# Lecture 8. HISTORY OF NON-EUCLIDEAN GEOMETRY

In this lecture, we will retrace the history of the creation of non-Euclidean geometry by Gauss, Lobachevsky, and Bolyai (and their predecessors and followers) and discuss the traditional axiomatic approach to the foundations of geometry. The story begins with Euclid's *Elements*, the brilliant first attempt to construct mathematics as a deductive science.

## §8.1. Euclid's fifth postulate

The Ancient Greeks realized that, in a deductive science, in order to deduce (prove) facts from other facts by logical reasoning, it is necessary to start from some facts which are not proved. Euclid called these facts *postulates* (we call them *axioms*) and explicitly formulated five of them. He also used several other axioms implicitly (without formulating them). Apparently, Euclid (and other Greek mathematicians) thought that the postulates should be self-evident (simple and so obvious that no doubt about their truth could arise).

Euclid's last axiom, the *fifth postulate*, however, is not simple and not obvious. Its modern equivalent it can be stated as follows:

(V+) *For any straight line and any point not on this line there is a unique parallel to this line passing through the given point.*

Here by a *parallel* to a given line one means a straight line that has no common points with the given line. In Euclid's formulation, the statement was more complicated and less obvious.

(V) *If a straight line falling on two straight lines makes the sum of the interior angles on one side less than two right angles, then the two straight lines, if extended indefinitely, meet on that side on which are the angles with sum less than two right angles.*

Presumably, Greek mathematicians (perhaps Euclid himself) tried to deduce the fifth postulate from the other axioms. In any case, in Euclid's *Elements*, the application of the fifth postulate is postponed as much as possible: it occurs for the first time in the proof of Proposition 27 of Book 1 (there are 48 propositions, i.e., theorems in our terminology, in that book).

After Euclid, for two thousand years, many scientists tried to prove the fifth postulate, and many "succeeded", usually by proving statements equivalent to (V) by means of arguments based on additional axioms which were not explicitly formulated.

## §8.2. Statements equivalent to the fifth postulate

We have already mentioned one such statement, namely (V+). Here are some more (in square brackets [ ], we indicate the mathematician who used this approach to "prove" the fifth postulate).

(1) *The sum of the three angles of any triangle is equal to* $\pi$ (*to two right angles, in Euclid's terminology*). [This statement appears in Euclid's *Elements* as Proposition 32, and was proved by using the fifth postulate; Legendre gave a "proof" in 1805 without the fifth postulate.]

(2) *A line intersecting one of two parallel lines intersects the other.* [Proclus, 5th century]

(3) *Similar but not congruent triangles exist.* [John Wallis, 1663]

(4) *The fourth angle of a quadrilateral with three right angles is also a right angle.* [Nasiraddin, 13th century, Saccheri, 1679, Lambert, 1776] . Such a quadrilateral was later called a *Saccheri quadrilateral.*

Trying to prove the fifth postulate, most mathematicians (including those mentioned above) argued by contradiction. As a rule, they considered two cases, assuming that the sum of angles of a triangle is (a) more than $\pi$ or (b) less than $\pi$ (equivalently, that the fourth angle of the Saccheri quadrilateral is more (less) than $\pi/2$, or that there are no parallels, respectively more than one parallel, through a given point to a given line). In the first case, it is possible to correctly obtain a contradiction using the Euclidean axioms. In the second case, a contradiction does not follow, but the desire to prove the fifth postulate was so strong, that the mathematicians working on the problem usually produced an incorrect proof.

## §8.3. Gauss

Carl Friedrich Gauss (1777–1855) first began working on the fifth postulate in 1796, at the age of nineteen, and argued by contradiction, like his predecessors, but went much further in developing the theory in case (b). It is not clear when he came to the conclusion that no contradiction would arise. In a famous letter (1824) to his friend F.A.Taurinus, he explained that in the case $\alpha + \beta + \gamma < \pi$ one obtains a "thoroughly consistent curious geometry", which he called "non-Euclidean". He concluded his letter by asking Taurinus not to tell anyone about his "private communication", which he was thinking of publishing at "some future time" .

Later, in 1832, he learned from his friend Farkas Bolyai, that the latter's son, Janos, had arrived at the same conclusions, and, in 1841, that

Lobachevsky had done the same. Gauss even learned Russian to read Lobachevsky's early work, but never directly communicated with either Janos Bolyai or Lobachevsky.

The most amazing thing, however, is that Gauss, when he was not thinking about number theory or the fifth postulate, had constructed the differential geometry of surfaces, including surfaces of constant negative curvature, which are, in fact, a model (at least locally) of hyperbolic geometry. All these years, he had this model before his eyes, but never made the obvious connection with non-Euclidean geometry. He died without suspecting that a proof of the consistency of hyperbolic case was at his finger tips!

### §8.4. Lobachevsky

N.I.Lobachevsky (1793–1856), like everybody else, tried to prove the fifth postulate by contradiction. As he progressed further in the case $\alpha+\beta+\gamma < \pi$, he became convinced that the theory was consistent. In an unpublished textbook, written in 1823, he mentions that all attempts to prove the fifth postulate were erroneous. In 1926, Lobachevsky read a paper in Kazan about a "new geometry" (which he later called imaginary), and published (in Russian) a memoir about it in the Kazan Bulletin, which was unnoticed abroad. Trying to gain recognition, he published his work in German (*Geometrische Untersuchugen*, 1840) and in French (*Pangéométrie*, 1855), but without success.

N.I.Lobachevsky was not only the President (rector, in the Russian terminology) of Kazan University, but also its Head Librarian. The Kazan library received many scientific periodicals, including the most famous mathematical journal of the time, *Crelle's Journal*. Library cards (which have come down to us) show that Lobachevsky read every issue of Crelle's Journal that reached Kazan, except two successive issues in the 1830ies. These two issues contained two papers by Mindling, in which the author obtained, on surfaces of constant negative curvature, trigonometric formulas identical to the trigonometric formulas previously obtained by Lobachevsky on the hyperbolic plane. Had Lobachevsky seen one of these papers, he would have immediately observed that they constituted a proof of the consistency of hyperbolic geometry!

### §8.5. Bolyai

Janos Bolyai (1802–1860) was the son of a mathematician, Farkas Bolyai, who had "proved" the fifith postulate (his friend Gauss had pointed out his error). Janos first followed in his father's footsteps by trying to prove the fifth postulate by contradiction, but soon realized that he was obtaining a consistent geometry. In 1823 he wrote to his father: "Out of nothing I have created a strange new universe". But it was only in 1932 (three years after Lobachevsky) that his investigations were published in an Appendix to his father's book *Tentamen*.

Farkas sent the book to Gauss, asking to comment on the Appendix. Instead of praising and encouraging Janos, Gauss wrote that this would be "praising myself", since he had discovered the same things thirty years before, and the Appendix "spared him the effort" of writing up his discovery. Discouraged, Janos Bolyai published nothing more...

### §8.6. Beltrami, Helmholtz, Lie, Cayley, Klein, and Poincaré

The first proof of the consistency of hyperbolic geometry is attributed to Beltrami, who showed (1868) that its axioms and theorems hold (at least locally) on surfaces of constant negative curvature. The physicist Helmholtz was probably the first to try to prove the consistency of hyperbolic geometry, but his arguments were regarded as not sufficiently rigorous by mathematicians. Sophus Lie improved the arguments of Helmhotz and was the first to stress the role of transformation groups in mathematics. Klein gave the definition of geometry that we introduced in Lecture 1, and, simultaneously with Cayley (but independently of him), gave an elementary global model of hyperbolic geometry; he also coined the terms *hyperbolic, parabolic, elliptic* for the three geometries. Poincaré constructed the two models of hyperbolic geometry that we discussed in the previous two lectures.

### §8.8. Hilbert

David Hilbert made the first successful attempt to give an axiomatic exposition of Euclidean (space) geometry, rigorous in the modern sense of the word. It consists of 21 axioms, three undefined concepts (*point, line, plane*), and several undefined relations. This approach is rarely used in modern times, because today Euclidean geometry is a branch of linear algebra over the real numbers which can be defined very easily (using the fact that the straight line is "isomorphic" to the real numbers $\mathbb{R}$).