GEOMETRY

Lectures in Simple English by A.B.Sossinsky (IUM, Fall 2003)

Lecture 1: SYMMETRY GROUPS and GEOMETRIES

§1.1. What is a geometry?

This course is called "Geometry". But geometry is *not* the object of any university course in mathematics. Geometry is *not* even a branch of mathematics. For example, "plane geometry" (studied in school) is formally a part of *linear algebra*: the object of plane geometry is the two-dimensional linear space over $\mathbb{R}$ supplied with a positive definite quadratic form.

Many mathematicians believe that geometry does not have an object, that *geometry is just a way of thinking, a way of looking at mathematics.*

Other mathematicians will tell you that there are many geometries, that *each geometry is a set with a transformation group acting on it* (=множество с действием группы преобразований на нем). They will say, for example, that "plane geometry is nothing but (=ничто иное как) the set $\mathbb{R} \times \mathbb{R}$ of pairs of real numbers with the group $GL(2, \mathbb{R})$ acting on it (*real affine plane geometry*) or the group $O(2)$ acting on that set (*Euclidean plane geometry* (=*планиметрия*))".

In this lecture we will formalize the notion of geometry as *a set X with a transformation group G acting on it* (denoted $(X : G)$) and give many examples of such actions. This (algebraic!) notion is one of the most important in geometry, mathematics, physics, and the natural sciences. It expresses (mathematically) the fundamental idea of *symmetry.*

In general, the notions (=definitions), theorems, and proofs studied in this course all come from algebra. But we will be studying them from the geometric point of view. We will see that geometries are not separate objects: geometries form a *category,* two geometries can be *equivalent,* a geometry can be included into another or projected onto some other geometry. The corresponding abstract definitions will be difficult to understand at first. But the aim of this course is to

(1) teach you to understand and use modern abstract mathematics by giving a visual, intuitive, geometric meaning to abstract notions and arguments;

(2) develop your geometric intuition while giving simple and rigorous formal definitions to visual geometric images.

§1.2. Symmetries of some figures

(1) *Symmetries of the square.* Consider all the isometries of the unit square $\Box = ABCD$, i.e., all the distance-preserving mappings (сохраняющие расстояния отображения) of the square to itself. Denote by s_H, s_V, and s_{ac}, s_{bd} the line symmetries in the horizontal and vertical mid-lines, and in the diagonals AC, BD, respectively. Denote by r_0, r_1, r_2, r_3 the rotations about the center of the square by 0, 90, 180, 270 degrees, respectively. These eight transformations are all called *symmetries* of the square and denoted

$$\mathrm{Sym}(\Box) = \{r_0, r_1, r_2, r_3, s_H, s_V, s_{ac}, s_{bd}\}.$$

The set $\mathrm{Sym}(\Box)$ is closed under the operation of *composition* (also called *multiplication*), i.e., the composition of two symmetries is a symmetry. For example, $s_{ac} * s_v = r_1$, $s_V * s_V = r_0$, $r_1 * r_2 = r_3$; here if we write, say, $s_{ac} * r_1$, then s_{ac}

is performed (=выполняется) first, and r_1 is performed second (this is important, because composition is not commutative). Other results appear in the following *multiplication table*:

$*$	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_0	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_1	r_1	r_2	r_3	r_0	s_{ac}	s_{bd}	s_V	s_H
r_2	r_2	r_3	r_0	r_1	s_V	s_H	s_{bd}	s_{ac}
r_3	r_3	r_0	r_1	r_2	s_{bd}	s_{ac}	s_H	s_V
s_H	s_H	s_{bd}	s_V	s_{ac}	r_0	r_2	r_3	r_1
s_V	s_V	s_{ac}	s_H	s_{bd}	r_2	r_0	r_1	r_3
s_{ac}	s_{ac}	s_H	s_{bd}	s_V	r_1	r_3	r_0	r_2
s_{bd}	s_{bd}	s_V	s_{ac}	s_H	r_3	r_1	r_2	r_0

Here (for example) the element s_V at the intersection of the sixth column and the fourth row is the composition of r_2 and s_H in that order, $s_H = r_2 * s_V$ (first the transformation r_2 is performed, then s_V). Composition is *noncommutative.*

Obviously, composition is *associative.* The set $\mathrm{Sym}(\square)$ contains the *identity* transformation r_0 (also denoted id or $\mathbf{1}$). Any element X of $\mathrm{Sym}(\square)$ has an *inverse* X^{-1}, i.e., an element such that $X * X^{-1} = X^{-1} * X = \mathbf{1}$.

The set $\mathrm{Sym}(\square)$ supplied with the composition operation is called the *symmetry group of the square.*

(2) *Symmetries of the cube.* Let

$$I^3 = \{(x, y, z) \in \mathbb{R}^3 | \; -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\}$$

be the unit cube. A *symmetry* of the cube is defined as any isometric transformation of I^3. The composition of two symmetries (of I^3) is a symmetry. There is a total of 48 symmetries of the cube (list them!). The set of all symmetries of the cube supplied with the composition operation is called the *symmetry group of the cube* and is denoted by $\mathrm{Sym}(I^3)$. This group is also associative, noncommutative, has an identity, and all its elements have inverses.

(3) *Symmetries of the circle.* Let $S^1 := \{(x, y) \in \mathbb{R}^2 | \; x^2 + y^2 = 1\}$ be the unit circle. Denote by $\mathrm{Sym}(S^1)$ the set of all isometries of the circle. The elements of $\mathrm{Sym}(S^1)$ are of two types: rotations r_α, $\alpha \in [0, 2\pi)$, and axial symmetries s_α, $\alpha \in [0, 2\pi)$. The composition of rotations is given by the formula

$$r_\alpha * r_\beta = r_{(\alpha+\beta) \bmod 2\pi}.$$

There is also a nice formula for the composition of symmetries (find it!). The set of all isometries of the circle supplied with the composition operation is called the *symmetry group of the circle* and is denoted by $\mathrm{Sym}(S^1)$. The group $\mathrm{Sym}(S^1)$ has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

(4) *The projective plane.* Consider the set $X = Ant(S^2)$ of all pairs of diametrically opposed points on the sphere. (Thus elements of X are *not* ordinary points, but *pairs of points.*) Let G be the group of distance-preserving mappings of S^2 to itself. Then $X = Ant(S^2)$, the set of diametrically opposed points on S^2 under the action of the isometry group of the sphere is called the *projective plane* and denoted by $\mathbb{R}P^2$. (In your linear algebra course, another definition of $\mathbb{R}P^2$, equivalent to

the strange definition above, will be given; then the term "projective plane" for this classical object will become clear.)

The projective plane $\mathbb{R}P^2$ contains another classical geometric object – the Möbius band. Can you find it there?

§1.3. Main definitions

Let X be a set (finite or infinite) of arbitrary elements called *points*. By definition, a *transformation group G acting on X* (=группа преобразований G действующая на X) is a (nonempty) set G of bijections of X supplied with the composition operation $*$ and satisfying the following conditions:

(i) G is closed under composition, i.e., for any transformations $g, g' \in G$ the composition $g * g'$ belongs to G;

(ii) G is closed under taking inverses, i.e., for any transformation $g \in G$, its inverse g^{-1} belongs to G.

These conditions immediately imply that G contains the identity transformation. Indeed, take any $g \in G$; by (ii), $g^{-1} \in G$; by (i), $g^{-1} * g \in G$; but $g^{-1} * g = \mathrm{id}$ (by definition of inverse element). Note also that composition in G is associative (because the composition of mappings (=отображений) and functions is always associative).

If $x \in X$ and $g \in G$, then by xg we denote the image of the point x under the transformation g. (The more usual notation $g(x)$ is not convenient (=неудобно): we have $x(g * h) = (xg)h$, but $(g * h)(x) = h(g(x))$.)

A mapping of transformation groups $\alpha : G \to H$ is called a *homomorphism* if α respects (=уважает) the product structure, i.e., $\alpha(g_1 * g_2) = \alpha(g_1) * \alpha(g_2)$ for all $g_1, g_2 \in G$; a homomorphism α is a *monomorphism* (an *epimorphism*) if α is injective (surjective, respectively (=соответственно)).

Two transformation groups G and H acting on two sets X and Y ($X = Y$ is not forbidden) are called *isomorphic* if there exists a bijective mapping $\phi : G \to H$ such that $\phi(g * g') = \phi(g) * \phi(g')$ for all $g, g' \in G$.

In this course, a pair $(X : G)$, where G is a transformation group acting on X, will be called the *geometry of G on X*. The four examples in §2 define the geometry of the square, the geometry of the cube, the geometry of the circle, and the geometry of the projective plane. Another example is the set $\mathrm{Bij}(X)$ of *all* bijections of any set X. Note that $\mathrm{Bij}(X)$ contains any transformation group G acting on X.

If X is a subset of $\mathbb{R}^n$ and $(X : G)$ is a geometry, then a subset $F \subset X$ is called a *fundamental region* of this geometry (or of the action of G on X) if

- F is open;
- $F \bigcap g(F) = \emptyset$ for any $g \in G$;
- $X = \bigcup_{g \in G} \mathrm{Clos}(g(F))$, where $\mathrm{Clos}(.)$ denotes the closure of a set.

For example, in the case of the square, a fundamental region of the action of $\mathrm{Sym}(\square)$ is the triangle AOM, where O is the center of the square and M is the midpoint of side AB; of course $\mathrm{Sym}(\square)$ has many other fundamental regions. Thus fundamental regions are not necessarily unique. Moreover, fundamental regions don't always exist: for instance, $\mathrm{Sym}(S^1)$ (and other "continuous" geometries) do not have a fundamental region.

§1.4. Order, generators, subgroups

By definition, the *order of an element* g of a group G acting on a set X is the smallest natural number k such that

$$g^k = \underbrace{g * g * \cdots * g}_{k} = \mathbf{1};$$

notation: $\mathrm{ord}(g)$. The *order of the group* G itself is the number of its elements and is denoted by $|G|$.

For example, $|Sym(\square)| = 8$, $\mathrm{ord}(r_1) = 4$, $\mathrm{ord}(s_B) = 2$. Also, $|\mathrm{Sym}(S^1)| = \infty$, and $\mathrm{Sym}(S^1)$ contains elements of infinite order and elements of any finite order (find an element of order 17!).

Suppose G is a transformation group acting on some set X. Then a subset H of G is called a *subgroup of* G if H is closed with respect to composition (i.e., $h, h' \in H \Rightarrow h * h' \in H$) and closed with respect to inverse elements (i.e., $h \in H \Rightarrow h^{-1} \in H$)). Any subgroup $H \subset G$ contains the identity element $\mathbf{1}$ (prove this!). For example, the set of rotations

$$\mathrm{Rot}(\square) = \{r_0,\, r_1\, r_2,\, r_3\} \subset \ \mathrm{Sym}(\square)$$

is a subgroup (of order 4) of the symmetry group of the square, but the set of $\{s_H,\, s_V,\, s_{ac},\, s_{bd}\}$ is not a subgroup (why?). Are there any subgroups of order 2 in $\mathrm{Sym}(\square)$?

Let G be a transformation group (finite or infinite); a family $\{g_1, \ldots, g_k\} \subset G$ is (by definition) a set of *generators* of G if any $g \in G$ is the composition of some elements of this family. For example, if a is an element of order 5 in some transformation group G, then $\{a, a^2, a^3, a^4, a^5 = \mathrm{id}\}$ is a subgroup of G generated by the one-element set $\{a\}$. Another example: the two elements r_1 and s_H generate the group $\mathrm{Sym}(\square)$ (check this!).

§1.5. The category of geometries

Category theory, or “abstract nonsense” is a very general formal algebraic language; it will not be studied or used in this course. However, you should know that a category is a class of “objects” and “morphisms” satisfying certain axioms; for example, the category of sets is the class of sets and their mappings, the category of groups is the class of groups and their homomorphisms.

Similarly (аналогично), the category of geometries is the class of all geometries (in the sense of §1.3) and their morphisms; by definition, a *morphism* of a geometry $(X : G)$ to a geometry $(Y : H)$ is a correspondence (=соответствие) between them respecting their geometric structure (the group action); more precisely, a morphism is a pair (ϕ, α), where $\phi : X \to Y$ is a mapping and $\alpha : G \to H$ is a homomorphism such that $\phi(xg) = (\phi(x))(\alpha(g))$ for all $x \in X$ and $g \in G$. A morphism (ϕ, α) of geometries is an *embedding* (=вложение) if ϕ and α are injective; it is a *surjection* if ϕ and α are surjective.

Two geometries $(X : G)$ and $(Y : H)$ are called *isomorphic*, if there exist a bijection $\phi : X \to Y$ and an isomorphism $\alpha : G \to H$ such that $\phi(xg) = (\phi(x))(\alpha(g))$ for all $x \in X$ and all $g \in G$. This is a typical definition in the style of algebraic “abstract nonsense”. It is so trivial and so tautological that it is almost impossible to understand. To understand it, try to prove that $\alpha(\mathbf{1}) = \mathbf{1}$, if the corresponding geometries are equivalent.

§1.5*. Some philosophical remarks

The examples in §2 (square, cube, circle) were taken from elementary school geometry. This was done *to motivate* the choice of the action of the corresponding transformation group. Thus, in the example of the cube, let us forget school geometry: instead of the cube I^3 with its vertices, edges, faces, angles, interior points and other structure, consider the abstract set of points $\{A, B, C, D, A', B', C', D'\}$ and define the "isometries" of the "cube" as a set of 48 bijections; for example, the "rotation by 270°" about the vertical axis is the bijection

$$A \mapsto B,\ B \mapsto C,\ C \mapsto D,\ D \mapsto A,\ A' \mapsto B',\ B' \mapsto C',\ D' \mapsto A',$$

and the 47 other "isometries" are defined similarly. Then (still forgetting school geometry), we can *define* vertices, edges (AB is an edge, but AC' is not), faces, prove that all edges are congruent, all faces are congruent, the "cube" can "rotate" about each vertex, etc.). The result is the *intrinsic* (=внутренняя) geometry of the set of vertices of the cube.

Above we called $Q^{(3)} := \mathrm{Sym}(I^3)$ the geometry of the cube; $Q^{(3)})$ is not the same geometry as the *geometry of the vertex set of the cube*

$$Q^{(0)} := \mathrm{Sym}(A, B, C, D, A', B', C', D').$$

Of course the group G acting in these two geometries is the *same group* of order 48, but it acts on two *different sets*: the (infinite) set of points of the cube I^3 and the (finite) set of its 8 vertices $A, B, C, D, A', B', C', D'$. Thus the algebra of the two situations is the same, but the geometry is different. The geometry of the solid cube I^3 is of course much richer than the geometry of the vertex set of the cube. For example, we can define line segments inside the cube, establish their congruence, etc. (Note that segments of the same length inside the cube are not necessarily congruent in the geometry of the cube!)

Another example: the set of three points $\{A, B, C\}$ with two transformations, namely the identity and the "symmetry"

$$A \mapsto A,\ B \mapsto C,\ C \mapsto B.$$

What should this geometry be called? Yes, it should be called the intrinsic geometry of the vertex set of the isosceles (=равнобедренный) triangle (why?).