The Classical and Adelic Point of View in Number Theory — 10/13/2010

K. Conrad — Problem Set 4

1. a) Show $J_{\mathbf{Q}(i)} = \mathbf{Q}(i)^\times \cdot (\mathbf{C}^\times \times \prod_{v\neq\infty} \mathbf{Z}[i]_v^\times)$ and $\mathbf{Q}(i)^\times \cap (\mathbf{C}^\times \times \prod_{v\neq\infty} \mathbf{Z}[i]_v^\times) = \{\pm 1, \pm i\}$. This is an analogue of the formula $J_{\mathbf{Q}} = \mathbf{Q}^\times \cdot (\mathbf{R}^\times \times \prod_p \mathbf{Z}_p^\times)$, where $\mathbf{Q}^\times \cap (\mathbf{R}^\times \times \prod_p \mathbf{Z}_p^\times) = \{\pm 1\}$.

   b) Using the fact that $\mathbf{R}^\times = \{\pm 1\} \times \mathbf{R}_{>0}$, the formula $J_{\mathbf{Q}} = \mathbf{Q}^\times \cdot (\mathbf{R}^\times \times \prod_p \mathbf{Z}_p^\times)$ can be refined to $J_{\mathbf{Q}} = \mathbf{Q}^\times \times (\mathbf{R}_{>0} \times \prod_p \mathbf{Z}_p^\times)$. Let's try to generalize this to $J_{\mathbf{Q}(i)}$ using the decomposition for $J_{\mathbf{Q}(i)}$ in part a. *Assuming* there is a direct product decomposition $\mathbf{C}^\times = \{\pm 1, \pm i\} \times H$ for some subgroup H of $\mathbf{C}^\times$, show we can write $J_{\mathbf{Q}(i)} = \mathbf{Q}(i)^\times \times G$ for some group G.

   Then show the assumption is wrong: $\mathbf{C}^\times \neq \{\pm 1, \pm i\} \times H$ for any subgroup H because $\mathbf{C}^\times$ has no subgroup of index 4, and in fact $\mathbf{C}^\times$ has no subgroup of any finite index greater than 1.

2. Let K be a number field. We want to show J_K admits a direct product decomposition which is compatible with the idelic norm. Choose an archimedean absolute value v_0 on K. Write down the indices (absolute values) of J_K so the v_0-component comes first. Define an embedding $f_0\colon \mathbf{R}_{>0} \to J_K$, depending on whether the chosen v_0 is real or complex, by the formula

$$f_0(t) := \begin{cases} (t, 1, 1, 1, \dots), & \text{if } v_0 \text{ is real,} \\ (\sqrt{t}, 1, 1, 1, \dots), & \text{if } v_0 \text{ is complex.} \end{cases}$$

   Show the multiplication map $M\colon \mathbf{R}_{>0} \times J_K^1 \to J_K$ where $M(t, \mathbf{x}) = f_0(t)\mathbf{x}$ is an isomorphism of topological groups and it fits into the commutative diagram below, where the maps along the top row are the standard inclusion and projection for subgroups of a direct product of groups and the maps along the bottom row are the inclusion of J_K^1 into J_K and the idelic norm on J_K.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & J_K^1 & \longrightarrow & \mathbf{R}_{>0} \times J_K^1 & \longrightarrow & \mathbf{R}_{>0} & \longrightarrow & 1 \\ & & \big\downarrow \scriptstyle{\text{id.}} & & \big\downarrow \scriptstyle{M} & & \big\downarrow \scriptstyle{\text{id.}} & & \\ 1 & \longrightarrow & J_K^1 & \longrightarrow & J_K & \xrightarrow[\|\cdot\|]{} & \mathbf{R}_{>0} & \longrightarrow & 1 \end{array}$$

(**Remark**. Unless K has exactly one archimedean absolute value, *i.e.*, unless K is $\mathbf{Q}$ or an imaginary quadratic field, the isomorphism $J_K \cong \mathbf{R}_{>0} \times J_K^1$ is not canonical since it depends on the choice of v_0.)

3. Let G be a locally compact Hausdorff group and μ be a Haar measure on G.

   a) If H is an open subgroup, show the restriction $\mu|_H$ is a Haar measure on H.

   b) Give an example of G and a subgroup H which is not open such that $\mu|_H$ is not a Haar measure on H.

4. A Borel measure μ on a topological space which satisfies the two conditions

$$\mu(A) = \inf_{\text{open } U \supset A} \mu(U), \qquad \mu(A) = \sup_{\text{cpt. } K \subset A} \mu(K),$$

   for all Borel sets is called a *regular* measure. A Borel measure satisfying the first condition for all Borel sets A and the second condition when A is open or σ-finite is called a σ*-regular* measure, so by definition a Haar measure is σ-regular. This exercise gives an example of a Haar measure which is not regular.

   Set $G = S^1 \times \mathbf{R}_d$, where $\mathbf{R}_d$ is the real line with the discrete topology. This is a direct product of two locally compact (abelian) groups, so G is locally compact with the product topology. Think about G as an infinitely long vertical cylinder whose horizontal slices are "topologically independent" circles. Let μ be a Haar measure on G and set $A = \{1\} \times [0, 1]$ (a vertical line segment of length 1).

   a) Show A is not open in G but it is closed.

   b) Show $\mu(K) = 0$ for all compact subsets K of A.

   c) Let U be any open set in G containing A. Therefore

$$U \supset \bigcup_{0 \le y \le 1} (1 - \varepsilon_y, 1 + \varepsilon_y) \times \{y\},$$

   where $0 < \varepsilon_y < 1$ for all y. Show there's an $\varepsilon > 0$ such that $\varepsilon_y \ge \varepsilon$ for infinitely many y. (Hint: Assume otherwise. There are uncountably many y in $[0, 1]$.)

   d) By part c, $U \supset (1 - \varepsilon, 1 + \varepsilon) \times \{y\}$ for infinitely many $y \in [0, 1]$. Use this to show $\mu(U) = \infty$ and conclude from σ-regularity that $\mu(A) = \infty$ so μ is not regular. (Page 1 of Rudin's book "Fourier Analysis on Groups" says any Haar measure is regular. This is wrong.)