Lecture 7. THE POINCARÉ HALF-PLANE MODEL

In this lecture, we will present another model of the hyperbolic plane, also due to Poincaré. This model is a geometry in the sense of Lecture 1, and we will prove that it is isomorphic (as a geometry) to the disk model studied in the previous lecture. This will allow us to return to the disk model and, in particular, to define a distance function on it, and study "hyperbolic trigonometry".

§7.1. Affine and linear-fractional transformations of $\overline{\mathbb{C}}$

In this lecture we will be studying the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ and some of its subsets under the action of the Möbius group (for the definition of this group, see below) and some of its subgroups.

A transformation of $\overline{\mathbb{C}}$ of the form $z \mapsto az + b$, $\infty \to \infty$, where $a, b \in \mathbb{C}$ and are not both zero, is called *affine*. Denoting $a = re^{i\varphi}$, $r > 0$, we can write

$$z \mapsto e^{i\varphi} z \mapsto r(e^{i\varphi} z) \mapsto (re^{i\varphi} z) + b = az + b,$$

which shows that any affine transformation is the composition of a rotation (by the angle φ), a homothety (with coefficient r), and a parallel translation (by the vector b). This implies the following theorem:

Theorem 1. *Affine transformations take straight lines to straight lines, circles to circles, and preserve angles and cross products.*

Recall that the *cross-product* of four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ is the number

$$\boxed{\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.}$$

The proof of the fact that the cross-ratio is preserved by affine transformations is an easy particular case of an exercise that will be discussed in the exercise class.

A transformation of $\overline{\mathbb{C}}$ given on $\mathbb{C}$ by

$$\boxed{z \mapsto \frac{az + b}{cz + d},}$$

where $ac - bd \neq 0$ is called *linear-fractional*. The set of all linear-fractional transformations form a group called the *Móbius group* and denoted by Möb.

Denoting $z_1 = az + b$, $z_2 = cz + d$, we can write

$$z \mapsto \frac{1}{z_2} \cdot z_1 = \overline{\left(\frac{1}{\overline{z_2}}\right)} \cdot z_1 = \frac{az+b}{cz+d},$$

which shows that any linear fractional transformation is the composition of affine transformations, reflections in the line $Imz = 0$ and an inversion. This, together with assertions (i)–(iv) of §6.1 of the previous lecture, implies the following statement:

Theorem 2. *Linear-fractional transformations take straight lines and circles to straight lines or circles, and preserve angles and cross products.*

The last statement of the theorem (preservation of the cross-ratio) follows from the exercise already mentioned above.

The two following examples of linear-fractional transformations are very important for what follows.

Example 1. The linear-fractional transformations that take the upper half plane to itself are given by the formulas

$$z \mapsto \frac{az+b}{cz+d} \quad \text{and} \quad z \mapsto \frac{a(-\overline{z})+b}{c(-\overline{z})+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ac - bd > 0$. This follows because for any z in the upper half-plane we have

$$Im\,\frac{az+b}{cz+d} = Im\,\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = Im\,\frac{adz+bc\overline{z}}{|cz+d|^2} = \frac{(ad-bc)Im\,z}{|cz+d|^2},$$

which is positive iff $ac - bd > 0$. The set of all such linear-fractional transformations obviously constitute a group, which we denote by $\mathbb{R}$Möb.

Example 2. The linear-fractional transformation

$$\Omega : z \mapsto i \cdot \frac{1+z}{1-z}$$

maps the unit disk $\mathbb{D}^2 := \{z \in \mathbb{C} : |z|^2 \leq 1\}$ to the upper half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : Im\,z > 0\}$.

This follows from Theorem 2 because the points $-1, 0, 1$ are mapped to $0, i, \infty$, respectively.

§7.2. The Poincaré half-plane model

The *Poincaré half-plane model* is a geometry consisting of the points $z \in \mathbb{C}$ such that $Im\, z > 0$, supplied with the transformation group $\mathbb{R}$Möb. In this geometry, *straight lines* are defined either as open half circles (in the upper half-plane) perpendicular to the line $Im\, z = 0$ (which is called the *absolute*) or as the open rays $\{z \in \mathbb{C} : Rez = x_0 \in \mathbb{R}, Im\, z > 0\}$.

Let us define the *Möbius distance* $\mu(A, B)$ between two points A, B of the upper half-plane by setting

$$\boxed{\mu(A, B) := |\ln(\langle A, B, X, Y\rangle)|,}$$

where X and Y are the intersection points of the line (AB) with the absolute if the points A, B have different real parts, and setting

$$\boxed{\mu(A, B) := |\ln(\langle A, B, \infty, X\rangle)|,}$$

if $Re(A) = Re(B) = x_0$ and X is the point of the absolute with x-coordinate equal to x_0.

Theorem 3. *The isometry group of the upper half plane with respect to the distance μ coincides with the group $\mathbb{R}$Möb described in Example 1.*

The proof is a straightforward verification that we omit.

§7.3. Isomorphism between the half-plane and disk models

It turns out that the half-plane model is isomorphic to the disk model studied in the previous lecture. More precisely, we have the following result:

Theorem 4. *The map Ω from Example 2 defines an isomorphism of the geometry $(\mathbb{D}^2 : Möb_0)$ (the disk model from Lecture 6) and the geometry $(\mathbb{C}_+ : \mathbb{R}Möb)$ (the Poincaré half-plane model) if we define the corresponding isomorphism (which we denote by Δ) of the groups $\mathbb{R}$Möb and Möb$_0$ by setting*

$$\mathrm{Mob}_0 \ni g \mapsto \Omega \circ g \circ \Omega^{-1} \in \mathbb{R}\mathrm{Mob}.$$

Proof. The map Ω is one-to-one because it has the obvious inverse given by the rule $w \mapsto (i - w)/(i + w)$. The isomorphism Δ is compatible with Ω by definition.

Now let us define the *Lobachevsky distance* λ between two points A, B of the disk $\mathbb{D}^2$ by setting

$$\boxed{\lambda(A, B) := |\ln(\langle A, B, X, Y\rangle)|,}$$

where X and Y are the intersection points of the line (AB) with the absolute.

Now Theorems 2, 3, and 4 immediately imply the following result:

Corollary. *The group of isometric transformations of the disk with respect to the distance λ coincides with the group $Möb_0$ generated by all reflections in the "straight lines" of the disk model.*

§7.4. Hyperbolic functions

The complex exponent e^z is related to the ordinary trigonometric functions by the beautiful *Euler formula*:

$$\boxed{e^{i\varphi} \;=\; \cos\varphi + i\sin\varphi,}$$

whose proof is obvious if we consider the trigonometric circle in the plane $\mathbb{C}$.

The hyperbolic functions sh, ch, th, cth (*hyperbolic sine, cosine, tangent, cotangent*, respectively) are defined by the formulas

$$\mathrm{sh}\, x := \frac{e^x - e^{-x}}{2} \qquad \mathrm{ch}\, x := \frac{e^x + e^{-x}}{2}$$

$$\mathrm{th}\, x := \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \mathrm{cth}\, x := \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

These functions satisfy formulas similar to the formulas for ordinary trigonometric functions (the proofs are straightforward verifications from the definitions). Here are some examples:

$$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1, \quad \mathrm{th}\, x\, \mathrm{cth}\, x = 1, \quad \mathrm{sh}(x \pm y) = \mathrm{sh}\, x\, \mathrm{ch}\, y \pm \mathrm{ch}\, x\, \mathrm{sh}\, y,$$

$$\mathrm{sh}\, 2x = 2\,\mathrm{sh}\, x\, \mathrm{ch}\, x, \quad \mathrm{ch}\, 2x = \mathrm{sh}^2 x + \mathrm{ch}^2 x, \quad \mathrm{ch}(x \pm y) = \mathrm{ch}\, x\, \mathrm{ch}\, y \pm \mathrm{sh}\, x\, \mathrm{sh}\, y,.$$

§7.5. Trigonometry on the hyperbolic plane

Because of Theorems 2 and 4, the elementary trigonometric formulas of hyperbolic geometry are exactly the same for the half-plane and the disk model. Their proof is quite straightforward (perhaps a little simpler in the case of the half-plane) and are omitted. We state them in the form of theorems. Below ABC is a triangle, α, β, γ are the angles opposite to A, B, C, respectively, and a, b, c the sides opposite to A, B, C, respectively.

Theorem 5. [Hyperbolic sine theorem] *The following formula holds:*

$$\boxed{\frac{\operatorname{sh} a}{\sin\alpha} = \frac{\operatorname{sh} b}{\sin\beta} = \frac{\operatorname{sh} c}{\sin\gamma}.}$$

Theorem 6. [Hyperbolic cosine theorem] *The following formula holds:*

$$\boxed{\operatorname{ch} a = \operatorname{ch} b \operatorname{ch} c - \operatorname{sh} c \operatorname{sh} b \cos\alpha.}$$

§7.6. The angle of parallelism and the Schweikart constant

Let (AB) be a line in hyperbolic geometry (we can use either one of the two models here) and C be a point not on (AB); let X and Y be the intersection points of the line (AB) with the absolute, so that the rays $[CX)$ and $[CY)$ are the parallels to (AB) passing through C; let $[CH]$, $H \in (AB)$, be the perpendicular lowered from C to (AB); let $d := \lambda(C, H)$ be the Lobachevsky distance between C and H; finally, let α be the measure of the angle XCH (or, which is the same, of YCH). Then it is not difficult to prove that α depends only on d (do it!); α is called the *angle of parallelism.*

Theorem 7. *The angle of parallelism α is given by the formula:*

$$\boxed{\operatorname{ctg} d \;=\; \cos\alpha.}$$

The proof is a calculation based on the fact that $d = |\ln\langle A, H, X, Y\rangle|$.

Now let O be the center of the disk model and let $[OA)$ and $[OB)$ be perpendicular rays issuing from O; let X and Y be the intersection points of the rays $[OA)$ and $[OB)$ with the absolute; let (CD) be the line intersecting the absolute at A and B; let $[OH]$, $H \in (CD)$, be the perpendicular lowered from O to (CD); let $\sigma := \lambda(O, H)$ be the Lobachevsky distance between O and H. The number σ is called the *Schweikert constant*; it is an absolute constant of the hyperbolic plane.