Lecture 4

DISCRETE SUBGROUPS OF THE ISOMETRY GROUP OF THE PLANE AND TILINGS

This lecture, just as the previous one, deals with a classification of objects, the original interest in which was perhaps more aesthetic than scientific, and goes back many centuries ago. The objects in question are regular tilings (also called tessellations), i.e., configurations of identical figures that fill up the plane in a regular way. Each regular tiling is a geometry in the sense of Klein; it turns out that, up to isomorphism, there are 17 such geometries; their classification will be obtained by studying the corresponding transformation groups, which are discrete subgroups (see the definition in Section 4.3) of the isometry group of the Euclidean plane.

4.1. Tilings in architecture, art, and science

In architecture, regular tilings appear, in particular, as decorative mosaics in the famous Alhambra palace (14th century Spain). Two of these are reproduced below in black and white (see Fig.4.1). Beautiful color photos of the Alhambra mosaics may be found at the website www.alhambra.com ???????

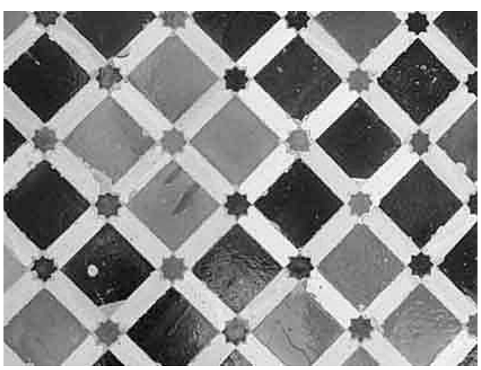
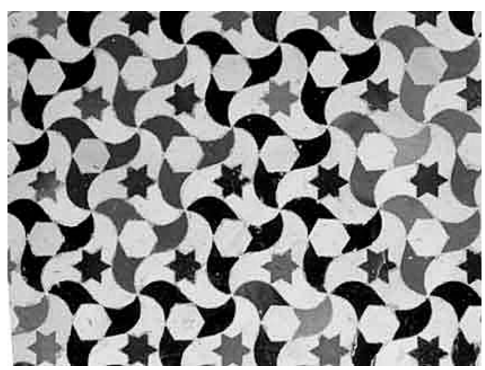

Figure 4.1. Two Alhambra mosaics

In art, the famous Dutch artist A.Escher, famous for his "impossible" paintings, used regular tilings as the geometric basis of his wonderful "periodic" watercolors. Two of those are shown Fig.4.2. Color reproductions of his work appear on the website www.Escher.com.

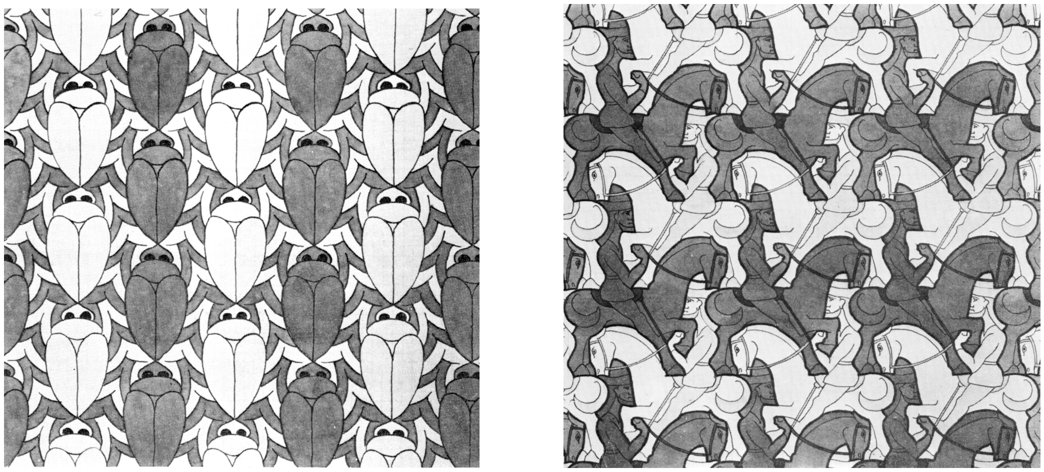

Fig.4.2. Two periodic watercolors by Escher

From the scientific viewpoint, not only regular tilings are important: it is possible to tile the plane by copies of one tile (or two) in an irregular (nonperiodic) way. It is easy to fill the plane with rectangular tiles of size say 10cm by 20cm in many nonperiodic ways. But that $\mathbb{R}^2$ can be filled irregularly by convex 9-gons is not obvious. Such an amazing construction, due to Vorderberg (1936), is shown in Fig.4.3. The figure shows how to fill the plane by copies of two tiles (their enlarged copies are shown separately; they are actually mirror images of each other) by fitting them together to form two spiraling curved strips covering the whole plane.

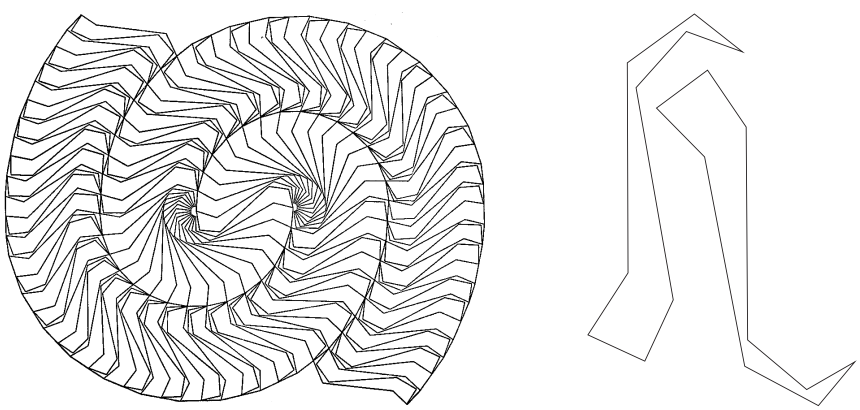

Fig.4.3. The Vorderberg tiling

Somewhat later, in the 1960ies, interest in irregular tilings was revived by the nonperiodic tilings due to the British mathematical physicist Roger Penrose, which are related to statistical models and the study of quasi-crystals. More recently, irregular tilings have attracted the attention of mathematicians, in particular that of the 2006 Fields medallist Andrey Okounkov in his work on three-dimensional Young diagrams.

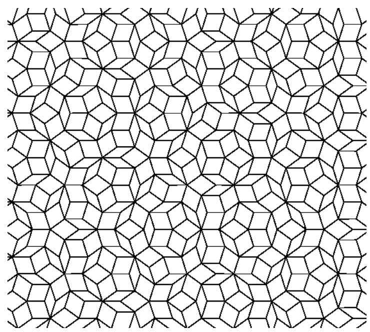

Figure 4.3. A Penrose tiling

4.2. Tilings and crystallography

The first proof of the classification theorem of regular tilings (defined below, see Sec.4.5.1) was obtained by the Russian crystallographer Fedorov in 1891. Mathematically, they are given by special discrete subgroups, called the *Fedorov groups*, of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane. As we mentioned above, there are 17 of them (up to isomorphism). The Fedorov groups act on the Euclidean plane, forming 17 different (i.e., nonisomorphic) geometries in the sense of Klein, which we call *tiling geometries*.

The proof given here, just as the one in the previous lecture, is group-theoretic, and is based on the study of discrete subgroups of the isometry group of the plane. In fact, the actual classification principle cannot be stated without using transformation groups (or something equivalent to them), and at first glance it is difficult to understand how it came about that the architects of the Alhambra palace, five centuries before the notion of group appeared in mathematics, actually found 11 of the 17 regular tilings. Actually, this is not surprising: a deep understanding of symmetry suffices to obtain answers to an intuitively clear question, even if one is unable to state the question in the terminology of modern mathematics.

Less visual, but more important for the applications (crystallography), is the generalization of the notion of regular tiling to three dimensions: configurations of identical polyhedra filling $\mathbb{R}^3$ in a regular way. Mathematically, they are also defined by means of discrete subgroups called *crystallographic groups* of the isometry group of $\mathbb{R}^3$ and have been classified: there are 230 of them. Their study is beyond the scope of this lecture.

We are concerned here with the two-dimensional situation, and accordingly we begin by recalling some facts from elementary plane geometry, namely facts concerning the structure of isometries of the plane $\mathbb{R}^2$.

4.3. Isometries of the plane

Recall that by $\mathrm{Sym}(\mathbb{R}^2)$ we denote the group of isometries (i.e., distance-preserving transformations) of the plane $\mathbb{R}^2$, and by $\mathrm{Sym}^+(\mathbb{R}^2)$ its group of motions (i.e., isometries preserving orientation). Examples of the latter are parallel translations and rotations, while reflections in a line are examples of isometries which are not motions (they reverse orientation).

(We consider an isometry orientation-reversing if it transforms a clockwise oriented circle into a counterclockwise oriented one. This is not a mathematical definition, since it appeals to the physical notion of “clockwise rotation”, but there is a simple and rigorous mathematical definition of orientation-reversing (-preserving) isometry; see the discussion about orientation in Appendix II.)

Below we list some well known facts about isometries of the plane; their proofs are relegated to exercises appearing at the end of the present chapter.

4.3.1. A classical theorem of elementary plane geometry says that any motion is either a *parallel translation* or a *rotation* (see Exercise 4.1).

4.3.2. A less popular but equally important fact is that any orientation-reversing isometry is a *glide reflection*, i.e., the composition of a reflection in some line and a parallel translation by a vector collinear to that line (Exercise 4.2).

4.3.3. The composition of two rotations is a rotation (except for the particular case in which the two angles of rotation are equal but opposite: then their composition is a parallel translation). In the general case, there is a simple construction that yields the center and angle of rotation of the composition of two rotations (see Exercise 4.3). This important fact plays the key role in the proof of the theorem on the classification of regular tilings.

4.3.4. The composition of a rotation and a parallel translation is a rotation by the same angle about a point obtained by shifting the center of the given rotation by the given translation vector (Exercise 4.4).

4.3.5. The composition of two reflections in lines l_1 and l_2 is a rotation about the intersection point of the lines l_1 and l_2 by an angle equal to twice the angle from l_1 to l_2 (Exercise 4.5).

4.4. Discrete groups and discrete geometries

The action of a group G on a space X is called *discrete* if none of its orbits possess accumulation points, i.e., there are no points of $x \in X$ such that any neighborhood of x contains infinitely many points belonging to one orbit. Here the word "space" can be understood as Euclidean space $\mathbb{R}^n$ (or as a subset of $\mathbb{R}^n$), but the definition remains valid for arbitrary metric and topological spaces.

A simple example of a discrete group acting on $\mathbb{R}^2$ is the group consisting of all translations of the form $k\,\vec{v}$, where v is a fixed nonzero vector and $k \in \mathbb{Z}$. The set of all rotations about the origin of $\mathbb{R}^2$ by angles which are integer multiples of $\sqrt{2}\,\pi$ is a group, but its action on $\mathbb{R}^2$ is not discrete (since $\sqrt{2}$ is irrational, orbits are dense subsets of circles centered at the origin).

4.5. The seventeen regular tilings

4.5.1 *Formal definition.* By definition, a *tiling* or *tessellation* of the plane $\mathbb{R}^2$ by a polygon T_0, the *tile*, is an infinite family $\{T_1, T_2, \dots\}$ of pairwise nonoverlapping (i.e., no two distinct tiles have common interior points) copies of T_0 filling the plane, i.e., $\mathbb{R}^2 = \bigcup_{i=1}^{\infty} T_i$.

For example, it is easy to tile the plane by any rectangle in different ways, e.g. as a rectangular lattice as well as in many irregular, nonperiodic ways. Another familiar tiling of the plane is the *honeycomb lattice*, where the plane is filled with identical copies of a regular hexagon.

A polygon $T_0 \subset \mathbb{R}^2$, called the *fundamental tile*, determines a *regular tiling* of the plane $\mathbb{R}^2$ if there is a subgroup G (called the *tiling group*) of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane such that

(i) G acts discretely on $\mathbb{R}^2$, i.e., all the orbits of G have no accumulation points;

(ii) the images of T_0 under the action of G fill the plane, i.e.,

$$\bigcup_{g \in G} g(T_0) = \mathbb{R}^2;$$

(iii) for $g, h \in G$ the images $g(T_0)$, $h(T_0)$ of the fundamental tile coincide if and only if $g = h$.

Actually, (ii) and (iii) imply (i), but we will not prove this (see the first volume of the book *Géométrie* by Berger, pp.37-38).

The action of a tiling group $G \subset \mathrm{Sym}(\mathbb{R}^2)$ on the plane $\mathbb{R}^2$ is, of course, a geometry in the sense of Klein that we call the *tiling geometry* (or *Fedorov geometry*) of the group G.

4.4.2 *Examples of regular tilings*

Six examples of regular tilings are shown in Fig.4.3.

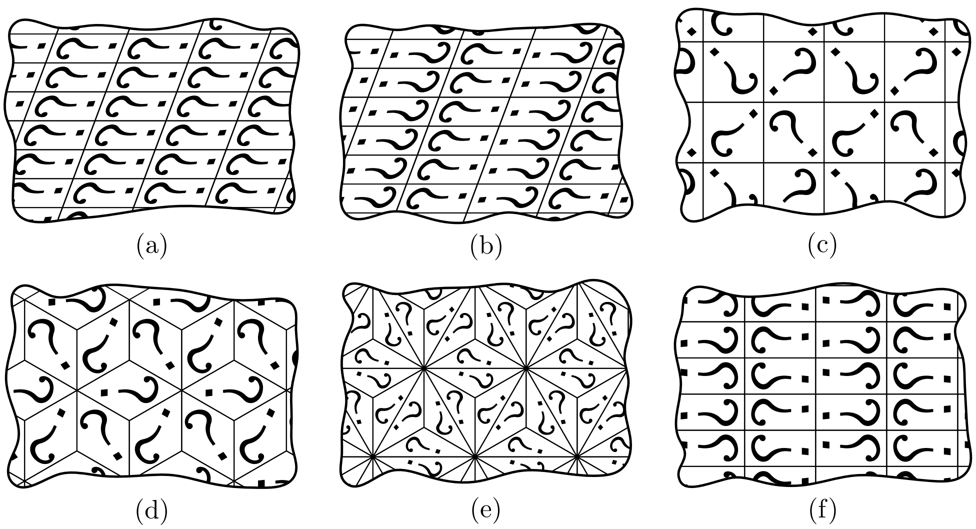

Figure 4.4. Six regular tilings of the plane

Given two tiles, there is one element of the transformation group that takes one to the other. The question marks show *how* the tiles are mapped to each other. (Without the question marks, the action of the transformation group would not be specified; see Exercise 4.16).

The first five tilings (a-e) are *positive*, i.e., they correspond to subgroups of the group $\mathrm{Sym}^+(\mathbb{R}^2)$ of motions (generated by all rotations and translations) of the plane (one-sided tiles slide along the plane). The sixth tiling (f) allows "turning over" the (two-sided) tiles.

Let us look at the corresponding tiling groups in more detail.

4.4.3. Theorem. [Fedorov, 1891]. *Up to isomorphism, there are exactly five different one-sided tiling geometries of the plane* $\mathbb{R}^2$. *They are shown in* Fig.4.4,a–e.

Proof. Let G be a group of positive tilings. Consider the subgroup $G_T \subset G$ of all parallel translations in G.

4.4.3. Lemma. *The subgroup G_T is generated by two noncollinear vectors v and u.*

Proof. Arguing by contradiction, suppose that G_T is trivial (there are no parallel translations except the identity). Let r, s be any two (nonidentical) rotations with different centers. Then $rsr^{-1}s^{-1}$ is a nonidentical translation (to prove this, draw a picture). A contradiction. □

Now suppose that all the elements of G_T are translations generated by (i.e., proportional to) one vector v. Then it is not difficult to obtain a contradiction with item (ii) of the definition of regular tilings. □

Now if G contains no rotations, i.e., $G = G_T$, then we get the tiling (a). Further, If G contains only rotations of order 2, then it is easy to see that we get the tiling (b).

4.4.5. Lemma. *If G contains a rotation of order $\alpha \geq 3$, then it contains two more rotations (of some some orders β and γ) such that*

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.}$$

Sketch of the proof. Let A be the center of a rotation of order α. Let B and C be the nearest (from A) centers of rotation not obtainable from A by translations. Then the boxed formula follows from the fact that the sum of angles of triangle ABC is π. The detailed proof of this lemma is one of the problems for the exercise class. □

Since the three rotations are of order greater or equal to 3, it follows from the boxed formula that only three cases are possible.

	$1/\alpha$	$1/\beta$	$1/\gamma$
case 1	1/3	1/3	1/3
case 2	1/2	1/4	1/4
case 3	1/2	1/3	1/6

Studying these cases one by one, it is easy to establish that they correspond to the tilings (c,d,e) of Fig.4.3.

This concludes the proof of Theorem 4.4.3. □

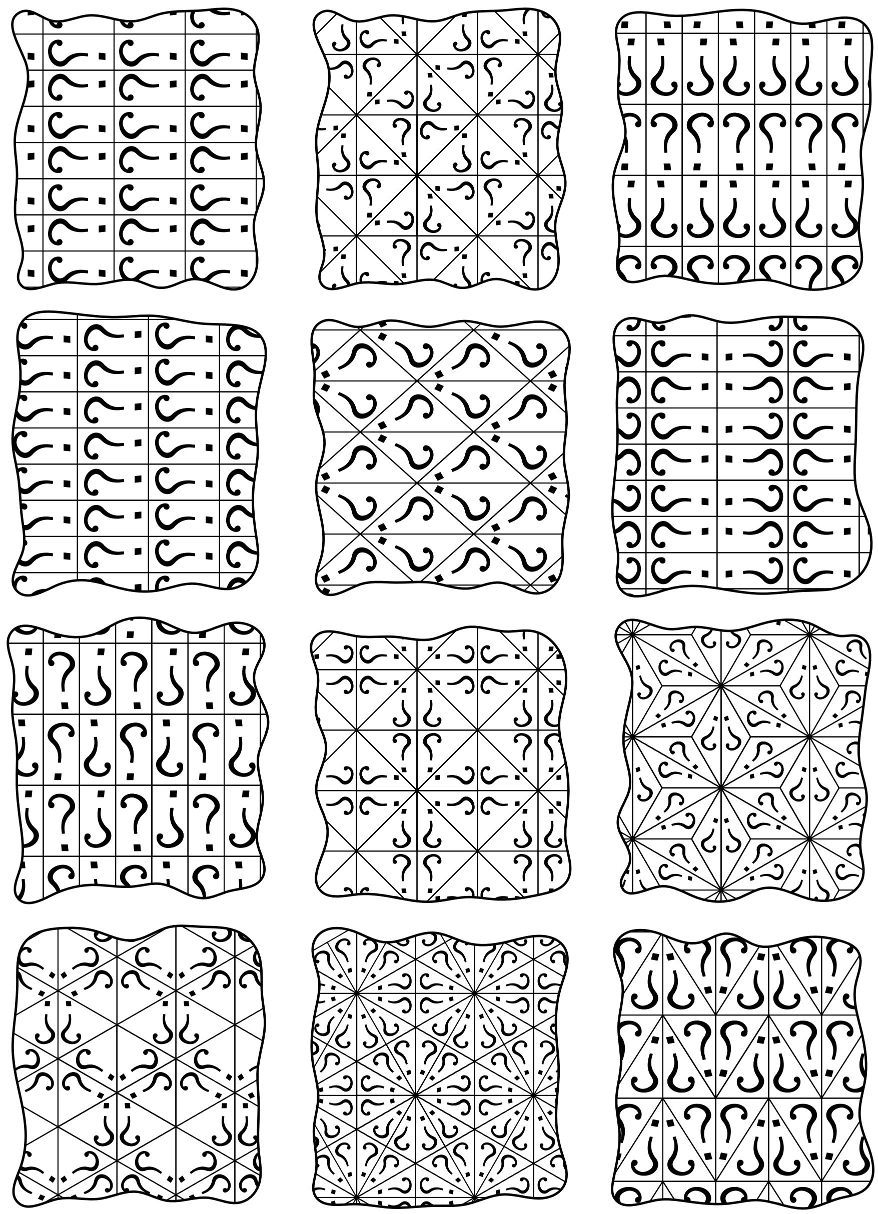

Figure 4.4. Two-sided regular tilings

In the general case (all tilings, including those by two-sided tiles), there are exactly seventeen nonequivalent tilings. This was also proved by Fedorov. The 12 two-sided ones are shown on the previous page.

We will not prove the second part of the classification theorem for regular plane tilings (it consists in finding the remaining 12 regular tilings, for which two-sided tiles are required). However, we will look at some examples of these 12 tilings in the exercise class. Note that there is a nice web site with beautiful examples of decorative patterns corresponding to the 17 regular tilings:

http://www2.spsu.edu/math/tile/symm/ident17.htm

4.5. The 230 crystallographic groups

The crystallographic groups are the analogs in $\mathbb{R}^3$ of the tiling groups in Euclidean space $\mathbb{R}^2$. The corresponding periodically repeated polyhedra are not only more beautiful than tilings, they are more important: the shapes of most of these polyhedra correspond to the shapes of real-life crystals. There are 230 crystallographic groups. The proof is very tedious: there are 230 cases to consider, in fact more, because many logically arising cases turn out to be geometrically impossible, and it lies, as we mentioned above, outside the scope of this course.

Those of you who would like to see some nontrivial examples of geometries corresponding to some of the crystallographic groups should look at Problem 4.5 and postpone their curiosity to the next lecture, where 4 examples of actual crystals will appear in the guise of Coxeter geometries. Another possibility is to consult the website http://webmineral.com/crystal.shtml

4.6. Problems

4.1. Prove that any motion of the plane is either a translation by some vector v, $|v| \geq 0$, or a rotation r_A about some point A by a nonzero angle.

4.2. Prove that any orientation-reserving isometry of the plane is a glide reflection in some line L with glide vector u, $|u| \geq 0$, $u||L$.

4.3. Justify the following construction of the composition of two rotations $r = (a, \varphi)$ and (b, ψ). Join the points a and b, rotate the ray $[a, b >$ around a by the angle $\varphi/2$, rotate the ray $[b, a >$ around b by the angle $-\psi/2$, and denote by c the intersection point of the two obtained rays; then c is the center of rotation of the composition rs and its angle of rotation is

$2(\pi - \varphi/2 - \psi/2)$. Show that this construction fails in the particular case in which the two angles of rotation are equal but opposite, and then their composition is a parallel translation).

4.4. Prove that the composition of a rotation and a parallel translation is a rotation by the same angle about the point obtained by shifting the center of the given rotation by the given translation vector.

4.5. Prove that the composition of two reflections in lines l_1 and l_2 is a rotation about the intersection point of the lines l_1 and l_2 by an angle equal to twice the angle from l_1 to l_2.

4.6. Indicate a finite system of generators for the transformation groups corresponding to each of the tilings shown in Figure 4.3 a), b),...,f).

4.7. Is it true that the transformation group of the tiling shown on Figure 4.3 (b) is a subgroup of the one of Figure 4.1 (c)?

4.8. Indicate the points that are centers of rotation subgroups of the transformation group of the tiling shown in Figure 4.3(c).

4.9. Write out a presentation of the isometry group of the plane preserving

(a) the regular triangular lattice;
(b) the square lattice;
(c) the hexagonal (i.e., honeycomb) lattice.

4.10. For which of the five Platonic bodies can a (countable) collection of copies of the body fill Euclidean 3-space (without overlaps)?

4.11. For the five pictures in Fig.4.5 on the next page (three of which are Alhambra mosaics and two are Escher watercolors) indicate to which of the 17 Fedorov groups they correspond.

4.12. Exactly one of the 17 Fedorov groups contains a glide reflection but no reflections. Which one?

4.13. Which two of the 17 Fedorov groups contain rotations by $\pi/6$?

4.14. Which three of the 17 Fedorov groups contain rotations by $\pi/2$?

4.15. Which five of the 17 Fedorov groups contain rotations by π only?

4.16. Rearrange the question marks in the tiling (c) so as to make the corresponding geometry isomorphic that of the tiling (a).