambda$, then $N(\lambda) = \dim L_\lambda$. But L_λ makes sense in a more general context. It can be defined for an arbitrary self-adjoint operator as the image of its spectral projection E_λ which enters into the spectral decomposition

$$A = \int \lambda dE_\lambda.$$

Then under appropriate conditions we can define a generalized spectrum distribution function by the formula

$$\text{“}N(\lambda)\text{”} = \text{“}\dim\text{”} L_\lambda.$$

A function which we get in such a way for periodic or almost periodic operators in $\mathbb{R}^n$ coincides with so called integrated density of states which is well known and useful in physics of solids and disordered media (see [**Sh4, Sh5**]). Such a function can be also defined for the self-adjoint Γ-invariant elliptic operators on regular coverings (in the context described above). In particular, semi-classical asymptotics of this function [**Sh7**] can be established in a context which generalizes the one that proved to be very useful in topology beginning with a remarkable paper by E. Witten [**Wi**]. Under appropriate conditions these asymptotics imply the existence of arbitrarily many gaps in the spectrum of an elliptic Γ-periodic operator.

L^2 Betti numbers

Let X be a compact closed manifold, M its regular covering, so that $X = M/\Gamma$ where Γ is a discrete group which acts freely on M. In particular, M can be the universal cover of X, then $\Gamma = \pi_1(X)$ (the fundamental group of X) acting on M by deck transformations. In this case M. Atiyah [**A**] introduced L^2 Betti numbers $b_p^{(2)}(X)$ as von Neumann dimensions of the spaces of L^2 harmonic forms of degree p on M. Here L^2 harmonic forms are solutions of the equation $\Delta_p\omega = 0$ where $\Delta_p = dd^* + d^*d$ is the Laplacian on p-forms on M, taken with a Γ-invariant Riemannian metric. The numbers $b_p^{(2)}(X)$ depend on the choice of the regular covering manifold M, but for the sake of simplicity we skip this dependence in the notation of these numbers. A particular case of the Atiyah index theorem is that

$$\sum_{p=0}^{\dim X} (-1)^p b_p^{(2)}(X) = \chi(X),$$

where $\chi(X)$ is the Euler characteristic of X.

It was proved by J. Dodziuk [**Do**] that the L^2 Betti numbers are homotopy invariants.

It occurs that in the Morse inequalities for Morse functions on X the usual Betti numbers can be replaced by L^2 Betti numbers (see [**NS1**] and an analytic proof in [**Sh7**]). Sometimes these generalized Morse inequalities give stronger result than the standard ones.

Spectra near zero invariants

The Laplacians Δ_p on infinite regular coverings M of a compact manifold X often have a continuous spectrum with 0 as its accumulation point. If we introduce $N_p(\lambda)$ as the von Neumann spectrum distribution function of Δ_p, then this means that $N_p(\lambda)$ can be non-constant on $(0, \epsilon)$ for any $\epsilon > 0$. It can be proved that $N_p(+0) = b_p^{(2)}(X)$ and the function $\lambda \mapsto N_p(\lambda)$ is increasing. So the previous condition can be reformulated by saying that $N_p(\lambda) - b_p^{(2)}(X) > 0$ for all $\lambda > 0$. It was noted in [**NS1, NS2, NS3**] that the spectra near zero carry a topological information. In particular, the numbers

$$\alpha_p = \liminf_{\lambda \to +0} \frac{\log(N_p(\lambda) - b_p^{(2)}(X))}{\log \lambda}$$

do not depend on the choice of the Riemannian metric on X. (They are mostly called Novikov-Shubin invariants now.) It was proved in [**GS1**] that in fact they are even homotopy invariants. So these numbers provide homotopy invariants of X which are defined in terms of the behavior of the von Neumann spectrum distribution function of Δ_p near 0, the phenomenon which can be non-trivial only due to the non-compactness of M, even though the result is a topological invariant of X.

Prerequisites

The necessary prerequisites for Chapters 1-3 are a subset of the most standard material from Analysis, Algebra and Topology. I tried to make the book as self-contained as possible. In particular, many facts from Functional Analysis, which are not entirely standard, are provided with detailed proofs or sketches of the proof. The same holds for pseudo-differential operators and Sobolev spaces. Some relevant facts are formulated as exercises which can help the reader to master the material.

The difficulty of the material more or less increases with the number of the Chapter. Beginning with Chapter 4 the reader will find considerably less details provided, and I refer the reader to original sources if more details are needed.

Apology

Unfortunately it is certainly impossible to cover all the material connected with any sufficiently developed topic in a book of a reasonable volume. Therefore a lot of important material which is closely related with the topic of this book, is nevertheless left out of it. As usual the choice of the material is connected with the personal taste of the author. If I could extend the volume somewhat more, my personal choice would be to include results on torsion-type invariants and foliations. But even this, still very limited extension would require another book.

However I feel that the mission of a mathematical book is not to cover a huge amount of material, but to teach the readers an important and beautiful mathematics. Therefore I tried hard to enable this book for achieving this goal.

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