Lecture 8: COVERING SPACES

In this lecture we conclude our excursion in topology: we define and classify covering spaces $p : E \to B$ by using the fundamental group $\pi_1(B)$. Covering spaces appear naturally in topology, complex analysis, and geometry.

§8.1. Definition and examples of covering spaces

In this lecture, as in Lecture 7, by a space we mean a path connected subset of some Euclidean space $\mathbb{R}^N$ and denote such subsets by $X, Y, \ldots$. By a map $f : X \to Y$ we mean a continuous map (unless indicated otherwise).

We will need two preliminary definitions. The *Cartesian product* (= декартово произведение) of two spaces X and Y is the space $X \times Y$ obtained in the following way: if $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$, then

$$X \times Y := \{(x_1, \ldots, x_{N+M}) \in \mathbb{R}^{N+M} \,|\, (x_1, \ldots, x_N) \in X,\ (x_{N+1}, \ldots, x_{N+M}) \in Y\}.$$

In short notation, we write $X \times Y = \{(x, y) \,|\, x \in X,\ y \in Y\}$. The *projection on the first factor* (= проекция на первый сомножитель) $\mathrm{pr}_1 : X \times Y \to X$ is defined by the rule $(x, y) \mapsto x$; the *projection on the second factor* pr_2 is defined in a similar way. Examples: $[0, 1] \times [0, 1]$ is the square, $S^1 \times S^1$ is the torus.

A subset Γ of $\mathbb{R}^N$ is called *discrete* if it has no accumulation points in $\mathbb{R}^N$.

A *covering space* or *covering* (= накрытие) (p, E, B, Γ) is a (continuous) map $p : E \to B$ of a (path connected) space E onto (= на) a (path connected) space B such any point $x \in B$ has a neighborhood (= окрестность) $U_x \subset B$ for which there is a homeomorphism $h_x : p^{-1}(U) \to U \times \Gamma$ (where Γ is a fixed discrete space) such that $\mathrm{pr}_1 \circ h_x = p$, where $\mathrm{pr}_1 : U \times \Gamma \to U$ is the projection to the first factor. For a covering (p, E, B, Γ), p is called the *projection*, E is the *covering space* (=накрывающее пространство), B is the *base* (=база), and Γ is the *fiber* (=слой).

A trivial example of a covering space is the projection $\mathrm{pr}_1 : B \times \Gamma \to B$ on the first factor of the product of a space B by a discrete set Γ. The simplest nontrivial example is the map $w_2 : S^1 \to S^1$ of degree 2 given by the formula $w_2(e^{i\phi}) = e^{2i\phi}$; geometrically, this is the same as the natural projection of the boundary of the Möbius strip on its midline. Another example was considered in Lecture 6 – the exponential map $\exp\colon \mathbb{R} \to S^1$ given by $\exp(t) = e^{it}$.

More typical is the example of the geometry $(S^1, \mathbb{Z}_{/3})$. There is a canonical map of S^1 to the orbit space (also isomorphic to S^1) of the action of $\mathbb{Z}_{/3}$ on S^1. This map $w_3 : S^1 \to S^1$ is the covering projection given by the formula $w_3(e^{i\phi}) = e^{3i\phi}$. In that case the induced homomorphism of fundamental groups

$$(w_3)_\# : \pi_1(S^1) \to \pi_1(S^1)$$

is clearly the monomorphism $\mathbb{Z} \to \mathbb{Z}$ determined by the correspondence $1 \mapsto 3$.

Thus we have the "short exact sequence of groups":

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0.$$

Note that the group $\Gamma := \mathbb{Z}/3\mathbb{Z}$ acts on the space S^1 and each inverse image $(w_3)^{-1}(s)$, $s \in S^1$, is an orbit of Γ and consists of three isolated points (i.e., is isomorphic to Γ as a discrete space).

§8.2. The covering homotopy theorem

We now formulate a technical theorem. This theorem plays the key role in our study of coverings. In this theorem, and in the rest of this lecture, all spaces will be assumed "nice"; this means that they are *locally connected* and *locally simply connected*; the definitions of these two notions are given in topology courses; for example, polyhedra and the so-called "cell spaces" (=клеточные пространства) are nice.

Theorem 1 [Covering Homotopy Theorem]. *For any homotopy $f_t : X \to B$ of a map $f = f_0 : X \to B$ to the base B of a covering $p : E \to B$ and any lift F of f (i.e., any map $F : X \to E$ for which $p \circ F = f$) there exists a unique homotopy $F_t : X \to E$ such that $F_0 = F$ and $p \circ F_t = f_t$ for all t.*

Sketch of the proof. We begin with a classical lemma.

Lemma [Path Lifting Lemma]. *For any path $s : [0,1] \to B$ to the base B of a covering $p : E \to B$ and any point $\widetilde{x}$ for which $p(\widetilde{x}) = s(0)$ there exists a unique path $\widetilde{s} : [0,1] \to E$ such that $\widetilde{s}(0) = \widetilde{x}$ and $p \circ \widetilde{s} = s$.*

Proof of the lemma. By the definition of coverings, for each point $t \in [0,1]$ there is a neighborhood U_t such that $p^{-1}(U_t)$ is homeomorphic to $U_t \times \Gamma$. By the compactness of $[0,1]$, we can choose a finite number of such neighborhoods $U_1, \dots, U_n$ such that $U_i \supset s([t_i, t_{i+1}])$, where $0 = t_1 < t_2 < \cdots < t_{n+1}$. In the set $p^{-1}(U_1)$ choose the component $\widetilde{U}_1$ that contains $\widetilde{x}$ and consider the set $(p_{|\widetilde{U}_1})^{-1}(s([t_1,t_2]))$ (this is the first piece of the covering that we are constructing; we denote its endpoint by y_2). Now in the set $p^{-1}(U_2)$ choose the component $\widetilde{U}_2$ that contains y_2 and consider the set $(p_{|\widetilde{U}_2})^{-1}(s([t_2,t_3]))$ (this is the second piece of the covering that we are constructing). Continuing in the same way (and noting that at each step the construction is unique), we obtain the required lifted path. □

Now Theorem 1 easily follows from the Lemma. Indeed, let $x \in X$ be an arbitrary point. Then the formula $t \mapsto f_t(x)$ defines a path $s_x : [0,1] \to B$ which can be uniquely lifted (by the Lemma) to a path $\widetilde{s}_x : [0,1] \to E$. Now the required homotopy is given by the formula $F_t(x) := \widetilde{s}_x(t)$. □

§8.3. Role of the fundamental group for covering maps

In this section we will see how the fundamental group behaves (=ведет себя) under covering projections and learn that it classifies coverings.

Theorem 2. *The homomorphism of fundamental groups $p_\# : \pi_1(E) \to \pi_1(B)$ induced by any covering $p : E \to B$ is a monomorphism.*

Proof. Let pt and $\widetilde{\mathrm{pt}} \in p^{-1}(\mathrm{pt})$ be the base points of the fundamental groups.

We argue by contradiction (=от противного): let $\alpha : [0,1] \to E$ be a nonzero loop in $\pi_1(E)$ which is mapped to a zero loop $p_\#(\alpha)$ in $\pi_1(B)$. Then there exists a homotopy that contracts (=стягивает) the loop $p_\#(\alpha)$ to the base point pt. By the homotopy covering theorem (Theorem 1), this homotopy can be lifted to a homotopy that will contract α to the base point $\widetilde{\mathrm{pt}}$. A contradiction. $\square$

Theorem 3. *For any subgroup $G \subset \pi_1(B)$ of the fundamental group of a space B there exists a covering space $p : E \to B$ such that $p_\#(\pi_1(E)) = G$.*

Sketch of the proof. Consider the set $E(B, \mathrm{pt})$ whose points are all the paths in B issuing from the base point pt, $\alpha : [0,1] \to B$, $\alpha(0) =$pt. In this set, consider the equivalence relation $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $\alpha\beta^{-1} \in G$ and then take the quotient (=фактор) of this set by this equivalence relation: $E := E(B, \mathrm{pt})/_\sim$. Define the map $p : E \to B$ by assigning to each equivalence class of paths their common endpoint (=с общим концом). Then E is a space (in a certain sense), p is a covering projection and one can show that $p_\#(\pi_1(E)) = G$. $\square$

§8.4. The category of regular covering spaces

A covering $p : E \to B$ is called *regular* if the subgroup $p_\#(\pi_1(E)) \subset \pi_1(B)$ is normal. Then each set $p^{-1}(b)$, $b \in B$ is an orbit of the action of the quotient group $G := \pi_1(B)/p_\#(\pi_1(E))$ on E; thus (E, G) is a geometry in the sense of Lecture 1.

A *morphism* of regular covering spaces $p : E \to B$, $q : F \to C$ is a pair of maps $\varphi : B \to C$, $\widetilde{\varphi} : E \to F$ such that $\varphi \circ p = q \circ \widetilde{\varphi}$. Thus the regular coverings form a category (which is a subcategory of the category of all geometries). Two regular coverings are called *equivalent* if there is a morphism between them such that the two maps φ, $\widetilde{\varphi}$ are bijective.

If we fix the base B in the category of regular coverings, we obtain a subcategory (of all regular coverings with base B). This subcategory has a natural order relation (=отношение порядка): we say that the covering $p' : E' \to B$ is *higher* than the covering $p : E \to B$ if there exists a covering $q : E' \to E$ such that $p \circ q = p'$. Is there a highest covering, i.e., is there a covering that "covers" any other covering? The (positive) answer is given by the following theorem.

Theorem 4. *For an arbitrary (nice) space B there exists a highest covering space, namely any covering $p : E \to B$ for which $\pi_1(E) = 0$.*

Sketch of the proof. To construct the required covering (called the *universal covering*) apply the construction in the proof of Theorem 3 with $G = 0$. $\square$