11.5. Problems

11.1. Prove that $\mathbb{S}^p \times \mathbb{S}^q = (\mathbb{S}^p \wedge \mathbb{S}^q) \bigcup_f \mathbb{D}^{p+q}$, where f is some map $f : \mathbb{S}^{p+q} \to \mathbb{S}^p \wedge \mathbb{S}^q$ non homotopic to the constant map.

11.2. In the notation of Sec.11.2 of the lecture, let $f, g : \widehat{K}^n \to Y$ be two maps that coincide on $\widehat{K}^{n-1}$. Denote by $d^n(f, g) \in C^n(K^n, A; \pi_n(Y))$ the cochain defined on a simplex Δ^n as follows: take two copies of Δ^n, glue then along their boundary, map one of the copies via f and the other via g to Y, thus determining an element of $\pi_n(Y)$. Prove that

$$d^n(f, g) = c^n(g) - c^n(f).$$

Use this fact to prove the Eilenberg Theorem.

11.3. Prove that $d^n(f, g) + d^n(g, h) + d^n(h, f) = 0$, where d^n is the *distinguishing cochain* defined in Problem 11.2.

11.4. Prove that $\mathbb{S}^1$ is a $K(\mathbb{Z}, 1)$ space.

11.5. Let M^2 be a closed surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Prove that M^2 is a $K(\pi_1(M^2), 1)$ space.

11.6. Prove that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$ space.

11.7. Prove that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ space.

11.8. Prove that L_m^∞ is a $K(\mathbb{Z}_m, 1)$ space, where the *infinite-dimensional lens space* L_m^∞ is defined as follows: consider $\mathbb{S}^\infty$ as the subspace of $\mathbb{C}^\infty$ of all points $(z_1, z_2, \dots)$ such that $\sum |z_i^2| = 1$; introduce the equivalence relation $(z_1, z_2, \dots) \sim (\varepsilon z_1, \varepsilon z_2, \dots)$, where $\varepsilon = \exp(2\pi i/m)$; then $L_m^\infty := \mathbb{S}^\infty / \sim$.

11.9. Prove that

$$H_n(L^\infty) = \begin{cases} \mathbb{Z}_m & \text{for odd } n; \\ 0 & \text{for even } n. \end{cases}$$

11.10. Let M^3 be a closed 3-manifold such that $\pi_1(M^3)$ is infinite and $\pi_2(M^3) = 0$. Prove that M^3 is a $K(\pi, 1)$ space for some π.

11.11. Let X be a finite-dimensional simplicial $K(\pi, 1)$ space. Prove that π has no finite-order elements.