Lecture 7: THE FUNDAMENTAL GROUP

In this lecture we continue our excursion in topology: we define the fundamental group $\pi_1(X)$ and study its properties. The group $\pi_1(X)$ is one of the most important topological invariants. It was invented by Poincaré over a 100 years ago. It is extensively used in analysis, algebra, and geometry.

§7.1. Definition of $\pi_1(X)$

In this lecture, as in Lecture 5, we consider subsets of some Euclidean space $\mathbb{R}^N$, denote such subsets by $X, Y, \dots$, and often call them "spaces". A space $X \subset \mathbb{R}^N$ is called *path connected* (=линейно связным) if for any pair of points $x, y \in X$ there exists a (continuous) map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. In this lecture all spaces are assumed path connected; also, the word "map" will always mean "continuous map".

Consider a (path connected) space X with a distinguished point (=отмеченной точкой) pt$\in X$, called the *base point*. A *loop* (=петля) on $(X$,pt$)$ is a (continuous) map $f : [0,1] \to X$ such that $f(0) = f(1) =$pt. Two loops f, g are called *equivalent* if there exists a homotopy $F_t : X \to X$, $t \in [0,1]$, joining f to g (i.e., $F_0 = f$ and $F_1 = g$) that preserves the base point (i.e., $F_t(\mathrm{pt}) =$pt for all $t \in [0,1]$). The set of all equivalence classes of loops is denoted by $\pi_1(X, \mathrm{pt})$ and its elements are denoted by $[f]$, $[g]$, etc., and are still called loops (by abuse of language).

Let us define a product operation on the set $\pi_1(X, \mathrm{pt})$. If $[f]$ and $[g]$ are two loops, their *product* is the loop given by

$$(f * g)(t) = \begin{cases} f(2t) & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1, \end{cases}$$

and $[f] * [g] := [(f * g)]$. It is not hard to prove that $[f] * [g]$ is well defined (=корректно определено), i.e., does not depend on the choices $f \in [f]$ and $g \in [g]$.

Proposition 1. *The set $\pi_1(X, pt)$ supplied with the product operation is a group, called the fundamental group of X with base point* pt.

Proof. The proof is a straightforward verification (=непосредственная проверка) of the axioms of groups. □

Proposition 2. *The fundamental groups of the same (path connected) space with different base points are isomorphic:*

$$\pi_1(X, \mathrm{pt}) \cong \pi_1(X, \mathrm{pt}').$$

Proof. The space X is path connected, therefore there is a path $h : [0,1] \to X$ joining the base points pt and pt$'$. Let us construct a map φ from $\pi_1(X, \mathrm{pt})$ to $\pi_1(X, \mathrm{pt}')$. Take an element $[f] \in \pi_1(X, \mathrm{pt})$ and put:

$$\varphi(f)(t) = \begin{cases} h(3t) & \text{if} \quad 0 \le t \le \frac{1}{3}, \\ f(3t-1) & \text{if} \quad \frac{1}{3} \le t \le \frac{2}{3}, \\ h(3-3t) & \text{if} \quad \frac{2}{3} \le t \le 1. \end{cases}$$

The proof of the fact that φ is well defined (i.e., does not depend on the choice of $f \in [f]$) and that φ is an isomorphism is a straightforward verification of the definitions. □

From now on (=с этого момента) we will often write $\pi_1(X)$ instead of $\pi_1(X, \mathrm{pt})$ because Proposition 2 tells us that the choice of the base point pt does not change the isomorphism class of the fundamental group.

§7.2. Induced homomorphisms

The fundamental group is actually a "functor": not only does it assign (=ставит в соответствие) a group to each space with base point, but it also assigns a homomorphism to each (continuous) map of spaces preserving base points. Suppose $f : X \to Y$ is a map such that $f(\mathrm{pt})=\mathrm{pt}'$, where pt and pt$'$ are the corresponding base points. Let us construct a homomorphism $f_\# : \pi_1(X, \mathrm{pt}) \to \pi_1(Y, \mathrm{pt}')$. Let $[\alpha] \in \pi_1(X, \mathrm{pt})$. Put $(f_\#(\alpha))(t) = f(\alpha(t))$ for any $t \in [0, 1]$. Obviously, we obtain a loop with base point pt$'$.

Proposition 3. *The correspondence $\pi_1(X, pt) \ni [\alpha] \mapsto f_\#(\alpha) \in \pi_1(Y, pt')$ defines a homomorphism $f_\# : \pi_1(X, pt) \to \pi_1(Y, pt')$.*

The homomorphism $f_\#$ is called *induced* (by f). The proof of the fact that $f_\#$ is well defined and is a indeed a homomorphism is a straightforward verification of definitions.

§7.3. Homotopy invariance of π_1

The fundamental group is a *homotopy functor*, i.e., π_1 satisfies the following homotopy invariance theorem:

Theorem 1. *Homotopy equivalent spaces have isomorphic fundamental groups and homotopic maps induce the same homomorphism.*

The proof of this theorem is not very difficult, but is omitted in this course.

Corollary. *The fundamental group is a topological invariant.*

§7.4. The fundamental group of the circle

The first nontrivial example of the fundamental group is given by the following theorem:

Theorem 2. *The fundamental group of the circle is isomorphic to the group of integers:*

$$\boxed{\pi_1(S^1) \cong \mathbb{Z}.}$$

Sketch of the proof. The isomorphism is constructed by assigning to each loop the "number of times the loop goes around the circle in the positive direction". To define this number, we apply the construction used to define the degree of circle maps in Lecture 5. When this is done, the rest is a straightforward verification of definitions. □

Theorem 3. *For any map $f : S^1 \to S^1$ the homomorphism $f_\# : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by the integer $deg(f)$.*

The proof is an (easy) exercise (if you understand the proof of Theorem 2.)

§7.5. Some more information about the fundamental group

In this section, we present some important information about the fundamental group (the detailed proofs will be given in topology courses).

Proposition 3. *The fundamental groups of Euclidean space* $\mathbb{R}^n$*, complex space* $\mathbb{C}^n$*, the sphere* S^n*, the special orthogonal group* $SO(n)$*, the projective plane* $\mathbb{R}P^2$*, and the torus* $S^1 \times S^1$ *are the following:*

(i) $\pi_1(\mathbb{R}^n) \cong \pi_1(\mathbb{C}^n) \cong 0$ *for all* n;

(ii) $\pi_1(S^n) \cong 0$ *for all* $n \geq 2$;

(iii) $\pi_1(\mathbb{R}P^2) \cong \pi_1(SO(n)) \cong \mathbb{Z}_2$;

(iv) $\pi_1(SO(n)) \cong \mathbb{Z}_2$;

(v) $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \bigoplus \mathbb{Z}$.

Although we give no proofs, the statements (i)-(v) should be intuitively clear. Thus (i) and (ii) hold because any loop in the corresponding spaces can contracted to a point. To see that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, recall that $\mathbb{R}P^2$ contains the Möbius band and use this to determine the generator of $\mathbb{Z}_2$. Finally, to see the isomorphism (v), recall, first, that the generators of π_1 of the torus $S^1 \times S^1$ are the parallel (p) and the meridian (m) and, second, that the torus can be obtained by identifying the opposite sides of the square.

Theorem 4. *For any group presentation* $G = \langle g_1, \dots, g_l : R_1, \dots, R_m \rangle$ *there exists a space* $X \subset \mathbb{R}^N$ *with fundamental group isomorphic to* G*:* $\pi_1(X) \cong G$.

The construction of X from a given presentation is effective. We do not describe the construction in the general case, but give two examples.

Given the presentation $\langle m, p : mpm^{-1}p^{-1} \rangle$, take two unit circles C_m and C_p passing through the the origin of the two subspaces $\mathbb{R}_1^2 \bigoplus \mathbb{R}_2^2 = \mathbb{R}^4$, divide the boundary C of the disk D^2 into four equal arcs and glue (=склеить) the arcs successively to C_m and to C_p in the positive direction, to C_m and to C_p in the negative direction. After the gluing, the disk will become a torus (do you see that?) with the required fundamental group (see Proposition 3, (iv)).

Given the presentation $\langle m : m^2 \rangle$ (of the group $\mathbb{Z}_2$), take a unit circle in $\mathbb{R}^4$, divide the boundary C of the disk D^2 into two arcs and glue each arc successively along the whole circle C. After the gluing, the disk will become $\mathbb{R}P^2$ (this is not easy to see, but try!); the projective plane $\mathbb{R}P^2$ has the required fundamental group (see Proposition 3, (iii)).