1. Let G be a locally compact group and μ be a left-invariant Borel measure on G. (There are *no* regularity assumptions about the measure.) Show the following conditions are equivalent:

   a) $\mu(K) < \infty$ for all compact subsets K of G,

   b) $\mu(U_0) < \infty$ for some nonempty open subset U_0 of G.

   Therefore in the definition of a left Haar measure, instead of requiring all compact subsets have finite measure we can require there is some nonempty open subset with finite measure.

2. For each number field K, show $\mathbf{A}_K$ and J_K are both σ-compact: each can be written as the union of countably many compact subsets.

3. Let $\mu_{p^\infty} = \{z \in \mathbf{C} : z^{p^n} = 1 \text{ for some } n\}$, but we view it as a *discrete* group, which is not its usual topology as a subset of $\mathbf{C}$. Prove $\widehat{\mu_{p^\infty}} \cong \mathbf{Z}_p$ as topological groups using an explicit isomorphism. (Do not use Pontryagin duality!!).

4. a) In the group $\mathbf{Q}/\mathbf{Z}$, show the subgroup of elements of p-power order is isomorphic to $\mathbf{Q}_p/\mathbf{Z}_p$ by an explicit isomorphism. (There is no topology here.)

   b) The function $r \mapsto e^{2\pi i r}$ induces an isomorphism of $\mathbf{Q}/\mathbf{Z}$ with the group μ of all roots of unity in $\mathbf{C}$ as abstract groups. This isomorphism and part a gives us an isomorphism $f\colon \mathbf{Q}_p/\mathbf{Z}_p \to \mu_{p^\infty}$ as abstract groups.[1] Show the composite map

$$\mathbf{Q}_p \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\ f\ } \mu_{p^\infty} \hookrightarrow S^1$$

   is precisely the standard character $x \mapsto e^{2\pi i \{x\}_p}$ on $\mathbf{Q}_p$.

5. Let F be a finite extension of $\mathbf{Q}_p$.

   a) Use a $\mathbf{Q}_p$-basis of F to show a topological group isomorphism $\widehat{\mathbf{Q}_p} \cong \mathbf{Q}_p$ implies a topological group isomorphism $\widehat{F} \cong F$. This shows F is self-dual without using a "natural" isomorphism.

[1] Since the quotient topology on $\mathbf{Q}_p/\mathbf{Z}_p$ is discrete, $\mathbf{Q}_p/\mathbf{Z}_p$ is the correct model for μ_{p^∞} as a topological group with the discrete topology.

b) For each $y \in F$, define $\chi_y \colon F \to S^1$ by $\chi_y(x) = e^{2\pi i \{\mathrm{Tr}_{F/\mathbf{Q}_p}(xy)\}_p}$. Show $y \mapsto \chi_y$ is a topological group isomorphism $F \cong \widehat{F}$, using part a. Here we are giving a "natural" self-duality of F.

(Hint: On any separable field extension L/K, every K-linear map $L \to K$ has the form $x \mapsto \mathrm{Tr}_{L/K}(xy)$ for a unique $y \in L$.)

c) Set $F = \mathbf{Q}_3(\sqrt{6})$ and $\chi(a + b\sqrt{6}) = e^{2\pi i \{a\}_3} e^{-2\pi i \{5b\}_3}$. For which explicit number $y \in F$ do we have $\chi(x) = e^{2\pi i \{\mathrm{Tr}_{F/\mathbf{Q}_3}(xy)\}_3}$ for all $x \in F$?