Lecture 4: COXETER GEOMETRIES

In this lecture, we study the geometries determined by reflections (=отражениями) in systems of hyperplanes in $\mathbb{R}^d$. Our goal (=цель) is to specify a reasonable class of such geometries (we will call them Coxeter geometries) and to describe (=описать) their classification. We will study their transformation groups (these groups are discrete, like the transformation groups for pavings defined in Lecture 2).

Many ideas of this (geometric) theory are similar to the very important (algebraic) theory of group representation (studied in advanced algebra courses and used in modern physics). In particular, the famous (=знаменитые) Dynkin diagrams are similar to Coxeter schemes (defined below).

We will first study Coxeter geometries on the plane and space ($d = 2, 3$) and classify them. You will see that the proofs are based on the same ideas as for pavings. The classification theorems are conveniently stated (=удобно формулируются) in terms of certain graphs (called Coxeter schemes).

§4.1. An example: the kaleidoscope

The kaleidoscope is a children's toy (=детская игрушка): bright little pieces of glass (=яркие стекляшки) are placed inside a regular triangular prism and are multiply reflected by three mirrors forming the lateral faces of the prism (=боковые грани призмы). Looking into the prism, you see a colorful (infinitely) repeated pattern (=узор): the picture in the triangle and its mirror image alternate, forming a hexagon (the union of six equilateral triangles), see Fig.1,b, surrounded by more equilateral triangles *ad infinitum.*

Mathematically, this is a two-dimensional phenomenon: the equilateral triangle forming the base of the prism is the fundamental region of a discrete group acting on the plane of the base.

Now if the kaleidoscope is deformed (e.g., the angles between the faces are slightly changed), then the picture becomes fuzzy (=-гв-гс), no pattern can be seen. In such a situation, the images of the base triangle overlap infinitely many times (see Fig.1,a), the transformation group acting on the triangle is not discrete; we will not study this "bad" case: we only study the "nice" kaleidoscope case generalized to any dimension.

§4.2. Coxeter polygons and polyhedra

Consider a dihedral angle $\alpha < \pi/2$ formed by two plane two-sided mirrors S_1, S_2. What will the observer (=наблюдатель) O see? Any picture P inside the angle will be reflected by S_1; its image P' will be in turn be reflected by the image of S_1 by S_2, and so on. At the same time, the picture P inside the angle will be reflected by S_2; its image P'' will be in turn be reflected by the image of $S2$ by S_1, etc. (See Fig.2, where only a two-dimensional section (=двумерное сечение) of our situation is shown.) Two cases are possible: either the reflections coming from different sides will overlap (Fig.1,a) or the reflected pictures will coincide (Fig.1,b). Obviously (=очевидно), the pictures will coincide if (and only if) the angle α is of the form π/k, where $k = 2, 3, \ldots$.

Mathematically, this situation is the following (=ситуация такова). On the Euclidean plane, we take two straight lines forming the angle α and consider the group G of all transformations of the plane generated by the reflections in these two planes. Let F be the region bounded (=ограниченный) by the two half planes forming the angle α. Obviously, no two regions $g(F)$ and $h(F)$, $g, h \in G$, $g \neq h$, overlap iff $\alpha = \pi/k$, where $k = 2, 3, \ldots$. In that case, G is the dihedral group $\mathbb{D}_k$.

Now suppose we are given a convex polygon F in the plane with vertex angles less than or equal to $\pi/2$. Consider the group G_F of transformations of the plane generated by reflections in the lines containing the sides of F. We say that G_F is *acts transitively on* F if the images $g(F)$, $g \in G_F$, never overlap. A necessary condition for the transitive action of G_F on F is that all the vertex angles of F be of the form π/k for various values of k; this follows from the argument in the previous paragraph (=абзац). Obviously, this condition is not sufficient.

The previous arguments are the motivation for the following definition. A convex polygon is called a *Coxeter polygon* if all its vertex angles are of the form π/k for various values of $k = 2, 3, \ldots$. Coxeter polygons will be classified below (=ниже) – there are only four.

The above (=вышесказанное) can be generalized to three-dimensional space. The corresponding definition is the following: a convex polyhedron is called a *Coxeter polyhedron* if all its dihedral angles are of the form π/k for various values of $k = 2, 3, \ldots$. Coxeter polyhedra will be classified below (there are seven).

§4.3. Coxeter geometries on the plane

Let F be a Coxeter polygon in the plane $\mathbb{R}^2$. The *Coxeter geometry* with fundamental region F is the geometry $(\mathbb{R}^2, G_F)$, where G_F is the group of transformations of the plane generated by the reflections in the lines containing the sides of the polygon F. The goal of this section is to classify all Coxeter geometries on the plane.

Theorem 1. *There are four Coxeter geometries in the plane; their fundamental polygons are the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with acute angles $\pi/3$ and $\pi/6$ (see Fig.2)*

Proof. Let F be the fundamental polygon of a Coxeter geometry. If it has n sides, then the sum of its angles is $\pi(n-2)$ and so the average value of its angles is $\pi(1-2/n)$. Now n cannot be greater than 4, because F would then have an obtuse (=тупой) angle (and this contradicts the definition of Coxeter polygon). If $n = 4$, then all angles of F are $\pi(1-2/4) = \pi/2$ and F is a rectangle. Finally, if $n = 3$, and the angles of the fundamental triangle are $\pi/k, \pi/l, \pi/m$, then (since their sum is π), we obtain a Diophantine equation for k, l, m:

$$\boxed{\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1}.$$

This equation has three solutions: $(3,3,3)$, $(2,4,4)$, $(2,3,6)$. These solutions correspond to the three triangles listed (=перечисленных) in the theorem. $\square$

§4.4. Coxeter geometries in Euclidean space $\mathbb{R}^3$

In this section we study the Coxeter geometries in $\mathbb{R}^3$. A Coxeter polyhedron $F \subset \mathbb{R}^3$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^3$) with dihedral angles of the form π/k for various values of $k = 2, 3, \ldots$. A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined just as in the case $d = 2$ (see §4.2).

Theorem 2. *There are seven Coxeter geometries in three-dimensional space; their fundamental polyhedra are the four right prisms over the the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with acute angles $\pi/3$ and $\pi/6$, and the three (nonregular) tetrahedra shown in Fig.3.*

It is not very difficult to prove that the seven polyhedra listed in the theorem indeed (=в самом деле) define Coxeter geometries. To prove that there are no other geometries, nontrivial information from linear algebra (in particular, the notion of Gramm matrix) is needed.

Therefore, we omit the proof (it can be found in the article by E.B.Vinberg in *Математическое просвещение*,Ser.3, # 7, pp.39-63).

A remark about terminology. The term "Coxeter geometry" is not a standard term. E.B.Vinberg uses the term "kaleidoscope" instead. Also, we do *not* use the term "Coxeter group" for the transformation group of a Coxeter geometry. This is because "Coxeter group" is standardly used in a more general sense (=смысл) than "transformation group of a Coxeter geometry".

Coxeter geometries are not only abstract mathematical objects, they are also important models in crystallography. For example, the polyhedron in Fig.3,b is the crystal of ordinary salt (=поваренная соль), while the polyhedron in Fig.3,c is a diamond (=алмаз) crystal.

§4.5. Coxeter schemes and the classification theorem

In this section we study the general case of a Coxeter geometry in $\mathbb{R}^d$ for an arbitrary positive integer d. A Coxeter polyhedron $F \subset \mathbb{R}^d$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^d$) with dihedral angles of the form π/k for various values of $k = 2, 3, \dots$. (The definition of the measure of a dihedral angle in Euclidean space of arbitrary dimension d appears in the linear algebra course.) A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined exactly like in the cases $d = 2$ and $d = 3$ (see §§4.2, 4.3).

A *Coxeter scheme* is a graph (with integer weights on the edges) encoding a Coxeter polyhedron (in particular, polygons) in any dimension d. The scheme of a given Coxeter polyhedron is constructed as follows: its vertices correspond to the faces of the polyhedron, two vertices whose (=змЁ) corresponding faces form an angle of π/m, $m \geq 3$, are joined by an edge (=соединены ребром) with the weight $m - 2$, if two faces are parallel, the corresponding vertices are joined by an edge with weight ∞. (Note that vertices corresponding to perpendicular edges are *not* joined by an edge.)

Graphically, instead of writing the weights 2,3,4 on the edges of a scheme, we draw double, triple, quadruple edges; instead of writing ∞ on an edge, we draw a thick (=толстый) edge.

For example, the Coxeter scheme of the rectangle consists of two components, each of which (=каждая из которых) has two vertices joined by an edge with weight ∞, while the scheme of an equilateral triangle has three vertices joined cyclically by three edges with weights 1.

Theorem 3. *The Coxeter geometries in all dimensions $d \geq 2$ are classified by the connected components of their Coxeter schemes listed in Fig.3.*

We omit the proof (see the article by V.O.Bugaenko in *Математическое просвещение*, Ser.3, # 7, pp. 82-106).