Lecture 6: DEGREE OF MAPS OF THE CIRCLE

This lecture, as well as the next ones, is not about geometries: it is about some basic facts of topology. We will define an important invariant of continuous maps (= mappings = отображений) of the circle $f : S^1 \to S^1$, called the degree and denoted by $\deg(f)$. In this lecture the degree is used to prove the Brouwer fixed point theorem and the Fundamental Theorem of Algebra. In the next lecture, the degree of maps of the circle will be a first step in the study of the fundamental group.

§6.1. Continuous maps

In this lecture, we consider subsets of some Euclidean space $\mathbb{R}^N$, denote such subsets by $X, Y, \dots$, and sometimes, by abuse of language, call them "spaces". For any point $x_0 \in X$ and any $\varepsilon > 0$, by $O(x, \varepsilon)$ we denote the ε-neighborhood of x in X, i.e., the set of all points x of X such that the distance from x to x_0 is less than ε. A map $f : X \to Y$ is called *continuous* if for any point $x \in X$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(O(x, \delta)) \subset O(f(x), \varepsilon)$. By definition, a map $h : X \to Y$ is a *homeomorphism* if it is bijective, continuous, and the inverse map h^{-1} is also continuous.

In topology, homeomorphic sets are considered equivalent and often called *topologically equivalent* (just as isomorphic groups are considered equivalent). For example, the circle and the boundary of the square are topologically equivalent.

We assume that the following notions (usually introduced in Calculus courses (курсах анализа))are known: open set, closed set, boundary point, compact set, connected set, connected component. The following two statements are simple and classical theorems, proved in Calculus courses:

- *the composition of two continuous maps is continuous;*
- *the image of a compact set under a continuous map is compact.*

§6.2. Homotopy

Let X and Y be subsets of $\mathbb{R}^N$. Two maps $f, g : X \to Y$ are called *homotopic* (notation $f \simeq g$) if they can be joined by a *homotopy*, i.e., by a map $F : X \times [0, 1] \to Y$ such that $F(x, 0) \equiv f(x)$ and $F(x, 1) \equiv g(x)$ (here $\equiv$ means for all $x \in X$). If we change the notation from $F(x, t)$ to $F_t(x)$, we can restate the previous definition by saying that there exists a family $\{F_t(x)\}$ of maps, parametrized by $t \in [0, 1]$, continuously changing from $f \equiv F_0$ to $g \equiv F_1$.

It is easy to prove that

1. $f \simeq f$ for any $f : X \to Y$ (*reflexivity*);
2. $f \simeq g \implies f \simeq g$ for all $f, g : X \to Y$ (*symmetry*);
3. $f \simeq g$ and $g \simeq h \implies f \simeq h$ for all $f, g, h : X \to Y$ (*transitivity*).

For example, to prove transitivity (3), we obtain a homotopy joining f and h by setting

$$F(x,t) = \begin{cases} F'_{2t}(x) & \text{for } 0 \le t \le 1/2, \\ F''_{2t-1}(x) & \text{for } 1/2 \le t \le 1, \end{cases}$$

where F'_t and F''_t are homotopies joining f to g, and g to h, respectively.

Therefore the homotopy of maps is an equivalence relation, so that the set $\mathrm{Map}(X, Y)$ of all (continuous) maps of X to Y splits into equivalence classes, called *homotopy classes*; the set of these equivalence classes is denoted $[X, Y]$.

Two spaces X and Y are called *homotopy equivalent* if there exist two maps $f : X \to Y$, $g : Y \to X$ (called *homotopy equivalences*) such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Obviously, *homeomorphic spaces are homotopy equivalent* (the homotopy equivalences are provided by any homeomorphism and its inverse). The converse statement is not true: for example, the point is homotopy equivalent to the 2-disk, but these two spaces are not homeomorphic.

Thus homotopy equivalence is a weaker equivalence relation than homeomorphism, so that homotopy classification is rougher (and hence easier – there are less classes) than topological classification. Its importance in topology is due to the fact that most topological invariants are homotopy invariants (this is the case of the so-called fundamental group, homology groups and related invariants such as the Euler characteristic).

§6.3. Definition of the degree of maps of S^1 to S^1

In this section we consider (continuous) maps $f : S^1 \to S^1$ of the circle into itself. Examples are the maps $w_k : S^1 \to S^1$ given by the rule $e^{i\varphi} \mapsto e^{ik\varphi}$, where S^1 is modelled by unimodular complex numbers: $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We are interested in the classification of such maps up to homotopy.

Theorem 1. *There is a natural bijection between homotopy classes of maps of the circle into itself and the integers,*

$$\boxed{[S^1, S^1] \quad \overset{(1-1)}{\longleftarrow\!-\!-\!\longrightarrow} \quad \mathbb{Z}.}$$

Proof. Consider the map $\exp : \mathbb{R} \to S^1$ given by the rule $\mathbb{R} \ni \varphi \mapsto e^{i\varphi} \in S^1$. The map exp is not a bijection; for example, it takes all points of the form $2k\pi$ to $1 \in S^1$ (see Fig.1). However, exp is a *local homeomorphism*, i.e., any point has a neighborhood U (e.g. any open interval of length less than 2π containing the point) such that the restriction $\exp|_U$ of exp to U is a homeomorphism.

Now any map $S^1 \to S^1$ can be regarded as a map $f : [0,1] \to S^1$ such that $f(0) = f(1) = 1 \in S^1$. For any such map there exists a unique map $\widetilde{f} : [0,1] \to \mathbb{R}$, called the *lift* of f, such that $\exp \circ \widetilde{f} = f$. Indeed, let us take a subdivision of $[0,1]$ into segments $[0,a_1], [a_1,a_2], \ldots, [a_m,1]$, so small that none of the images of these segments covers S^1; then, using the fact that exp is homeomorphic on intervals shorter than 2π, we first construct the (unique!) lift of $[0,a_1]$ from the point $0 \in \mathbb{R}$, then the lift of $[a_1,a_2]$ from the image of a_1 in the previous lift, and so on. Now we define the *degree* of the map f, denoted $\deg(f)$, as the integer $\widetilde{f}(1)/2\pi$.

Now we are ready to prove the theorem, i.e., establish the bijection (1). We do it by assigning $\deg(f)$ to every homotopy class $[f] \in [S^1, S^1]$. To prove that this assignment is well defined (i.e., does not depend on the choice $f \in [f]$) and is bijective, it suffices to prove that any f is homotopic to some w_k.

Let $f \in [f] \in [S^1, S^1]$ be an arbitrary map, considered as a map from $[0,1]$ to S^1 such that $f(0) = f(1) = 1 \in S^1$. Then the lift $\widetilde{f}$ is a (continuous) function defined on $[0,1]$ with values in $\mathbb{R}$. Let $\deg(f) = k$, i.e., $\widetilde{f}(1) = 2k\pi$. The graph of $\widetilde{f}$ is shown on Fig.2. The graph of w_k is a straight line joining the points $(0,0)$ and $(1, 2k\pi)$. There is an obvious homotopy (shown in the figure by arrows) joining w_k and $\widetilde{f}$; denote it by $G(x,t)$. Then $F(x,t) := \exp(G(x,t))$ is the required homotopy between f and w_k. $\square$

Corollary. *The identity map of the circle is not homotopic to the constant map* $S^1 \to 1 \in S^1$.

Remark. The notion of degree of a map can be generalized from maps of the circle to maps of the sphere S^n for any n, and even to arbitrary n-dimensional "oriented manifolds". Although the definition is not difficult, it is hard to prove in the general case (i.e., for any n) that the degree is well defined and depends only on the homotopy type of the map. To do that properly, you need *homology theory*, which lies outside the scope of this course.

§6.4. Brouwer fixed point theorem

The theorem proved in the previous section has numerous important corollaries, several of which will eventually be discussed in this course. In this section we give an illustration of this, namely the Brouwer fixed point theorem (for $n = 2$). Other more general fixed point theorems lie at the basis of fundamental existence theorems in ODE and PDE and in applications to engineering and economics problems, but they require homology theory for their proofs.

Theorem 2 [Brouwer]. *Any continuous map of the disk has a fixed point, i.e., if* $f : D^2 \to D^2$ *is continuous, then there exists a point* $x \in D^2$ *such that* $f(x) = x$.

Proof. For the proof, we will need a definition and a lemma. Let $A \subset X \subset \mathbb{R}^N$; then a continuous map $r : X \to A$ is called a *retraction* if r restricted to A is the identity.

Lemma. *There is no retraction of the 2-disk on its boundary circle.*

The proof of the lemma is an exercise in constructing elementary homotopies and applying Theorem 1 from §6.3.

To show that the fixed point theorem follows from the lemma, assume that the theorem is false. For any $x \in D^2$, we have $f(x) \neq x$, and so the intersection point $r(x)$ of the ray $[f(x), x)$ with the boundary circle is well defined and the map $x \mapsto r(x)$ is a continuous (why?) retraction of D^2 onto its boundary circle (to see this, look at Fig.3). But this contradicts the lemma. $\square$

§6.5. Degree of a point with respect to a curve

Consider a curve $f : S^1 \to \mathbb{R}^2$ and a point $p \in \mathbb{R}^2$ in its complement, $p \notin f(S^1)$. Let φ be the angular parameter on S^1 and V_φ be the vector joining the points p and $f(\varphi)$. As φ varies from 0 to 2π, the unit vector $V_\varphi/|V_\varphi|$ moves along the unit circle S_0 centered at p, defining a map $\gamma_f : S_0 \to S_0$. The *degree of the point* p *with respect to the curve* f is defined as the degree (in the sense §6.3) of the map γ, i.e., $\deg(p, f) := \deg(\gamma_f)$.

It is easy to prove that $\deg(p, f)$ does not change when p varies inside a connected component of $\mathbb{R}^2 \backslash f(S^1)$ (do it!). If the point p is "far from" $f(S^1)$ (i.e., in the connected component of $\mathbb{R}^2 \backslash f(S^1)$ with noncompact closure), then $\deg(p, f) = 0$ (why?).

There is a convenient method for computing the degree of any point p in the case when the curve f is immersed: join p by a (nonclosed) curve with a far away point a and move from a to p along that curve, adding one to the degree when you cross $f(S^1)$ in the positive direction (i.e., so that the tangent vector looks to the right) and subtracting one when you cross it in the negative direction. The proof of the fact that you will obtain $\deg(p, f)$ when your reach p is an exercise.

§6.6. The fundamental theorem of algebra

The fundamental theorem of algebra says that *any polynomial*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0, \quad n > 0,$$

has a least one (possibly complex) root; here the coefficients a_i may be real or complex. We will prove this theorem assuming that $a_n = 1$ and $a_0 \neq 0$; this does not restrict generality (why?).

Consider the curve $f_n : S^1 \to \mathbb{R}^2$ given by the formula $e^{i\varphi} \mapsto R_0^n e^{in\varphi}$, where R_0 is a (large) positive number that will be fixed later. Further, consider the family of curves $f_{p,R} : S^1 \to \mathbb{R}^2$ given by the formula $e^{i\varphi} \mapsto p(Re^{i\varphi})$, where $R \leq R_0$. We can assume that the origin O does not belong to $f_{p,R_0}(S^1)$ (otherwise the theorem is proved).

Lemma. *If R_0 is sufficiently large, then* $\deg(O, f_{p,R_0}) = \deg(O, f_n) = n$.

Before proving the lemma, let us show that it implies the theorem.

By the lemma, $\deg(O, f_{p,R_0}) = n$. Let us continuously decrease R from R_0 to 0. If for some value of R the curve $f_{p,R}(S^1)$ passes through the origin, the theorem is proved. So we can assume that $\deg(O, f_{p,R})$ changes continuously as $R \to 0$; but since the degree is an integer, it remains constant and equal to n. However, if R is small enough, the curve $f_{p,R}(S^1)$ lies in a small neighborhood of a_0; but for such an R we have $\deg(O, f_{p,R}) = 0$. This is a contradiction, because $n \geq 1$.

It remains to prove the lemma. The equality $\deg(O, f_n) = n$ is obvious. To prove the other equality, it suffices to show that for any φ the difference Δ between the vectors $V_p(\varphi)$ and $V_n(\varphi)$ that join the origin O with the points $f_p(R_0 e^{i\varphi})$ and $f_n(R_0 e^{i\varphi})$, respectively, is small in absolute value (as compared to $R_0^n = |V_p(\varphi)|$) if R_0 is large enough. Indeed, by the definition of degree, if the mobile vector is replaced by another mobile vector whose direction always differs from the direction of the first one by less than $\pi/2$, the degree will be the same for the two vectors.

Clearly, $|\Delta| = |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$. Let us estimate this number, putting $z = R_0 e^{\varphi}$ (we assume that $R_0 > 1$) and $A = \max\{a_{n-1}, a_{n-2}, \ldots, a_0\}$. We have

$$|\Delta| = |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \leq |A(R_0^{n-1} + R_0^{n-2} \cdots + 1)| \leq A \cdot n \cdot R_0^{n-1}.$$

Now if we put $R_0 := K \cdot A$, where K is a large positive number, we will obtain $|\Delta| \leq nA(KA)^{n-1} = nK^{n-1}A^n$. Let us compare this quantity to R_0^n; the latter equals $R_0^n = K^n A^n$, so for K large enough the ratio $|\Delta|/R_0^n$ is as small as we wish. This proves the lemma and concludes the proof of the theorem. □