10.6. Problems

10.1. Let $\varphi : C_k(M) \to C^{n-k}(M_*)$ be the natural homomorphism between the simplicial chain group of a smooth closed manifold M^n and the cochain group of the dual cell decomposition M_* of M^n. Prove that $\delta_{n-k} \circ \varphi_k = \varphi_{k-1} \circ \partial_k$, where ∂ and δ are the boundary and coboundary operators.

10.2. Prove that if $H_1 M^3 = 0$ for a closed 3-manifold M^3, then M^3 is a homology 3-sphere, i.e., its homology groups as $\mathbb{S}^3$.

10.3*. Give an example of a homology 3-sphere which id not homeomorphic to the 3-sphere.

10.4. Let the torus be presented as a CW-space with four cells, let c^1 be the cochain taking one of the 1-cells to 1 and the other to 0, while c_1 is the chain with coefficient 1 at the second 1-cell and 0 at the other. Compute $c^1 \frown c_1$.

10.5. Prove the Leibnitz rule for the cap product.

10.6. Prove the following properties of the cap product:

(i) $f_*(f^*(b) \frown a) = b \frown f_*(a)$, where $f : X \to Y$ is a simplicial map, $a \in H_*X$, and $b \in H^*Y$;

(ii) $\langle a^q, b^p \frown c_{p+q} \rangle = \langle a^q \frown b^p, c_{p+q}$;

(iii) $a^p \frown (b^q \frown c_{p+q+r}) = (a^p \sigma)$

10.7.

10.8.