LECTURE 4. SPHERICAL GEOMETRY

So far we have studied finite and discrete geometries, i.e., geometries in which the main transformation group is either finite or discrete. In this lecture, we begin our study of infinite continuous geometries with spherical geometry $(S^2 : O(3))$, the geometry of the isometry group of the sphere, which is in fact $O(3)$, the subgroup of all isometries of $\mathbb{R}^3$ that map the origin to itself ($O(3)$ is known as the *orthogonal group* in linear algebra courses). But first we list the most important classical continuous geometries.

§4.1. A list of classical continuous geometries

• The *finite-dimensional vector spaces* over the field of real numbers are actually geometries in the sense of the main definition of Lecture 1. From that point of view, they can be written as

$$\boxed{(V^n : GL(n))},$$

where V^n denotes n-dimensional vector space over $\mathbb{R}$ and $GL(n)$ is the *general linear group*, i.e., the group of all non-degenerate linear transformations of V^n to itself. These "geometries" are rather algebraic and will be studied in your linear algebra course.

• The *affine spaces* are, informally speaking, vector spaces "without a fixed origin". This means that their transformation groups $Aff(n)$ contain, besides $GL(n)$, all parallel translations of V^n. We denote the corresponding geometry by

$$\boxed{(V^n_\bullet : Aff(n))};$$

here the dot $\bullet$ in the subscript simply means that the elements of V^n are now called *points*, not vectors. This is a more geometric notion than that of vector space, but will also be studied in your linear algebra course.

• The *Euclidean spaces* are geometries that we denote

$$\boxed{(\mathbb{R}^n : O(n))};$$

here $O(n)$ is the isometry group of $\mathbb{R}^n$ (called the *orthogonal group* in linear algebra courses). We assume that you are familiar with Euclidean space from school and know the structure of the isometry group $O(n)$ for $n = 2, 3$.

• The *hyperbolic spaces* $Lob(n)$ (called *Lobachevsky spaces* in Russia) are "metric spaces of constant negative curvature" (you will learn what this means much later, in differential geometry courses) with transformation group the isometry group of the hyperbolic space (i.e., the group of transformations preserving the distance). We will only study the hyperbolic space of dimension $n = 2$, i.e., the hyperbolic plane ("*ploskost' Lobachevskogo*" in Russian) in the form of the *Poincaré disk model*, which we denote

$$\boxed{(D^2 : Moeb);}$$

here $D^2 := \{(x, y) \in \mathbb{R}^2 | x^2+y^2 \leq 1\}$ is the unit disk and $Moeb$ is the group of *Möbius transformations* (which will be defined in Lecture 6). Another lecture will be devoted to the axiomatic construction of hyperbolic plane geometry and the dramatic history of its creation by Gauss, Lobachevsky and Bolyai.

• The *projective spaces* $\mathbb{R}P^n$ are obtained from affine spaces by "adding points at infinity" in a certain way, and taking, for the transformation group, a group of linear transformations on the so-called "homogeneous coordinates" of points $(x_1 : \cdots : x_n : x_{n+1}) \in \mathbb{R}P^n$. We can write this geometry as

$$\boxed{(\mathbb{R}P^n : Proj(n)).}$$

For arbitrary n, projective geometry is usually studied in linear algebra courses. We will only study the *projective plane* $\mathbb{R}P^2$ in this course (see Lecture 7).

• The *elliptic spaces* are "metric spaces of constant positive curvature" (you will learn what this means later, in differential geometry courses) with transformation group the isometry group of the elliptic space. We will only study a particular case of elliptic geometry, namely *spherical geometry*, which is the topic of this lecture.

$$\boxed{(S^2 : O(3)).}$$

§4.2. Lines, distances, polars, and perpendiculars on S^2

Let S^2 be the unit sphere in $\mathbb{R}^3$: $S^2 := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. By a *line* on the sphere we mean a great circle, i.e., the intersection of S^2 with a plane passing through its center. Any two lines intersect in two *opposite* points, i.e., in two points symmetric with respect to the center of S^2. Therefore *there are no parallel lines in spherical geometry.* It is proved in differential geometry courses that *spherical lines are geodesics*, i.e., they are the shortest paths between two points; we do not need this fact and omit the proof (based on the differential calculus).

If two points A, B are not opposite, then there is only one line joining them and one shortest line segment with endpoints at A and B. For opposite points, there is an infinity of lines joining them (for the North and South pole, these lines are the meridians).

By the *distance* between two points on S^2 we mean the length of the shortest line segment joining the points; for example, the distance between the North and South poles on the unit sphere is equal to π. Clearly, elements of the group $O(3)$ map lines to lines and preserve the distance between points.

Given any line l, the two points of intersection of the straight line in $\mathbb{R}^3$ perpendicular to the plane containing l with S^2 are called the *poles* of l. Conversely, the line l is called the *polar* of each of these points.

Two lines are called *perpendicular* if the planes that they determine in $\mathbb{R}^3$ are perpendicular. If P is a point and l is a line, then there is a unique line passing through P and perpendicular to l, provided l is not the polar of P; if l is the polar of P, then there are infinitely many perpendiculars (for example, all meridians pass through the North pole and are perpendicular to the equator).

§4.3. Angles and triangles on S^2

The measure of the *angle* between two half lines with common end point is defined as the measure of the angle (in $\mathbb{R}^3$) between the two rays tangent to the half lines (recall that lines are great circles, so that the tangent rays lie in the plane of the corresponding great circle); angles are measured in radians.

Let A, B, C be three distinct points of S^2, no two of which are opposite. The union of the shortest line segments joining the points A and B, B and C, C and A is called the *triangle* ABC. For a triangle ABC, we always denote by α, β, γ the measure of the angles opposite to A, B, C and by a, b, c the lengths of the sides opposite to A, B, C (note that the length a is equal to the measure of the angle BOC in $\mathbb{R}^3$).

Theorem 1. *The area S_{ABC} of a spherical triangle with angles α, β, γ is equal to*

$$\boxed{S_{ABC} = \alpha + \beta + \gamma - \pi}.$$

Proof. First we note that the area of a spherical *biangle* (the spherical region bounded by two spherical lines) is equal to 2φ, where φ is the angle between the lines. There are 12 spherical biangles formed by pairs of lines AB, BC, CA. Choose six of them, namely those that contain triangle ABC or the opposite triangle $A_1B_1C_1$ formed by the three points opposite to A, B, C. Each point of the triangles ABC and $A_1B_1C_1$ is covered by exactly three of the chosen six biangles, while the other points of the sphere are covered by exactly one such biangle (we ignore the points on the lines). Therefore,

$$4(\alpha+\beta+\gamma) = 4\pi + 2S_{ABC} + 2S_{A_1B_1C_1}.$$

But the two triangles have the same area (because they are congruent), so that the previous formula implies the required equality. □

This theorem has the following fundamental consequence.

Corollary. *The sum of angles of any triangle is more than π.*

Theorem 2 (The " sine theorem")

$$\boxed{\frac{\sin a}{\sin\alpha} + \frac{\sin b}{\sin\beta} + \frac{\sin c}{\sin\gamma}.}$$

Proof. Let K be the projection of A on the plane ABC, let L and M be the projections of A on the lines OB and OC. Then L and M coincide with the projections of K on OB and OC. Therefore,

$$AK = LA\sin\beta = \sin c \sin\beta, \quad AK = MA\sin\gamma = \sin b \sin\gamma.$$

Thus, $\sin b : \sin\beta = \sin c : \sin\gamma$. Similarily, $\sin b : \sin\beta = \sin a : \sin\alpha$. This immediately implies the required equality. □

Theorem 3. (The first "cosine theorem".)

$$\boxed{\cos a = \cos b\cos c + \sin b\sin c\cos\alpha}$$

The proof of this theorem and all the subsequent ones will be discussed in the exercise class.

Theorem 4. (The second "cosine theorem".)

$$\boxed{\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos c}$$

Theorem 5. *The medians of any triangle intersect at a single point.*

Theorem 5. *The altitudes of any triangle intersect at a single point.*

§4.4. Circles on the sphere S^2

A *circle* of radius r centered at the point A is defined as the set of points P for which the distance AP is equal to r. It is easy to prove that the circle lies in a plane perpendicular to OA (where O is the center of S^2) and is therefore a Euclidean circle (the intersection of S^2 with that plane). Note that the Euclidean radius of the circle is smaller than its spherical radius, and there exist circles of maximal radius (equal to $\pi/2$): they are actually spherical lines (the polars of their centers).

§4.5. Coxeter triangles on the sphere S^2

We will not develop Coxeter geometry on the sphere S^2 in full generality, but only consider *Coxeter triangles*, i.e., spherical triangles all of whose angles are of the form π/m, $m = 2, 3, \ldots$. It follows from Theorem 1 that any spherical Coxeter triangle $(\pi/p, \pi/q, \pi/r)$, where $p \le q \le r$, must satisfy the Diophantine equation

$$1/p + 1/q + 1/r = 1.$$

This equation has the following solutions

$$(2,3,3), (2,3,4), (2,3,5), (2,2,n) \quad \text{for} \quad n = 2, 3, \ldots.$$

If a Coxeter triangle is chosen of the appropriate size, then the group generated by reflections with respect to its sides is finite and defines a *paving* of the sphere whose fundamental domain is this triangle. In Lecture 2, we found all the finite subgroups of $O(3)$, so that it easy to determine all pavings of S^2 (there are infintely many of them, because there infinitely many dihedral groups). Some of the nicest pavings of the sphere are shown on the pictures.

§4.6. The n-dimensional case: spherical and elliptic geometries

The geometry of the n-dimensional sphere $(S^n : O(n+1))$ is similar in many ways to the geometry of S^2; lines, distances, angles, etc. are defined in the same way, but of course there are "multidimensional" objects which have no analogs for $n = 2$.

Spherical geometry is closely related to the *elliptic geometry* invented by Riemann. Elliptic geometry is obtained from spherical geometry by "identifying opposite points of S^n". For some reason, elliptic geometry is less popular than the hyperbolic geometry of Lobachevsky, and we do not study it in this course.

However, you should understand that hyperbolic and elliptic geometries are in a sense opposite to each other (for example, the sum of angles of triangles in one is less than π, and more than π in the other), while Euclidean geometry (which in this context should be called *parabolic*) is the limit between them (thus the sum of angles of triangles in exactly equal to π).