Lecture 6. HYPERBOLIC GEOMETRY: THE DISK MODEL

In this lecture we describe the disk model of hyperbolic geometry (=) due to Henri Poincaré and discuss the physical and philosophical implications of non-Euclidean geometry.

§6.1. Inversion and its properties

The main tool (=) that we will need in this lecture is inversion, a classical transformation from elementary plane geometry. Consider the plane $\mathbb{R}^2$ with an added extra point (called the *point at infinity* and denoted by ∞); the set $\mathsf{R} = \mathbb{R}^2 \cup \infty$ is often called the *Riemann sphere*. An *inversion* of center $O \in \mathbb{R}^2$ and radius $r > 0$ is the transformation that maps each point M to the point N on the ray (=) OM such that $|OM| \cdot |ON| = r^2$ and transposes the points O and ∞. Sometimes inversions are also called symmetries with respect to the *circle of inversion*, i.e., the circle of radius r centered at O. If the Riemann sphere is interpreted as the complex plane $\mathbb{C}$ with ∞ added, then an example of an inversion is the map $z \mapsto 1/\overline{z}$, where the bar over z denotes complex conjugation (=). Any inversion is a bijection of R and possesses the following properties.

(i) Each inversion maps any circle or straight line into a circle or a straight line.

(ii) Each inversion preserves angles.

(iii) Each inversion maps each circle (straight line) orthogonal to the circle of inversion onto itself.

(iv) Each inversion preserves the double ratio (=).

The proofs of these statements will be discussed in the exercise class.

§6.2. Points and straight lines in the disk model

The disk model of *hyperbolic geometry* is a geometry (H^2, M) whose points are the points of the Riemann sphere contained in the open disk

$$H^2 := \{(x, y) | x^2 + y^2 < 1\}$$

(the group action of M on H^2 will be defined later). We will often call H^2 the *hyperbolic plane*. The boundary circle $x^2 + y^2 = 1$ (which is not part of the hyperbolic plane H^2) is called the *absolute*. By definition, the *straight lines* of H^2 are the intersections with H^2 of all the circles orthogonal to the absolute and of all the diameters of the absolute.

Theorem 1. *One and only one straight line passes through any pair of distinct points of the hyperbolic plane.*

Sketch of the proof. The proof mainly follows from the elementary fact of Euclidean geometry that there is one and only one circle passing through two given points and orthogonal to a given circle. □

§6.3. Perpendiculars

Two lines are called *perpendicular* if they are orthogonal in the sense of elementary Euclidean geometry. (When both are diameters, this means they are perpendicular in the usual sense, when both are arcs of circles, this means that their tangents at the intersection point are perpendicular, when one is an arc and the other is a diameter, this means that the Euclidean line containing the diameter passes through the center of the circle containing the arc.)

Theorem 2. *There is one and only one line passing through a given point perpendicularly to a given line.*

Sketch of the proof. Translated to the language of circles in the Euclidean plane, this statement becomes an exercise in the elementary geometry of circles. □

§6.4. Parallels

Let l be a line and P be a point of the hyperbolic plane H^2 not contained in the line l. Denote by A and B the points where the continuation of l intersects the absolute. Consider the lines $k = PA$ and $m = PB$, and denote their second intersection points with the absolute by A' and B'. Clearly, the lines k, m do not intersect l. Moreover, any line passing through P between k and m (i.e., any line through P joining the arcs AA' and BB') does not intersect l. The lines APA' and BPB' are called *parallels* to l passing through P, the lines between them are called *nonintersecting* lines to l through P. We have proved the following statement.

Theorem 3. *There is more than one line passing through a given point P not intersecting a given line l if $P \notin l$. These lines are all located between the two parallels to l through P.* □

This is the first non-Euclidean theorem in our model; from the point of view of the history of mathematics, it is the main one.

§6.5. Triangles

Consider three points A, B, C not on one line. The three segments AB, BC, CA (called *sides*) form the *triangle* with *vertices* A, B, C. The *angles* of the triangle, measured in radians, are defined as the angles between the tangents (in the Euclidean sense) to the sides of the triangle at the vertices.

Theorem 4. *The sum of angles of any triangle is less than π.*

This will be proved in the exercise class.

By abuse of language, we will also call triangle the configuration determined by three points not on one line when one, two, or even all three vertices (called *improper* (=)) lie on the absolute. Note that the angle at an improper vertex is zero. Thus there exists a "very big" triangle whose sum of angles is zero!

§6.6. The isometry or Möbius group of H^2

Here we will specify (=) the transformation group M of the Poincaré geometry H^2. By definition, M is the group generated by reflections in all the lines of H^2; it will be called the *Möbius group* of the disk H^2. (Here by a reflection in the line l, when l is an arc, we mean the symmetry in the circle containing the arc, and when l is a diameter of H^2, we mean the symmetry with respect to the straight line containing the diameter. The fact that these reflections are bijections of H^2 onto itself will be proved in the exercise class.)

It is an interesting exercise in the elementary Euclidean geometry of circles to define the notions of *parallel translation* and of *rotation* for the model under consideration and study their properties.

Since M is indeed a transformation group acting on H^2 by bijections, the pair (H^2, M) is a geometry. This geometry is called the *hyperbolic plane* (=).

In the next lecture we will define the distance between points of H^2 and learn that M is actually the isometry group of H^2 with respect to this distance.

§6.7. The physics of H^2 according to Poincaré

In his famous book *Science and Hypothesis*, Henri Poincaré describes the physics of a small "universe" and the physical theories that its inhabitants will create. The universe (=) is Euclidean, flat (=), and has the form of a unit disk without boundary. Its temperature is 36° at the center of the disk and decreases along radii linearly to zero at its boundary. The lengths of all objects (including live objects) are proportional to temperature.

How will a little flat creature endowed with reason (=) and living in this disk describe the main physical laws of his universe? The first question he may ask is: Is the world finite or infinite? To answer this question an expedition is organized; but as the expedition moves towards the boundary of the disk, the legs of the explorers (=) become smaller, their steps shorter – they will never reach the boundary, and conclude that the world is infinite.

The next question may be: Does the temperature in the universe vary? Having constructed a thermometer (based on different expansion coefficients of various materials), scientists carry it around the universe and take measurements. However, since the lengths of all objects change similarly with temperature, the thermometer gives the same measurements all over the universe – the scientists conclude that the temperature is constant.

Then the scientists might study straight lines, i.e., investigate what is the shortest path between two points. We will not prove this now, but it can be shown that the shortest path will be not be the line segment joining the given points but the arc of the circle passing through these points and orthogonal to the boundary of the disk. This shortest line is unique.

For the same reasons as above, the inhabitants of Poincaré's universe will come to the conclusion that there is more than one parallel passing through a point outside a line and that the sum of angles of a triangle is less than π. Thus they will come to the conclusion that they live in an infinite flat universe with constant temperature governed by the laws of hyperbolic geometry. But this is not true – their universe is a finite disk, its temperature is variable (tends to zero towards the boundary) and the geometry is Euclidean, not hyperbolic!

The philosophical conclusion of Poincaré's argument is not agnosticism – he goes further. The example described above, according to Poincaré, shows not only that the truth about the universe cannot be discovered, but that it makes no sense to speak of any "truth" or approximation of truth in science – pragmatically, the inhabitants of H^2 are right to use hyperbolic geometry as the foundation (=) of their physics because it is convenient, and it is meaningless to search for any abstract Truth which only complicates matters.

However, there is a flaw (=) in Poincaré's argument. Perhaps you will discover it if you think about the design of thermometers and about what physicists call *dimension considerations* (=).