Lecture 9. PROJECTIVE GEOMETRY

In this lecture, we introduce the main ideas of projective geometry for the particular case of $\mathbb{R}P^2$, the projective plane. The general theory for projective spaces ($\mathbb{R}P^n$, $n \geq 1$) is usually studied by means of the homogeneous coordinate model, so we only give the corresponding definitions – the general theory is part of linear algebra, and we do not develop it here.

§9.1. The projective plane as a geometry

The *projective plane* $\mathbb{R}P^2$ is defined as the geometry ($\mathbb{R}P^2 : GL(3)$), whose elements (called *projective points*) are straight lines in $\mathbb{R}^3$ passing through the origin O and whose transformation group is the group of linear non-degenerate transformations of $\mathbb{R}^3$ (called *projective transformations*). The elements of $\mathbb{R}P^2$ (projective points) are Euclidean lines, nevertheless, we will often simply call them *points* (of our geometry). The *straight lines* (of our geometry) are defined as the (Euclidean) planes passing through the origin. The definitions immediately imply the two following assertions.

I. *One and only one "line" passes through any two distinct "points".*

II. *Any two distinct "lines" intersect in one and only one "point".*

(The quotation marks indicate that we are talking of points and lines *in our geometry*, not Euclidean points and lines.) Thus there are no parallel lines in our geometry, just as in spherical geometry. But we will see that the two geometries are very different; in particular, there is no natural metric in projective geometry (and hence no measure of angles, no perpendiculars, no areas, and so on) and projective lines intersect in one point, not two.

One may ask: Why is $\mathbb{R}P^2$ called a "plane", when its elements lie in the *space* $\mathbb{R}^3$ and the transformation group is also in three dimensions? We will give an intuitive answer now (and a rigorous one in the next section). You can imagine the projective plane as a Euclidean plane to which a "line at infinity" I has been added. When you move along a Euclidean line L to infinity in some direction you intersect the line at infinity at some point $P = L \cap I$; if you move along L in the opposite direction, you will reach I and intersect it *at the same point* P. Moreover, any line L' parallel to L will intersect I (and hence L!) at the same point P. So lines in $\mathbb{R}P^2$ are some kind of cycles (like "infinite circles"). The line at infinity, however, should not be regarded as a "special" or "exceptional" line, because most projective transformations transform it into an "ordinary" line.

§9.2. The homogeneous coordinate model

Returning to our geometry $(\mathbb{R}P^2 : GL(3))$, let us introduce coordinates for our points. Each point L (i.e., Euclidean line passing through the origin) is uniquely determined by its *direction vector*, i.e., by three coordinates (x_1, x_2, x_3), in the standard coordinates of $\mathbb{R}^3$, namely

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1).$$

Conversely, however, points *do not* uniquely determine the coordinates: if λ is a non-zero real number, then $(\lambda x_1, \lambda x_2, \lambda x_3)$ determines the same point as (x_1, x_2, x_3). In this situation, we call the two sets of coordinates *equivalent*, denote the corresponding equivalence class by $(x_1 : x_2 : x_3)$, and refer to $\chi(L) = (x_1 : x_2 : x_3)$ as the *homogeneous coordinates* of the point L.

Homogeneous coordinates make the computation of the action of elements $g \in GL(n)$ on points $L \in \mathbb{R}P^2$ very easy: the transformation g is given by a 3×3 matrix A_g, and

$$g(L) = A_g((x_1 : x_2 : x_3)) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

In linear algebra courses, the *projective space* $\mathbb{R}P^n$, $n \geq 1$, is defined in a similar way: its elements are homogeneous coordinates $(x_1 : \cdots : x_n)$, i.e., equivalence classes of n-tuples $(x_1, \ldots, x_n)$ of real numbers (not all equal to zero) up to multiplication by a non-zero constant. The group $GL(n)$ acts on each element by multiplication by $n \times n$ matrices corresponding to the transformations (linear operators). We leave the study of $\mathbb{R}P^n$ to the linear algebra course.

Now let us give a rigorous model of $\mathbb{R}P^2$ that will explain why $\mathbb{R}P^2$ is called the projective *plane*. In $\mathbb{R}^3$ consider the plane Π given by the equation $x_3 = 1$. Points of this plane have the coordinates $(x_1, x_2, 1)$. To the plane Π add the *line at infinity* Σ_∞ whose *points* are equivalence classes of Euclidean points $(x_1, x_2, 0)$ up to multiplication by a non-zero constant (notation $(x_1 : x_2 : 0)$). The set $\Pi \cup \Sigma_\infty$ is the set of points of the projective plane.

Note that the "points at infinity" $(x_1, x_2, 0) \in \Sigma_\infty$ are Euclidean straight lines in the plane $x_3 = 0$. Intuitively, you should think of these lines as "pointing to infinity" in a certain direction, so that the set Σ_∞ "surrounds" the plane Π.

The *lines* in this model of $\mathbb{R}P^2$ are the ordinary (Euclidean) lines in Π plus the "line" Σ_∞. There is an obvious bijection between the points and lines of $\mathbb{R}P^2$ (as defined in the previous section); in particular, the line Σ_∞ corresponds to the (Euclidean) plane $x_3 = 0$. Using this bijection, it is easy to define the action of $GL(3)$ in this model. But we abandon the study of this model: we only described it to explain why the projective plane is two-dimensional.

§9.3. Ratio and cross-ratio of collinear points

We mentioned above that there is no natural metric on the projective plane. Nevertheless, the affine structure in $\mathbb{R}^3$ allows us to define the ratio of oriented collinear segments (measure how much longer one is than the other). The definition is the following. Let A, B, C be collinear points in $\mathbb{R}P^2$; then the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are proportional, i.e., $\overrightarrow{AC} = \lambda\overrightarrow{BC}$; the real number λ (which may be negative) is called the *ratio* of the ordered triple of points A, B, C and is denoted by $\langle A, B, C\rangle$.

Now suppose A, B, C, D are four collinear points. Then the number

$$\langle A, B, C, D\rangle := \frac{\langle A, B, C\rangle}{\langle A, B, D\rangle}$$

is called the *cross-ratio* of the points A, B, C, D.

The ratio and cross ratio are easy to compute in coordinates. Suppose the collinear points A, B, C have the coordinates:

$$(x_A, y_A, 1),\ (x_B, y_B, 1),\ (x_C, y_C, 1).$$

Then, obviously,

$$\frac{x_C - x_A}{x_C - x_B} = \frac{y_C - y_A}{y_C - y_B} = \langle A, B, C\rangle,$$

and therefore

$$\langle A, B, C, D\rangle = \frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - y_B}.$$

Similar formulas hold if one or more of the points are at infinity, provided we learn to perform arithmetical operations involving ∞.

For example, if A is on the infinite line, then we put $x_A = \infty$ and, because "the infinities cancel", the cross ratio becomes

$$\frac{x_C - \infty}{x_C - x_B} : \frac{x_D - \infty}{x_D - x_B} = \frac{x_D - x_B}{x_C - x_B} = \langle B, C, D\rangle.$$

The details of the other cases are left to the reader.

§9.3. Projective transformations

One may want to ask: Why is our geometry called "projective", when the group defining it is the group $GL(3)$ of *linear* operators in $\mathbb{R}^3$? Let us try to answer this question. Let Π_1 and Π_2 be two planes in $\mathbb{R}^3$ and let $P \in \mathbb{R}^3$ be a point. The *projection* of Π_1 to Π_2 from P is the map that to each point $A \in \Pi_1$ assigns the point $A' \in \Pi_2$ where the line PA intersects Π_2. This assignment is not necessarily bijective: it may be undefined at some points A (if PA is parallel to Π_2), and nothing will be mapped on some points of Π_2(when?).

However, if we supply Π_1 and Π_2 with lines at infinity Σ^1_∞ and Σ^2_∞, and appropriately define the projection, then we obtain a bijection between the projective planes $\Pi_1 \cup \Sigma^1_\infty$ and $\Pi_2 \cup \Sigma^1_\infty$. The details are left to the reader.

A set of points $A_1, \dots, A_n$, $n \geq 3$ are said to be *in general position* if for any three of them A_k, A_l, A_m, the vectors $\overrightarrow{OA_k}, \overrightarrow{OA_l}, \overrightarrow{OA_m}$ constitute a basis of $\mathbb{R}^3$.

Theorem 1. *There exists one and only one projective transformation that takes four points $A, B, C, D \in \mathbb{R}P^2$ in general position to four other points $A', B', C' \in \mathbb{R}P^2$* in general position.

Proof. The proof is a simple exercise in linear algebra. By assumption, the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ constitute a basis of $\mathbb{R}^3$. Let (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{23}), (a_{13}, a_{23}, a_{33}) be the coordinates of the vectors $\overrightarrow{OA'}, \overrightarrow{OB'}, \overrightarrow{OC'}$ in that basis. Then any element g of $GL(3)$ given by a matrix of the form

$$g = \begin{pmatrix} \lambda a_{11} & \mu a_{12} & \nu a_{13} \\ \lambda a_{21} & \mu a_{22} & \nu a_{23} \\ \lambda a_{31} & \mu a_{32} & \nu a_{33} \end{pmatrix}$$

obviously takes the points A, B, C to the points (A', B', C') (and only linear transformation of this form do this). Denote by (d_1, d_2, d_3) the coordinates of the point D in the basis $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$. Computing $g(\overrightarrow{OD})$, using the fact that all the d_i are non-zero, it is easy to find (non-zero) values of λ, μ, ν such that $g(\overrightarrow{OD}) = OD'$. $\square$

Theorem 2. *The ratio of three and the cross-ratio of four collinear points is invariant under projective transformations.*

Proof. Linear transformations take collinear vectors to collinear vectors, therefore the ratio of three collinear points is preserved, and this in its turn implies that the cross-ratio of four collinear points is preserved. The case when one or more of the points are on the infinite line must be treated separately and is left to the reader. □

§9.5. Projective duality

Points and lines on the projective plane $(\mathbb{R}P^2 : GL(3))$ play, in a certain sense, symmetric roles. This will be easier to see if we introduce the notion of *incidence*: we will say that two lines a and b are *incident at the point* P if P is the intersection point of the lines a and b, and that the two points P and Q are *incident at the line* a if a passes through P and Q. Also, together with the standard term *collinear* (used for points all lying on one line) we will use the term *copunctal* for lines all passing through one and the same point.

Given an assertion of projective geometry formulated in this terminology, we can translate it into another statement, called *dual*, by replacing the word "line" by the word "point" (and "collinear" by "‘copunctal") and vice versa. For example, statement I from §9.1 can be expressed as: "One and only one line is incident to two distinct points"; its translation (i.e., the dual statement) will be "One and only one point is incident to two distinct lines", which is exactly the assertion of II (see §9.1). Another example: "Any projective transformation takes collinear points to collinear points" translates to "Any projective transformation takes copunctal lines to copunctal lines".

What is remarkable is that this kind of translation always translates true statements to true statements. To prove this, we will define the *dual geometry* to the geometry of $\mathbb{R}P^2$: it is the geometry $(D\mathbb{R}P^2 : GL(3))$ whose *points* are planes of $\mathbb{R}^3$ passing through the origin under the action of the group of linear non-degenerate transformations of $\mathbb{R}^3$. In $D\mathbb{R}P^2$, the intersections of two points (i.e., Euclidean planes) will be called the *line passing through the points* (it as actually a Euclidean line in Euclidean 3-space).

Theorem 1. *The geometries $(D\mathbb{R}P^2 : GL(3))$ and $(\mathbb{R}P^2 : GL(3))$ are isomorphic: there is a bijection, called duality and denoted by D, between the sets of points of the two geometries which, together with the identical homomorphism of $GL(3)$ onto itself, defines their isomorphism.*

Proof. To each "point" Π of $D\mathbb{R}P^2$, i.e., to each plane of $\mathbb{R}^3$ given by the equation $a_1x_1+a_2x_2+a_3x_3 = 0$, we assign the point of $\mathbb{R}P^2$ with homogeneous coordinates $(a_1 : a_2 : a_3)$ (which is of course the Euclidean line passing through the origin and perpendicular to the plane). If an element $g \in GL(3)$ takes the point $(a_1 : a_2 : a_3)$ to some point $(b_1 : b_2 : b_3)$, then the same element will take the plane Π to the plane given by $b_1x_1 + b_2x_2 + b_3x_3 = 0$ (why?). Thus the duality map $D : \mathbb{R}P^2 \to D\mathbb{R}P^2$ (which of course is bijective) is compatible with the action of $GL(3)$, so that we have constructed the required isomorphism.

Note that the duality correspondence is an *involution*, i.e., $D \circ D$ identically maps $\mathbb{R}P^2$ onto itself. Further, note that the isomorphism constructed above preserves incidence: if two points A, B of $\mathbb{R}P^2$ (i.e., two Euclidean lines passing through the origin O of $\mathbb{R}^3$) are incident to the line l (i.e., are contained in a Euclidean plane Π_l), then the two lines $D(A), D(B)$ in $D\mathbb{R}P^2$ intersect in the point (of $D\mathbb{R}P^2$) $D(l) = \Pi_l$. Thus we have the following statement.

Corollary: Duality Principle. *There is a bijection between the set of lines and the set of points of $\mathbb{R}P^2$ that preserves incidence and takes any theorem of projective geometry to a theorem of projective geometry.*

§9.6. Conics in $\mathbb{R}P^2$

The non-degenerate conic sections (or *conics* for short) in the Euclidean plane are, as is well known, the ellipse, the hyperbola and the parabola. In $\mathbb{R}P^2$, these three curves are projectively equivalent, so that *there exists only one conic* (*up to projective equivalence*).

A conic in $\mathbb{R}P^2$ can be defined as any set of points obtained from the curve C given by $(x_1)^2 + (x_2)^2 = 1$ (in the plane-with-line-at-infinity model described in §9.2, this curve is the Euclidean circle) by a projective transformation. Any projective transformation that takes the line at infinity l_∞ to itself transforms C into an ellipse; a projective transformation that takes one point of C to l_∞ transforms C into a parabola, and a projective transformation that takes two points of C to l_∞ transforms C into a hyperbola.

§9.7. The Pappus, Desargues, and Pascal theorems

We conclude this lecture with three beautiful classical theorems.

Desargues' Theorem. *The lines joining the corresponding vertices of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at one point S. Then the intersection points P_1, P_2, P_3 of the lines A_2B_2 and A_3B_3, A_1B_1 and A_3B_3, A_1B_1 and A_2B_2, respectively, are collinear.*

Proof. We can assume, without loss of generality, that the configuration of points $S, A_1, A_2, A_3, B_1, B_2, B_3, P_1, P_2, P_3$ lies in the plane $x_3 = 1$ in $\mathbb{R}^3$. On the line OB_2 (here and further "line" means "Euclidean line"), where O is the origin of coordinates, take a point B_2' such that B_2 is between O and B_2'; denote by A_2' the intersection point of the lines SB_2 and OA_2. Then the lines SB_1, SB_2', SB_3 define a trihedral angle in $\mathbb{R}^3$. The (Euclidean) planes $\Pi_1 := (A_1A_2'A_3)$ and $\Pi_2 := (B_1B_2'B_3)$ intersect in a line Λ. Obviously, the lines A_2B_2' and A_3B_3 intersect at a point of Λ, and so do the lines $A_1'B_1$ and A_3B_3, as well as the lines $A_1'B_1$ and A_2B_2'. Call these intersection points P_1', P_2', P_3', respectively. But, by construction, the projections of these points from O to the plane $x_3 = 1$ will be the points P_1, P_2, P_3, and so the points P_1, P_2, P_3 are collinear.

Pappus' Theorem. *The points A_1, A_2, A_3 are collinear, and the points B_1, B_2, B_3 are collinear. Let P_1, P_2, P_3 be the intersection points of the lines A_2B_3 and A_3B_2, A_1B_3 and A_3B_1, A_2B_1 and A_1B_2, respectively. Then the points P_1, P_2, P_3 are collinear.*

Pascal's Theorem. *The points A, B, C, D, E, F lie on a conic. Let P_1, P_2, P_3 be the intersection points of the lines AB and ED, AF and CD, CB and EF, respectively. Then the points P_1, P_2, P_3 are collinear.*

We omit the (not very difficult) proofs of these theorems.