Lecture 2: REGULAR POLYHEDRA and TILINGS

In this lecture we consider two special geometries and use them to construct some beautiful objects: regular polyhedra (=платоновы тела) and tilings of the plane (=паркеты).

All five regular polyhedra were known to the ancient Greeks (including Plato), they were studied from the astronomical point of view by Kepler (planetary orbits), from the physical point of view by Pythagoras and Heisenberg ("singing spheres") and from the artistic and philosophical point of view by Leonardo da Vinci. Mathematically, they are given by special finite subgroups of the isometry group $\mathrm{Sym}(S^2)$ of the sphere S^2 (see §2.2 below).

All seventeen possible tilings of the plane were known to the Arabian architects who constructed the famous Alhambra palace in Spain (14th century) and used the tilings to decorate the inner walls of the palace. The Dutch artist A.Escher used them for his wonderful "periodic" watercolors (=акварели). The first proof of the fact that there are no other tilings was obtained by the Russian crystallographer Fedorov (1891). Mathematically, they are given by special discrete subgroups of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane, called the *Fedorov groups*.

§2.1. Orbits, stabilizers, class formula

Let $(X : G)$ be some transformation group acting on a set X and let $x \in X$. Then the *orbit* of x is defined as

$$\mathrm{Orb}(x) := \{g(x) | g \in G\} \subset X,$$

and the *stabilizer* of x is

$$\mathrm{St}(x) := \{g \in G | g(x) = x\} \subset G.$$

For example, if $X = \mathbb{R}^2$ and G is the rotation group of the plane about the origin (=группа вращений вокруг начала), then the set of orbits consists of the origin and all concentric circles centered at the origin; the stabilizer of the origin is the whole group G, and the stabilizers of all the other points of $\mathbb{R}^2$ are trivial (i.e., they consist of one element – the identity id$\in G$).

Suppose $(X : G)$ is an action of a finite transformation group on a finite set. Then the number of points of G is given by

$$\boxed{|G| = |\mathrm{Orb}(x)| \times |\mathrm{St}(x)|}$$

for any $x \in X$. Now let $A \subset X$ be a set that intersects each orbit at exactly one point. Then the number of points of X is given by the formula

$$\boxed{|X| \;=\; \sum_{x \in A} \frac{|G|}{|St(x)|}},$$

called the *class formula*. These formulas follow immediately from definitions.

§2.2. Finite subgroups of $SO(3)$

Consider the geometry $(X : G)$ of the two-dimensional sphere

$$X = S^2 := \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$$

defined by the action of its isometry group $G = \mathrm{Sym} S^2$. (In linear algebra courses this group G is defined in a different (but equivalent) way and is usually denoted by $O(3)$.) The group $G = O(3)$ contains the rotation subgroup G^+ =Rot(S^2): each element of G^+ is a rotation of the sphere about some axis (=оси) passing through the origin by some angle ϕ, $0 \le \phi < 2\pi$. In linear algebra courses this group G^+ is defined in a different (but equivalent) way and is usually denoted by $SO(3)$.

Our goal is to find the finite subgroups of $G^+ = SO(3)$ and of $G = O(3)$. We begin with some examples.

(i) The *monohedral group* $\mathbb{Z}_{/n}$ for $n \ge 2$ (n elements); its elements are rotations about the vertical axis (i.e., the z-axis) by angles of $2k\pi/n$, $k = 0, \dots, n-1$.

(ii) The *dihedral group* $\mathbb{D}_n$ for some $n \ge 1$ ($2n$ elements); the group $\mathbb{D}_n$ is the isometry group of the regular n-gon (lying in the horizontal plane Oxy and inscribed in the sphere S^2); $\mathbb{D}_n$ consists of n rotations (by angles of $2k\pi/n$, $k = 0, 1, \dots, n-1$.) and n symmetries with respect to n the horizontal lines passing through the center, the vertices, and the midpoints of the sides (be careful: these lines are different when n is even or odd – draw a picture!). Note that $D(n) \subset SO(3)$ and the symmetries of $D(n)$ with respect to the horizontal lines are actually *rotations* in space (about these lines by angles of $180°$).

(iii) The *isometry group of the regular tetrahedron* inscribed in the sphere S^2 (24 elements), denoted by $\mathrm{Sym}(\Delta^3)$ (we will see later that $\mathrm{Sym}(\Delta^3)$ is isomorphic to the permutation group (=группа перестановок) $\mathbb{S}_4$ and the (12 element) rotation subgroup:

$$\mathrm{Rot}(\Delta^3) = \mathrm{Sym}^+(\Delta^3) \subset \mathrm{Sym}(\Delta^3),$$

consisting of 8 rotations about 4 axes (containing one vertex) by angles of $2\pi/3$ and $4\pi/3$, 3 rotations by π (describe them!) and the identity.

(iv) The *isometry group Sym(I^3) of the cube* (48 elements) and its (24 element) rotation subgroup:

$$\mathrm{Rot}(I^3) = \mathrm{Sym}^+(I^3) \subset \mathrm{Sym}(I^3)$$

(see the previous lecture). If we join (=соединяем) the center of each of the 6 faces of the cube by segments to the four neighboring (=соседние) centers, we obtain the carcass of *the octahedron dual to the cube* (see Fig.1,a). The octahedron has 6 vertices and 8 triangular faces; its isometry group is the same as that of the cube.

(v) The *isometry group Sym(Dod) of the dodecahedron* (120 elements) and its (60 element) rotation subgroup

$$\mathrm{Rot}(\mathrm{Dod}) = \mathrm{Sym}^+(\mathrm{Dod}) \subset \mathrm{Sym}(\mathrm{Dod}).$$

The dodecahedron is the (regular) polyhedron (inscribed in the sphere S^2) with 12 faces (congruent regular pentagons), 30 edges, and 20 vertices (see Fig.1,b). The existence of such a polyhedron will be proved in the exercise class. Joining the centers of the faces of the dodecahedron in a systematic way (see Fig.1,b), we get the *icosahedron dual to the dodecahedron*, which has the same group.

The following theorem states that there are no other finite subgroups.

Theorem 1. *Any finite subgroup of* $G^+ = \mathrm{Sym}^+(S^2)$ *is isomorphic to one of the following groups:*
(i) $\mathbb{Z}_{/n}$, (ii) $\mathbb{D}_n$, (iii)$\mathrm{Rot}(\Delta^3)$, (iv) $\mathrm{Sym}^+(I^3)$, (v) $\mathrm{Sym}^+(Dod)$.

Sketch of the proof of Theorem 1. It is known (see the linear algebra course) that any element of $SO(3)$ (and hence of G^+) is a rotation about a diameter of the sphere S^2 and has two fixed points (the ends of the diameter). Let F be the set of fixed points of the group G^+:

$$F = \{x \in S^2 \mid \exists g \in G^+ - \mathrm{id},\ g(x) = x\}.$$

Consider the (finite) geometry $(F : G)$ and let A be a set containing one point in each orbit of G in F. First we claim that the number of points in F is $|F| = |A||G| - 2(|G|-1)$. This will be proved in the exercise class. Using the class formula from §2.1, we can write $|F| = \sum_{x\in A} |G|/v(x)$, where $v(x) = |\mathrm{St}(x)|$. Note that $v(x)$ is the order of the rotation subgroup of G^+ determined by the diameter containing x. Replacing $|F|$ by its value found above and dividing by $|G|$, we obtain

$$\boxed{2 - \frac{2}{|G|} = \sum_{x\in A}\left(1 - \frac{1}{v(x)}\right)}.$$

The left-hand side of this formula is less than 2; hence the sum in the right-hand side can contain only 2 or 3 summands; therefore there can be only 2 or 3 orbits. Denote by x_1, x_2, x_3 points of these three orbits; denote by v_1, v_2, v_3 the values of $v(x)$ (in nondecreasing order). It is not difficult to see that only the following cases are allowed by the formula above:

	v_1	v_2	v_3	$\lvert G\rvert$
case 1	n	n	-	n
case 2	2	2	n	$2n$
case 3	2	3	3	12
case 4	2	3	4	24
case 5	2	3	5	60

The five cases correspond to (i)–(v), respectively. □

(For the details of the proof, see, for example, М.Бэрже, *Геометрия*, том 1, М.,Мир, с.39-41 or А.Н.Кострикин, *Ведение в Алгебру*, М., Наука, с. 102-108).

Corollary. *Any finite subgroup of G =Sym(S^2) is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_{/n}$, (ii) $\mathbb{D}_n$, (iii) $\mathbb{S}_4$, (iv) Sym(I^3), (v) Sym(Dod).

§2.3. The five regular polyhedra

A *regular polyhedron* is defined as a convex polyhedron (inscribed in the sphere S^2) such that

(i) all its faces are congruent regular polygons of k sides for some $k > 2$;

(ii) the endpoints of all the edges issuing from (=исходящие из) each vertex lie in one plane and form a regular l-gon.

Theorem 2. *There are exactly five different regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.*

Proof. This theorem follows from the Corollary to Theorem 1. Indeed, the definition implies that the isometry group of a regular polyhedron is finite and therefore must be one of the groups listed in Theorem 1. The two "series" (i) and (ii) do not give any (nondegenerate) polyhedra (why?). In case (iii), we get the tetrahedron (because its symmetry group is isomorphic to $\mathbb{S}_4$). In case (iv), we get the cube and its dual, the octahedron, in case (v), the dodecahedron and its dual, the icosahedron (see Fig.1). □

Thus we obtain five geometries with three different group actions (tetrahedron, cube $\sim$ octahedron, dodecahedron $\sim$ icosahedron). To understand the group actions in these geometries, it is useful to construct their fundamental regions.

§2.4. The seventeen regular tilings

By definition, a *tiling* of the plane $\mathbb{R}^2$ by a polygon T_0, the *tile* (=плитка), is an infinite family $\{T_1, T_2, \dots\}$ of pairwise nonoverlaping copies of T_0 (не налагающихся друг на друга экземпляров T_0) filling the plane (i.e., $\mathbb{R}^2 = \bigcup_{i=1}^{\infty} T_i$). For example, the plane can be tiled (=замощен) by any rectangle. A beautiful example of a tiling by a concave (=невыпуклый) polygon of 9 sides (discovered by Vordeberg, 1936) is shown in Fig.2.

A polygon $T_0 \subset \mathbb{R}^2$ (called the *fundamental tile*) determines a *regular tiling* (=паркет) of the plane $\mathbb{R}^2$ if there is a subgroup G (called the *tiling group*) of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane such that

(i) G acts discretely on $\mathbb{R}^2$, i.e., all the orbits of G have no accumulation points;

(ii) the images of T_0 under the action of G fill the plane, i.e., $\bigcup_{g \in G} g(T_0) = \mathbb{R}^2$;

(iii) for $g, h \in G$ the images $g(T_0)$, $h(T_0)$ of the tile can overlap only if $g = h$.

Actually, (ii) and (iii) imply (i), but we will not prove this.

Six examples of tilings are shown in Fig.3. The first five (a-e) are *positive*, i.e., they correspond to subgroups of the group $\mathrm{Sym}^+(\mathbb{R}^2)$ of motions (generated by all rotations and translations) of the plane (one-sided tiles slide along the plane). The sixth tiling (f) allows "turning over" (=переварачивать) the (two-sided) tiles.

Theorem 2 [Fedorov, 1891]. *There are exactly five different positive tilings of the plane $\mathbb{R}^2$. They are shown in* Fig.3,a–e.

In the general case (not only positive tilings, but all tilings), there are exactly seventeen nonequivalent tilings. This was also proved by Fedorov. They are shown, in particular, in the book by М.Бэрже *Геометрия*, том 1, М.,Мир, стр.35-36.)

§2.5. Sketch of the proof of Fedorov's theorem

Let G be a group of positive tilings. Consider the subgroup $G_T \subset G$ of all parallel translations in G. It can be proved that G_T is generated by two noncollinear vectors u and v. If G contains no rotations, then we get the tiling (a). If G contains only rotations of order 2, then we get the tiling (b). If G contains a rotation of order $\alpha \geq 3$, it can be proved that there are two more rotations (of order β and γ) such that

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.}$$

Since the three rotations are of order greater or equal to 3, only three cases are allowed by this formula

	$1/\alpha$	$1/\beta$	$1/\gamma$
case 1	1/3	1/3	1/3
case 2	1/2	1/4	1/4
case 3	1/2	1/3	1/6

It is easy to see that these cases correspond to the tilings (c–e) of Fig.3. $\square$

(For the details of the proof and for the general case (seventeen tilings), see, for example, М.Бэрже, *Геометрия*, том 1, М.,Мир, стр.29-34.)