Lecture 5
Reflection Groups and Kaleidoscopes

In this lecture, as in the previous one, we study geometries defined by certain discrete subgroups of the isometry group of the plane (and, more generally, of n-dimensional space), namely the subgroups generated by reflections (called Coxeter groups after the 20th century Canadian mathematician who invented them). These geometries are perhaps not as beautiful as those studied in the previous two lectures, but are more important in the applications (in algebra and topology). On the other hand, they do have an aesthetic origin: what one sees in a kaleidoscope (a child's toy very popular before the computer era) is an instance of such a geometry. Following E. B. Vinberg, we call these geometries (in the two-dimensional case) *kaleidoscopes*. We prove the classification theorem for them in dimension 2 and state the corresponding result without proof in higher dimensions (using the important notion of Coxeter scheme).

Many ideas of this (geometric) theory are similar to the very important (algebraic) theory of group representation (studied in advanced algebra courses and used in modern physics). In particular, the famous Dynkin diagrams are similar to Coxeter schemes (defined below).

§ 5.1. An example: the kaleidoscope. The kaleidoscope is a children's toy: bright little pieces of glass are placed inside a regular triangular prism and are multiply reflected by three mirrors forming the lateral faces of the prism. Looking into the prism, you see a colorful (infinitely) repeated pattern: the picture in the triangle and its mirror image alternate, forming a hexagon (the union of six equilateral triangles), see Fig. 5.1 (a), surrounded by more equilateral triangles *ad infinitum*.

Mathematically, this is a two-dimensional phenomenon: the equilateral triangle forming the base of the prism is the fundamental domain of a discrete group acting on the plane of the base.

Now if the kaleidoscope is deformed (e.g., the angles between the faces are slightly changed), then the picture becomes fuzzy, no pattern can be seen. In such a situation, the images of the base triangle overlap infinitely many times (see Fig. 5.1 (b)), the transformation group acting on the triangle is not discrete; we will not study this "bad" case: we only study the "nice" kaleidoscope case generalized to any dimension.

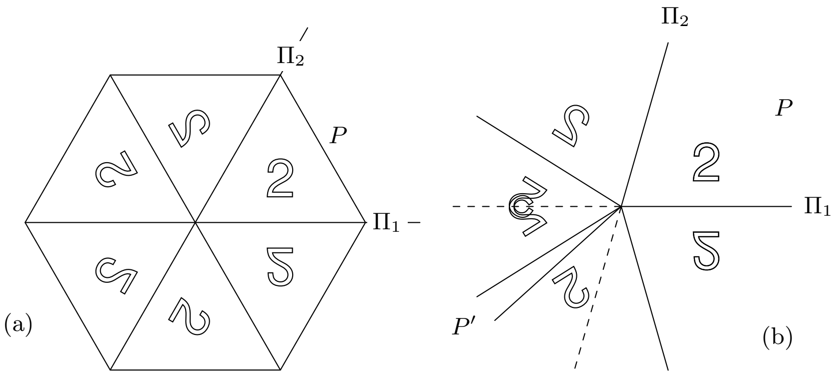


Fig. 5.1. The kaleidoscope

§ 5.2. Coxeter polygons and polyhedra. Consider a dihedral angle $\alpha < \pi/2$ formed by two plane two-sided mirrors Π_1, Π_2. What will the observer O see? Any picture P inside the angle will be reflected by Π_1; its image P' will be in turn be reflected by the image of Π_1 by Π_2, and so on. At the same time, the picture P inside the angle will be reflected by Π_2; its image P'' will be in turn be reflected by the image of Π_2 by Π_1, etc. Two cases are possible: either the reflections coming from different sides will overlap (Fig. 5.1 (b)) or the reflected pictures will coincide (Fig. 5.1 (a)). Obviously, the pictures will coincide if (and only if) the angle α is of the form π/k, where $k = 2, 3, \ldots$

Mathematically, this situation is the following. On the Euclidean plane, we take two straight lines forming the angle α and consider the group G of all transformations of the plane generated by the reflections in these two lines. Let F be the region bounded by the two half planes forming the angle α. Obviously, no two regions $g(F)$ and $h(F)$, $g, h \in G$, $g \neq h$, overlap if and only if $\alpha = \pi/k$, where $k = 2, 3, \ldots$ In that case, G is the dihedral group $\mathbb{D}_k$.

Now suppose we are given a convex polygon F in the plane with vertex angles less than or equal to $\pi/2$. Consider the group G_F of transformations of the plane generated by reflections in the lines containing the sides of F. We say that G_F is *acts transitively on* F if the images $g(F)$, $g \in G_F$, never overlap. A necessary condition for the transitive action of G_F on F is that all the vertex angles of F be of the form π/k for various values of k; this follows from the argument in the previous paragraph. Obviously, this condition is not sufficient.

The previous arguments are the motivation for the following definition. A convex polygon F is called a *Coxeter polygon* if all its vertex angles are of the form π/k for various values of $k = 2, 3, \ldots$ and it generates a transitive action of the group G_F. Coxeter polygons will be classified below — there are only four.

The above can be generalized to three-dimensional space. The corresponding definition is the following: a convex polyhedron is called a *Coxeter polyhedron* P if all its dihedral angles are of the form π/k for various values of $k = 2, 3, \ldots$ and it generates a transitive action of G_P, where G_P is the transformation group generated by the reflections in the planes containing the faces of P. Coxeter polyhedra will be classified below (there are seven).

§ 5.3. Coxeter geometries on the plane. Let F be a Coxeter polygon in the plane $\mathbb{R}^2$. The *Coxeter geometry* with fundamental region F is the geometry $(\mathbb{R}^2 : G_F)$, where G_F is the group of transformations of the plane generated by the reflections in the lines containing the sides of the polygon F. The goal of this section is to classify all Coxeter geometries on the plane.

Theorem 5.1. *There are four Coxeter geometries in the plane; their fundamental polygons are the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with angles* $\pi/3$ *and* $\pi/6$ (*see Fig.* 5.2).

P r o o f. Let F be the fundamental polygon of a Coxeter geometry. If it has n sides, then the sum of its angles is $\pi(n-2)$ and so the average value of its angles is $\pi(1-2/n)$. Now n cannot be greater than 4, because F would then have an obtuse angle (and this contradicts the definition of Coxeter polygon). If $n = 4$, then all angles of F are $\pi(1-2/4) = \pi/2$ and F is a rectangle. Finally, if $n = 3$, and the angles of the fundamental triangle are π/k, π/l, π/m, then (since their sum is π), we obtain a Diophantine equation for k, l, m:

$$\boxed{\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1}.$$

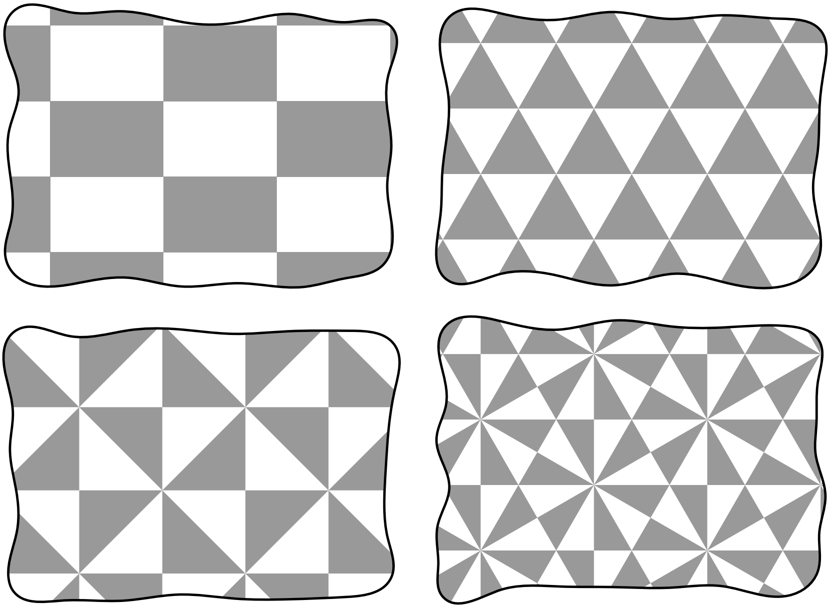


Fig. 5.2. The four plane Coxeter geometries

This equation has three solutions: $(3,3,3)$, $(2,4,4)$, $(2,3,6)$. These solutions correspond to the three triangles listed in the theorem. $\square$

§ 5.4. Coxeter geometries in Euclidean space $\mathbb{R}^3$. In this section, we study the Coxeter geometries in $\mathbb{R}^3$. A Coxeter polyhedron $F \subset \mathbb{R}^3$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^3$) with dihedral angles of the form π/k for various values of $k = 2, 3, \ldots$. A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined just as in the case $d = 2$ (see § 5.3).

Theorem 5.2. *There are seven Coxeter geometries in three-dimensional space; their fundamental polyhedra are the four right prisms over the the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with acute angles* $\pi/3$ *and* $\pi/6$, *and the three* (*nonregular*) *tetrahedra shown in Fig.* 5.3.

It is not very difficult to prove that the seven polyhedra (listed in the theorem) indeed define Coxeter geometries. To prove that there are no other

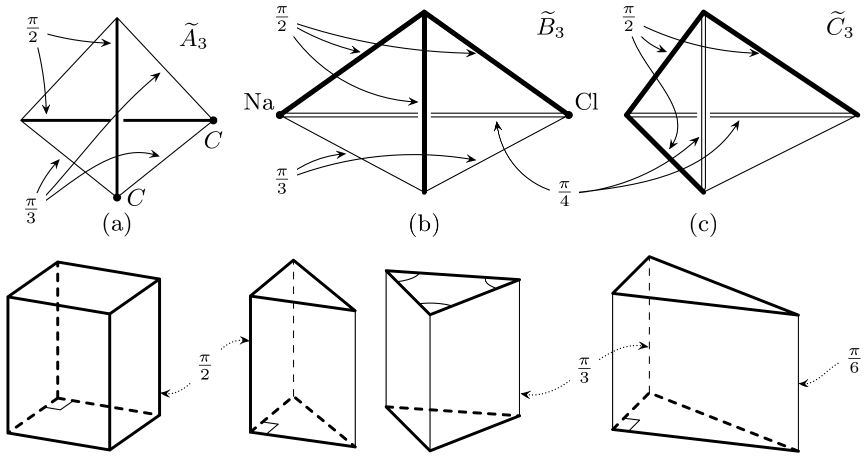


Fig. 5.3. The seven Coxeter geometries in 3-space

geometries, nontrivial information from linear algebra (in particular, the notion of Gramm matrix) is needed.

Therefore, we omit the proof (it can be found in the article by E. B. Vinberg in *Matematicheskoye Prosveshchenie*, Ser. 3, # 7, pp. 39—63).

A remark about terminology. The term "Coxeter geometry" is not a standard term. E. B. Vinberg uses the term "kaleidoscope" instead. Also, we do *not* use the term "Coxeter group" for the transformation group of a Coxeter geometry. This is because "Coxeter group" is standardly used in a somewhat different sense than "transformation group of a Coxeter geometry".

Coxeter geometries are not only abstract mathematical objects, they are also important models in crystallography. For example, the polyhedron in Fig. 5.3 (b) is the crystal of ordinary salt, while the polyhedron in Fig. 5.3 (a) is a diamond crystal.

§ 5.5. Coxeter schemes and the classification theorem. In this section, we study the general case of a Coxeter geometry in $\mathbb{R}^d$ for an arbitrary positive integer d. A Coxeter polyhedron $F \subset \mathbb{R}^d$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^d$) with dihedral angles of the form π/k for various values of $k = 2, 3, \ldots$ such that the reflections in the d-dimensionsal hyperplanes containing its faces generate a transatively acting group G_F. (The definition of the measure of a dihedral angle in Euclidean space of arbitrary dimension d appears in the linear algebra course.) A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined just as in the cases $d = 2$ and $d = 3$ (see §§ 5.2, 5.3).

A *Coxeter scheme* is a graph (with integer weights on the edges) encoding a Coxeter polyhedron (in particular, polygons) in any dimension d. The scheme of a given Coxeter polyhedron is constructed as follows: its vertices correspond to the faces of the polyhedron, two vertices whose corresponding faces form an angle of π/m, $m \geq 3$, are joined by an edge with weight $m - 2$; if two faces are parallel, the corresponding vertices are joined by an edge with weight ∞. (Note that vertices corresponding to perpendicular edges are *not* joined by any edge.)

Graphically, instead of writing the weights 2, 3, 4 on the edges of a scheme, we draw double, triple, quadruple edges; instead of writing ∞ on an edge, we draw a very thick edge.

For example, the Coxeter scheme of the rectangle consists of two components, each of which has two vertices joined by an edge with weight ∞, while the scheme of an equilateral triangle has three vertices joined cyclically by three edges with weights 1.

Theorem 5.3. *The Coxeter geometries in all dimensions are classified by the connected components of their Coxeter schemes listed in* Fig. 5.4.

We omit the proof (see the article by V. O. Bugaenko in *Matematicheskoye Prosveshchenie*, Ser. 3, # 7, pp. 82—106).

§ 5.6. Problems.

5.1. Three planes P_1, P_2, P_3 passing through the z-axis of Euclidean space $\mathbb{R}^3$ are given. The angles between P_1 and P_2, P_2 and P_3 are α and β, respectively.

(a) Under what conditions on α and β will the group generated by reflections with respect to the three planes be finite?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

5.2. Three straight lines L_1, L_2, L_3 in the Euclidean plane form a triangle with interior angles α, β, and γ.

(a) Under what conditions on α, β, γ will the group generated by reflections with respect to the three lines be discrete?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

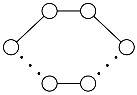
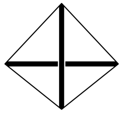
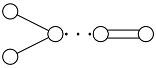
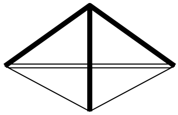
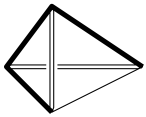
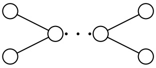
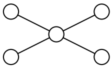
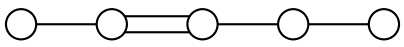
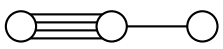
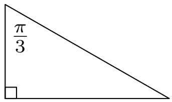
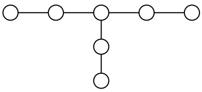
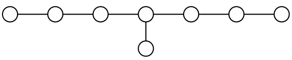
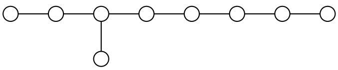
Name	Coxeter scheme	dim	#(faces)	view in $\mathbb{R}^3$
$\widetilde{A}_1$		1	2	
$\widetilde{A}_n$		$n-1$	n	
$\widetilde{B}_n$		$n-1$	n	
$\widetilde{C}_n$		$n-1$	n	
$\widetilde{D}_n$		$n-1$	$n \geqslant 5$	none!
$\widetilde{D}_4$		4	5	none!
$\widetilde{F}_4$		4	5	none!
$\widetilde{G}_2$		2	3	
$\widetilde{E}_6$				none!
$\widetilde{E}_7$				none!
$\widetilde{E}_8$				none!

Fig. 5.4. Coxeter schemes

5.3. Consider the six lines $L_1, \ldots, L_6$ containing the six sides of a regular plane hexagon and denote by G the group generated by reflections with respect to these lines. Does this group determine a Coxeter geometry? Justify your answer by using the classification theorem of plane Coxeter geometries and without using that theorem.

5.4. Let F be a Coxeter triangle, s_1, s_2, s_3 be the reflections with respect to its sides, and G_F the corresponding transformation group.

(a) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F that leave a chosen vertex of F fixed.

(b) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F which are parallel translations.

Consider the three cases of different Coxeter triangles separately.

5.5. Draw the Coxeter schemes of

(a) all the Coxeter triangles;

(b) all the three-dimensional Coxeter polyhedra.

5.6. Prove that there are exactly three edges at each vertex of any three-dimensional Coxeter polyhedron.

5.7. Let $(F : G_F)$ be a Coxeter Geometry of arbitrary dimension. Prove that

(a) if $s \in G_F$ is the reflection in a hyperplane P, then, for any $g \in G_F$, gsg^{-1} is the reflection in the hyperplane gP;

(b) any reflection from the group G_F is conjugate to the reflection in one of the faces of the polyhedron F,

5.8. Describe some four-dimensional Coxeter polyhedron other than the four-dimensional cube.

5.9. (a) Does the transformation group generated by the reflections in the faces of regular tetrahedron define a Coxeter geometry?

(b) Same question for the cube.

(c) Same question for the octahedron.

(d) Same question for the dodecahedron.