Lecture 1

Main Definitions and Toy Geometries

This lecture begins with the study of three toy examples of geometries (symmetries of the square, the cube, and the circle) and a model of the geometry of the projective plane. These examples are followed by the main definition of this course: a geometry in the sense of Klein is a set with a transformation group acting on it. We then define some useful general notions related to transformation groups. Finally, we study the relationships (called morphisms) between different geometries, thus introducing the category of all geometries. The notions introduced in this lecture are illustrated by some problems dealing with toy models of geometries that will be worked out in the exercise class.

§ 1.1. Symmetries of some figures. (1) *Symmetries of the square.* Consider all the isometries of the unit square $\square = ABCD$, i.e., all the

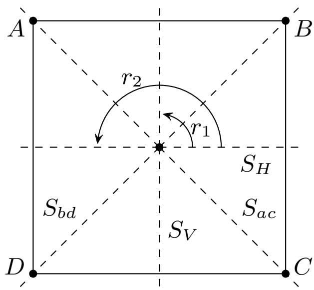

Fig. 1.1. Symmetries of the square

distance-preserving mappings of the square to itself. Denote by s_H, s_V, and s_{ac}, s_{bd} the line symmetries in the horizontal and vertical mid-lines, and in the diagonals AC, BD, respectively. Denote by r_0, r_1, r_2, r_3 the rotations about the center of the square by 0, 90, 180, 270 degrees, respectively. These eight transformations are all called *symmetries* of the square and denoted

$$\mathrm{Sym}(\square) = \{r_0,\ r_1,\ r_2,\ r_3,\ s_H,\ s_V,\ s_{ac},\ s_{bd}\}.$$

The set $\mathrm{Sym}(\square)$ is closed under the operation of *composition* (also called *multiplication*), i.e., the composition of two symmetries is a symmetry. For example, $s_{ac} * s_v = r_1$, $s_V * s_V = r_0$, $r_1 * r_2 = r_3$; here if we write, say, $s_{ac} * r_1$, then s_{ac} is performed first, and r_1 is performed second (this is important, because composition is not commutative).

Other results appear in the following *multiplication table*:

$*$	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_0	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_1	r_1	r_2	r_3	r_0	s_{ac}	s_{bd}	s_V	s_H
r_2	r_2	r_3	r_0	r_1	s_V	s_H	s_{bd}	s_{ac}
r_3	r_3	r_0	r_1	r_2	s_{bd}	s_{ac}	s_H	s_V
s_H	s_H	s_{bd}	s_V	s_{ac}	r_0	r_2	r_3	r_1
s_V	s_V	s_{ac}	s_H	s_{bd}	r_2	r_0	r_1	r_3
s_{ac}	s_{ac}	s_H	s_{bd}	s_V	r_1	r_3	r_0	r_2
s_{bd}	s_{bd}	s_V	s_{ac}	s_H	r_3	r_1	r_2	r_0

Here (for example) the element s_V at the intersection of the sixth column and the fourth row is $s_V = r_2 * s_H$, the composition of r_2 and s_H in that order (first the transformation r_2 is performed, then s_V). Composition is *noncommutative.*

Obviously, composition is *associative.* The set $\mathrm{Sym}(\square)$ contains the *identity* transformation r_0 (also denoted id or **1**). Any element X of $\mathrm{Sym}(\square)$ has an *inverse* X^{-1}, i.e., an element such that $X * X^{-1} = X^{-1} * X = \mathbf{1}$.

The set $\mathrm{Sym}(\square)$ supplied with the composition operation is called the *symmetry group of the square.*

(2) *Symmetries of the cube.* Let

$$I^3 = \{(x,\ y,\ z) \in \mathbb{R}^3 |\ -1 \le x \le 1,\ -1 \le y \le 1,\ -1 \le z \le 1\}$$

be the unit cube. A *symmetry* of the cube is defined as any isometric transformation of I^3. The composition of two symmetries (of I^3) is a symmetry. There is a total of 48 symmetries of the cube (list them!). The set of all symmetries of the cube supplied with the composition operation is called the *symmetry group of the cube* and is denoted by $\mathrm{Sym}(I^3)$. This group is also associative, noncommutative, has an identity, and all its elements have inverses.

(3) *Symmetries of the circle.* Let $S^1 := \{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 = 1\}$ be the unit circle. Denote by $\mathrm{Sym}(S^1)$ the set of all isometries of the circle. The elements of $\mathrm{Sym}(S^1)$ are of two types: rotations r_α, $\alpha \in [0, 2\pi)$, and axial symmetries s_α, $\alpha \in [0, 2\pi)$. The composition of rotations is given by the formula

$$r_\alpha * r_\beta = r_{(\alpha+\beta) \bmod 2\pi}.$$

There is also a nice formula for the composition of axial symmetries (find it!). The set of all isometries of the circle supplied with the composition operation is called the *symmetry group of the circle* and is denoted by $\mathrm{Sym}(S^1)$. The group $Sym(S^1)$ has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

(4) *Elliptic plane geometry.* Consider the set $X = \mathrm{Ant}(S^2)$ of all pairs of diametrically opposed points on the sphere. (Thus elements of X are *not* ordinary points, but *pairs of points.*) Then $X = \mathrm{Ant}(S^2)$, the set of diametrically opposed points on S^2 under the action of the orthogonal group $\mathrm{O}(3)$ given by $g\colon (x, -x) \mapsto (g(x), g(-x))$ (which is well defined because $g(-x) = -g(x)$) is called *elliptic plane geometry.* (Readers not familiar with the notion of orthogonal group are referred to the Appendix.)

The elliptic "plane" (the set X) contains another classical geometric object, namely the Möbius band. Can you find it there?

§ 1.2. Main definitions. Let X be a set (finite or infinite) of arbitrary elements called *points*. By definition, a *transformation group G acting on X* is a (nonempty) set G of bijections of X supplied with a composition operation denoted by $*$ and satisfying the following conditions:

(i) G is closed under composition, i.e., for any transformations $g, g' \in G$ the composition $g * g'$ belongs to G;

(ii) G is closed under taking inverses, i.e., for any transformation $g \in G$, its inverse g^{-1} belongs to G.

These conditions immediately imply that G contains the identity transformation. Indeed, take any $g \in G$; by (ii), $g^{-1} \in G$; by (i), $g^{-1} * g \in G$; but $g^{-1} * g = \mathrm{id}$ (by definition of inverse element). Note also that composition in G is associative (because the composition of mappings, in particular transformations, is always associative).

If $x \in X$ and $g \in G$, then by xg we denote the image of the point x under the transformation g. (The more usual notation $g(x)$ is not convenient: we have $x(g * h) = (xg)h$, but $(g * h)(x) = h(g(x))$.)

A mapping of transformation groups $\alpha\colon G \to H$ is called a *homomorphism* if α respects the product structure, i.e., $\alpha(g_1 * g_2) = \alpha(g_1) * \alpha(g_2)$ for all $g_1, g_2 \in G$; a homomorphism α is a *monomorphism* (an *epimorphism*) if α is injective (surjective, respectively).

Two transformation groups G and H acting on two sets X and Y ($X = Y$ is not forbidden) are called *isomorphic* if there exists a bijective mapping $\varphi\colon G \to H$ such that $\varphi(g * g') = \varphi(g) * \varphi(g')$ for all $g, g' \in G$.

In this course, a pair $(X : G)$, where G is a transformation group acting on X, will be called the *geometry* (*in the sense of Klein*) *of* G *on* X. The four examples in §2 define the geometry of the square, the geometry of the cube, the geometry of the circle, and the geometry of the projective plane. Another example is the set $\mathrm{Bij}(X)$ of *all* bijections of any set X. Note that $\mathrm{Bij}(X)$ contains any transformation group G acting on X.

If X is a subset of $\mathbb{R}^n$ and $(X : G)$ is a geometry, then a subset $F \subset X$ is called a *fundamental region* or *domain* of this geometry (or of the action of G on X) if

- F is open;
- $F \bigcap g(F) = \varnothing$ for any $g \in G$ (except $g =$id).;
- $X = \bigcup\limits_{g\in G} \mathrm{Clos}(g(F))$, where Clos(.) denotes the closure of a set.

For example, in the case of the square, a fundamental domain of the action of $\mathrm{Sym}(\square)$ is the triangle AOM, where O is the center of the square and M is the midpoint of side AB; of course $\mathrm{Sym}(\square)$ has many other fundamental regions. Thus fundamental regions are not necessarily unique. Moreover, fundamental regions don't always exist: for instance, $\mathrm{Sym}(S^1)$ (and other "continuous" geometries) do not have any fundamental regions.

§ 1.3. Order, generators, subgroups. By definition, the *order of an element* g of a group G acting on a set X is the smallest natural number k such that

$$g^k = \underbrace{g * g * \cdots * g}_{k} = \mathbf{1};$$

notation: $\mathrm{ord}(g)$. The *order of the group* G itself is the number of its elements and is denoted by $|G|$.

For example, $|\,\mathrm{Sym}(\square)| = 8$, $\mathrm{ord}(r_1) = 4$, $\mathrm{ord}(s_B) = 2$, $|\,\mathrm{Sym}(S^1)| = \infty$, and $\mathrm{Sym}(S^1)$ contains elements of infinite order and elements of any finite order (find an element of order 17).

Suppose G is a transformation group acting on some set X. Then a subset H of G is called a *subgroup of* G if H is closed with respect to composition (i.e., $h, h' \in H \Rightarrow h * h' \in H$) and closed with respect to inverse elements (i.e., $h \in H \Rightarrow h^{-1} \in H$)). Any subgroup $H \subset G$ contains the identity element **1** (prove this!). For example, the set of rotations

$$\mathrm{Rot}(\square) = \{r_0,\ r_1,\ r_2,\ r_3\} \subset \mathrm{Sym}(\square)$$

is a subgroup (of order 4) of the symmetry group of the square, but the set of $\{s_H,\ s_V,\ s_{ac},\ s_{bd}\}$ is not a subgroup (why?). Are there any subgroups of order 2 in $\mathrm{Sym}(\square)$?

Suppose that G is a transformation group (finite or infinite); a family $\{g_1, \dots, g_k\} \subset G$ is (by definition) a set of *generators* of G if any $g \in G$ is the composition of some elements of this family. For example, if a is an element of order 5 in some transformation group G, then $\{a, a^2, a^3, a^4, a^5 = \mathrm{id}\}$ is a subgroup of G generated by the one-element set $\{a\}$. Another example: the two elements r_1 and s_H generate the group $\mathrm{Sym}(\square)$ (check this!).

§ 1.4. The category of geometries. Category theory, or “abstract nonsense” is a very general formal algebraic language; it will not be studied or used in this course. However, you should know that a category is a class of “objects” and “morphisms” satisfying certain axioms; for example, the category of sets is the class of sets and their mappings, the category of groups is the class of groups and their homomorphisms.

Similarly, the category of geometries is the class of all geometries (in the sense of Klein, see §1.2) and their morphisms; by definition, a *morphism* (or *equivariant map*) of a geometry $(X : G)$ to a geometry $(Y : H)$ is a correspondence between them respecting their geometric structure (the group action); more precisely, a morphism is a pair (φ, α), where $\varphi\colon X \to Y$ is a mapping and $\alpha\colon G \to H$ is a homomorphism such that $\varphi(xg) = (\varphi(x))(\alpha(g))$ for all $x \in X$ and $g \in G$. A morphism (φ, α) of geometries is an *embedding* if φ and α are injective, and then $(X : G)$ and $(\varphi(X) : \alpha(G))$ is called a *subgeometry* of $(Y : H)$; it is a *surjection* if φ and α are surjective.

Two geometries $(X : G)$ and $(Y : H)$ are called *isomorphic*, if there exist a bijection $\varphi\colon X \to Y$ and an isomorphism $\alpha\colon G \to H$ such that

$$\varphi(xg) = (\varphi(x))(\alpha(g)) \quad \text{for all} \quad x \in X \quad \text{and all} \quad g \in G.$$

This is a typical definition in the style of algebraic “abstract nonsense”. It is so trivial and so tautological that it is almost impossible to understand.

To understand it, try to prove that $\alpha(\mathbf{1}) = \mathbf{1}$, if the corresponding geometries are equivalent.

§ 1.5. Some philosophical remarks. The examples in §1.1 (square, cube, circle) were taken from elementary school geometry. This was done *to motivate* the choice of the action of the corresponding transformation group. Now, in the example of the cube, let us forget school geometry: instead of the cube I^3 with its vertices, edges, faces, angles, interior points and other structure, consider the abstract set of points $\{A, B, C, D, A', B', C', D'\}$ and define the "isometries" of the "cube" as a set of 48 bijections; for example, the "rotation by 270°" about the vertical axis is the bijection

$$A \mapsto B,\ \ B \mapsto C,\ \ C \mapsto D,\ \ D \mapsto A,\ \ A' \mapsto B',\ \ B' \mapsto C',\ \ D' \mapsto A',$$

and the 47 other "isometries" are defined similarly. Then (still forgetting school geometry), we can *define* vertices, edges (AB is an edge, but AC' is not), faces, prove that all edges are congruent, all faces are congruent, the "cube" can "rotate" about each vertex, etc.). The result is the *intrinsic geometry of the set of vertices* of the cube.

Above we called $Q^{(3)} := \mathrm{Sym}(I^3)$ the geometry of the cube; $Q^{(3)})$ is not the same geometry as the *geometry of the vertex set of the cube*

$$Q^{(0)} := \mathrm{Sym}(A, B, C, D, A', B', C', D').$$

Of course the group G acting in these two geometries is the *same group* of order 48, but it acts on two *different sets*: the (infinite) set of points of the cube I^3 and the (finite) set of its 8 vertices $A, B, C, D, A', B', C', D'$. Thus the algebra of the two situations is the same, but the geometry is different. The geometry of the solid cube I^3 is of course much richer than the geometry of the vertex set of the cube. For example, we can define line segments inside the cube, establish their congruence, etc. (Note that segments of the same length inside the cube are not necessarily congruent in the geometry of the cube!)

Another example: the set of three points $\{A, B, C\}$ with two transformations, namely the identity and the "symmetry"

$$A \mapsto A,\ \ B \mapsto C,\ \ C \mapsto B.$$

What should this geometry be called? Yes, it should be called the intrinsic geometry of the vertex set of the isosceles triangle (why?).

§ 1.6. Problems.

1.1. List all the elements (indicating their orders) of the symmetry group (i.e., isometry group) of the equilateral triangle. List all its subgroups. How many elements are there in the group of motions (i.e., orientation-preserving isometries) of the equilateral triangle.

1.2. Answer the same questions as in Problem 1.1 for

(a) the regular n-gon (i.e., the regular polygon of n sides); consider the cases of odd and even n separately;

(b) the regular tetrahedron;

(c) the cube;

(e)* the dodecahedron;

(f)* the icosahedron;

(g) the regular pyramid with four lateral faces.

1.3. Embed the geometry of the square into the geometry of the cube, and the geometry of the circle into the geometry of the sphere.

1.4. For what n and m can the geometry of the regular n-gon be embedded in the geometry of the regular m-gon?

1.5. Let G be the symmetry group of the regular tetrahedron. Find all its subgroups of order 2 and describe their action geometrically.

1.6. Let G^+ be the group of motions of the cube. Indicate four subsets of the cube on which G^+ acts by all possible permutations.

1.7. Let G be the symmetry group of the dodecahedron. Indicate subsets of the dodecahedron on which G acts by all possible permutations.

1.8. Find a minimal system of generators for the symmetry group of

(a) the regular tetrahedron;

(b) the cube.

1.9. Embed the geometry of the cube in the geometry of the icosahedron. How many such (different) embeddings are there?

1.10. Describe fundamental domains of the symmetry group of

(a) the cube;

(b) the icosahedron;

(c) the regular tetrahedron.

1.11. Describe the Möbius band as a subset of $\mathbb{R}P^2$.

Lecture 2

Abstract Groups and Group Presentations

In order to study geometries more complicated than the toy models with which we played in the previous lecture, we need to know much more about group theory. Accordingly, in this lecture we study the relevant facts of this theory (which will constantly be used in what follows).

The theory of transformation groups began in the work of several great mathematicians: Lagrange, Abel, Galois, Sophus Lie, Felix Klein, Élie Cartan, Herman Weyl. At the beginning of the 20th century, algebraists decided to generalize this theory to the formal theory *abstract groups*. In this lecture, we will study this formal theory and learn that it is not a generalization at all: Cayley's Theorem says that all abstract groups are actually transformation groups. We will also learn that two important classes of groups (*free groups* and *permutation groups*) have certain universality properties. Finally, we will find out how *to present groups* by means of generators and relations; this allows to replace computations with groups by games with words.

§ 2.1. Abstract groups. By definition, an (abstract) *group* is a set G of arbitrary elements supplied with a binary operation $*$ (usually called *multiplication*) if it obeys the following rules:

- (*neutral element axiom*) there exists a unique element $e \in G$ such that $g * e = e * g = g$ for any $g \in G$;
- (*inverse element axiom*) for any $g \in G$ there exists a unique element $g^{-1} \in G$, called *inverse to* g, such that $g * g^{-1} = g^{-1} * g = e$;
- (*associativity axiom*) $(g * h) * k = g * (h * k)$ for all $g, h, k \in G$.

A group $(G, *)$ is called *commutative* or *Abelian* if $g * h = h * g$ for all $g, h \in G$ (in that case the operation is usually called a *sum* and the inverse element is usually denoted by $-g$ instead of g^{-1}). Let $(G, *)$, $(H, \circ)$ be groups, $\varphi\colon G \to H$ a mapping; φ is called a *homomorphism* (or a *morphism* *of groups*) if it preserves the operations, i.e., $\varphi(g_1 * g_2) = \varphi(g_1) \circ \varphi(g_2)$); a homomorphism φ is a *monomorphism* (respectively, an *epimorphism* or an *isomorphism*) if the mapping φ is injective (resp., surjective or bijective). From the point of view of abstract algebra, isomorphic groups are identical.

The notions of *order* (of elements of a group and of the group itself) and of *generator* for abstract groups are defined exactly like for transformation groups (see § 1.3).

Examples. Any transformation group is a group; the following groups are standard (you should know them):

- $\mathbb{Z}_m$, residues modulo m;
- S_n, permutations on n objects;
- $F(a_1, \ldots, a_k)$, the free group on k generators;
- $GL(n)$, nondegenerate linear operators on $\mathbb{R}^n$;
- $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, orthogonal and positive orthogonal operators on $\mathbb{R}^n$.

From now on we omit the group operation symbol, i.e., we write gh instead of $g * h$.

A *subgroup* H of a group G is a subset of G which satisfies the group axioms. A subgroup $H \subset G$ is *normal* if $gHg^{-1} = H$ for any $g \in G$. If H is a subgroup of G, then a *coset* $H_g \subset G$, for some $g \in G$, is the set of all elements of the form gh for $h \in H$. Two cosets either do not intersect or coincide. If H is normal, there is a well-defined operation in the family of cosets: the product of two cosets is the coset containing the product of any two elements of these cosets; the family of cosets supplied with this product operation satisfies the group axioms; it is called the *quotient group* of G by H and is denoted by G/H.

Example. in the additive group of integers $(\mathbb{Z}, +)$, elements of the form $5k$, $k \in \mathbb{Z}$, constitute a normal subgroup, denoted $5\mathbb{Z}$; the corresponding quotient group $\mathbb{Z}/5\mathbb{Z}$ is isomorphic to the group $\mathbb{Z}_5$.

§ 2.2. The Lagrange theorem. The elementary theorem proved below is the first structure theorem about abstract groups. It was proved (for transformation groups) almost two centuries ago by Lagrange.

Theorem 2.1. *If H is a subgroup of a finite group G, then the order of H divides the order of G.*

P r o o f. The cosets of H in G form a partition of the set of elements of G and all have the same number of elements as H. □

Corollary. *Any group G of prime order p is isomorphic to $\mathbb{Z}_p$.*

P r o o f. Let $g \in G$, $g \neq e$. Let m be the smallest positive integer such that $g^m = e$. Then $H := \{e, g, g^2, \ldots, g^{m-1}\}$ is a subgroup of G (why?) and by Theorem 1, m divides p. This is impossible unless $m = p$, but then $H = G$ is obviously isomorphic to $\mathbb{Z}_p$. $\square$

§ 2.3. Free groups and permutations. In this section, we study two classes of groups: the free groups (which have the "least structure") and the permutation groups (which have the "most structure").

Let $F := \{f_1, \ldots, f_k\}$ be a set of symbols. Then the set of formal symbols (called *letters*)

$$A := \{e, f_1, \ldots, f_k, f_1^{-1}, \ldots, f_k^{-1}\}$$

will be our *alphabet*. A string of letters from our alphabet will be called a *word*. Two words w_1 and w_2 are called *equivalent*, if one can be obtained from the other by using the following *trivial relations* $f_i f_i^{-1} = f_i^{-1} f_i = e$ for any i and $fe = ef = f$ for any $f \in A$; for example

$$f_1 f_3^{-1} \sim f_1 f_3^{-1} e \sim f_1 f_3^{-1} f_2 f_2^{-1} \sim f_1 f_3^{-1} f_2 e f_2^{-1}.$$

The *product* of two words is defined as their concatenation (i.e., writing then one after the other). The *free group* with generators $f_1, \ldots, f_k$ is defined as the set of equivalence classes of words supplied with the product (concatenation) operation and is denoted by $\mathbb{F} = \mathbb{F}[f_1, \ldots, f_k]$.

For example, $\mathbb{F}[f]$ is isomorphic to $(\mathbb{Z}, +)$, while $\mathbb{F}[f_1, f_2]$ is not commutative.

The *permutation group* on n objects is the family of all bijections of the set $\{1, 2, \ldots, n\}$ supplied with the operation of composition; it is denoted by S_n. It consists of $n!$ elements denoted by $[i_1, \ldots, i_n]$, where $i_k := \beta(k)$ and β is the bijection defining the given element.

Free groups and permutation groups have important "universality" properties.

Theorem 2.2. (i) *For any finite group G there exists a monomorphism of G into S_n for some n.*

(ii) *For any group G with a finite number of generators there exists an epimorphism of S_n (for some n) onto G.*

S k e t c h o f t h e p r o o f. (i) Let $|G| = n$ and $g_0 \in G$; then the mapping

$$G \ni g \mapsto g g_0 \in G$$

is a bijection of the n-element set G; this bijection can be identified with an element of S_n. Thus we have obtained a mapping $G \to S_n$; it is not difficult to prove that this mapping is a monomorphism.

(ii) Let $g_1, \ldots, g_n$ be a set of generators of G. Then it is not difficult to prove that the mapping

$$\alpha\colon \mathbb{F}[f_1, \ldots, f_n] \to G \quad \text{given by} \quad \alpha(f_i) = g_i,\ i = 1, \ldots, n,$$

is an epimorphism. □

§ 2.4. Group presentations. A presentation of a group is a way of defining the group by means of equations (called *defining relations*) in the generators of the group. The formal definition is the following. An expression of the form $G = \langle g_1, \ldots, g_n \ : R_1 = \cdots = R_k = \mathbf{1} \rangle$, where $R_1, \ldots R_k$ are words (*relators*) in the alphabet $A = \{g_1, \ldots, g_n g_1^{-1}, \ldots, g_n^{-1}\}$, is called a *presentation* of the group G; the group G is defined by its presentation as the quotient group

$$\mathbb{F}[g_1, \ldots, g_n]/\{R_1, \ldots, R_k\},$$

where $\{R_1, \ldots, R_k\}$ is the minimal (by inclusion) normal subgroup of the group $\mathbb{F}[g_1, \ldots, g_n]$ containing the elements (relators) $R_1, \ldots, R_k$.

This formal definition is difficult to understand. But the notion of group presentation is simple. It means that elements of G are words in the alphabet A defined up to the trivial relations (recall § 2.3 above) and all the relations $R_1 = e, \ldots, R_k = e$; the product is concatenation.

Here are some examples:

- $\mathbb{Z}_m = \langle a \ : a^m \rangle$;
- $\mathbb{F}[g_1, \ldots, g_n] = \langle g_1, \ldots, g_n \ : \quad \rangle$;
- $\mathbb{S}_3 = \langle s_1, s_2, s_3 \ : s_1^2,\ s_2^2,\ s_3^2,\ s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1},\ s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} \rangle$.

More details and examples will be given in the exercise class.

§ 2.5. Cayley's theorem. The following theorem (due to Cayley) shows that the notion of abstract group is not a real generalization: all groups are in fact transformation groups!

Theorem 2.3. *Any group G is a transformation group acting on the set G by right multiplication: $g \mapsto g g_0$ for any $g_0 \in G$.*

The proof is a straightforward verification.

Corollary. *Any group is a geometry in the sense of Klein (i.e., in the sense of formal definition given in* § 1.2*).*

This corollary shows that the definition of geometry given in § 1.2 is of course too general; additional restrictions on the set of elements and the transformation group are needed to obtain an object about which most mathematicians will agree that it is a *bona fide* geometry. However, there seems to be no formal agreement on this subject, so that the "additional restrictions" to be imposed are a matter of opinion, and we will not specify any (at least on the formal level) in this course.

§ 2.6. Problems.

2.1. Describe all the finite groups of order 6 or less and supply each with a geometric interpretation.

2.2. Describe all the normal subgroups and the corresponding quotient groups of

(a) the equilateral triangle;

(b) the motion group of the regular tetrahedron.

2.3. Let G be the motion group of the plane, P its subgroup of parallel translations, and R its subgroup of rotations with fixed center O. Prove that the subgroup P is normal and the quotient group G/P is isomorphic to R.

2.4. Prove that if the order of a subgroup is equal to half the order of the group (i.e., the subgroup is of *index* 2), then the subgroup is normal.

2.5. Find all the orbits and stabilizers of all the points of the group G generated by the permutation

$$(5\ 8\ 3\ 9\ 4\ 10\ 6\ 2\ 1\ 7) \in S_{10}$$

acting on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

2.6. Find the maximal order of elements in the group (a) S_5; (b) S_{13}.

2.7. Find the least natural number n such that the group S_{13} has no elements of order n.

2.8. Prove that the permutation group S_n is generated by the transposition $(1\,2)$ and the cycle $(1\,2\ldots n)$.

2.9. Present the symmetry group of the equilateral triangle by generators and relations in two different ways.

2.10. How many homomorphisms of the free group in two generators into the permutation group S_3 are there? How many of them are epimorphisms?

2.11. Prove that the group presented as follows

$$\langle a,\, b \mid a^2 = b^n = a^{-1}bab = 1\rangle$$

is isomorphic to the dihedral group $\mathbb{D}_n$ (defined in Lecture 3).

2.12. Show that if the elements a and b of a group satisfy the relations $a^5 = b^3 = 1$ and $b^{-1}ab = a^2$, then $a = 1$.

Lecture 3

Finite Subgroups of the Isometry Group of the Sphere and the Platonic Bodies

This lecture is devoted to the classification of regular polyhedra (the five "Platonic bodies", see Figure 3.1), whose aesthetic and scientific ap-

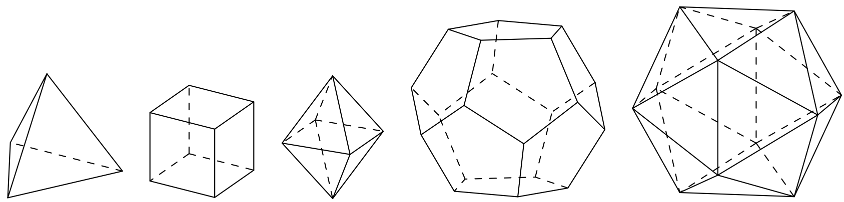

Fig. 3.1. The five Platonic bodies

peal has not weakened over the centuries, attracting, from the philosophical and artistic point of view, such great thinkers as Plato and Leonardo da Vinci (see his models of the dodecahedron and the icosahedron in Fig. 3.2), the astronomer Kepler (planetary orbits, see his weird engraving of inscribed Platonic bodies, supposedly indicating the distances from the planets to the Sun, Fig. 3.3), mathematicians and physicists such as Pythagoras and Heisenberg (the "singing spheres").

The proof that we give here is essentially group theoretic (we reduce the classification problem of regular polyhedra to classifying finite subgroups of the orthogonal group O(3), or, which is the same thing, the isometry group of the sphere $\mathbb{S}^2$). This proof is quite natural and more geometric, in a deeper sense, than the tedious and eclectic space geometry proof known from ancient times. Thus each of the Platonic bodies is a geometry in the sense of Klein with its own finite transformation group. The proof also serves as a beautiful illustration of the idea that transformation groups are the formalization of the idea of symmetry.

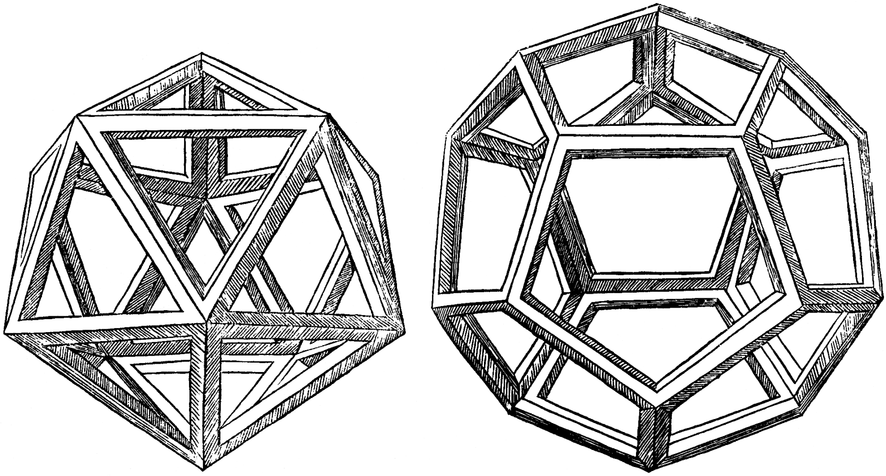

Fig. 3.2. Da Vinci engravings

§ 3.1. Orbits, stabilizers, class formula. Let $(X : G)$ be some transformation group acting on a set X and let $x \in X$. Then the *orbit* of x is defined as

$$\mathrm{Orb}(x) := \{g(x) | g \in G\} \subset X,$$

and the *stabilizer* of x is

$$\mathrm{St}(x) := \{g \in G | g(x) = x\} \subset G.$$

For example, if $X = \mathbb{R}^2$ and G is the rotation group of the plane about the origin, then the set of orbits consists of the origin and all concentric circles centered at the origin; the stabilizer of the origin is the whole group G, and the stabilizers of all the other points of $\mathbb{R}^2$ are trivial (i.e., they consist of one element — the identity $\mathrm{id} \in G$).

Suppose $(X : G)$ is an action of a finite transformation group on a finite set. Then the number of points of G is (obviously) given by

$$\boxed{|G| = |\mathrm{Orb}(x)| \times |\mathrm{St}(x)|}$$

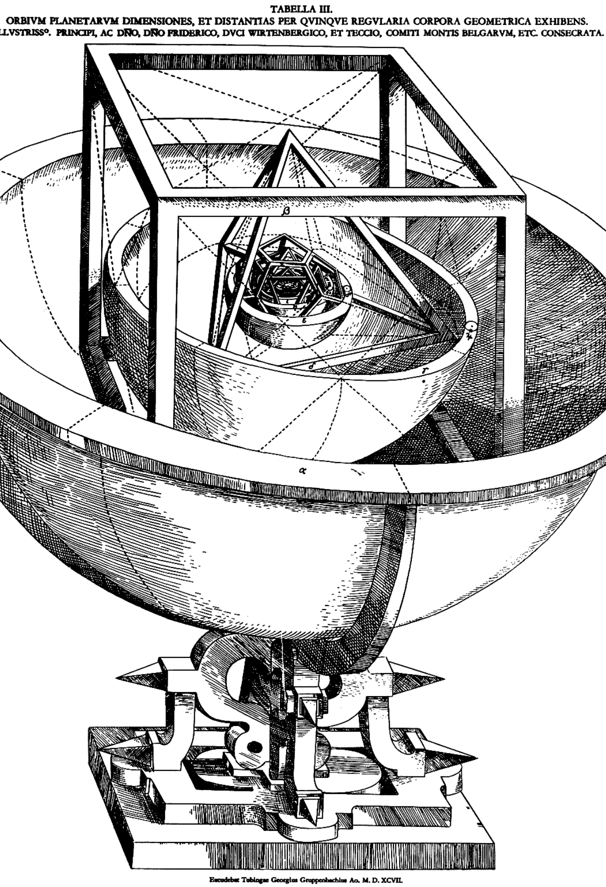

Fig. 3.3. Kepler's theory of planetary orbits

for any $x \in X$. Now let $A \subset X$ be a set that intersects each orbit at exactly one point. Then the number of points of X is given by the formula

$$\boxed{|X| \ = \ \sum_{x \in A} \frac{|G|}{|St(x)|}},$$

called the *class formula.* This formula, just as the previous one, follows immediately from definitions.

§ 3.2. Finite subgroups of SO(3). Consider the geometry $(X : G)$ of the two-dimensional sphere

$$X = \mathbb{S}^2 := \{(x,\, y,\, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$$

defined by the action of its isometry group $G = \operatorname{Sym} S^2$. (In linear algebra courses this group G is defined in a different (but equivalent) way and is usually denoted by O(3).) The group $G = \mathrm{O}(3)$ contains the rotation subgroup $G^+ = \mathrm{Rot}(\mathbb{S}^2)$: each element of G^+ is a rotation of the sphere about some axis passing through the origin by some angle φ, $0 \le \varphi < 2\pi$. In linear algebra courses this group G^+ is defined in a different (but equivalent) way and is usually denoted by SO(3).

Our goal is to find the finite subgroups of $G^+ = \mathrm{SO}(3)$ and of $G = \mathrm{O}(3)$. We begin with some examples.

(i) The *monohedral group* $\mathbb{Z}_n$ for $n \ge 2$ (n elements); its elements are rotations about the vertical axis (i.e., the z-axis) by angles of $2k\pi/n$, where $k = 0, \ldots, n-1$. (N.B.: the term monohedral is not standard.)

(ii) The *dihedral group* $\mathbb{D}_n$ for some $n \ge 1$ ($2n$ elements); the group $\mathbb{D}_n$ is the isometry group of the regular n-gon (lying in the horizontal plane Oxy and inscribed in the sphere $\mathbb{S}^2$); $\mathbb{D}_n$ consists of n rotations (by angles of $2k\pi/n$, $k = 0, 1, \ldots, n-1$) and n symmetries with respect to the horizontal lines passing through the center, the vertices, and the midpoints of the sides (be careful: these lines are different when n is even or odd — look at the figure!). Note that $\mathbb{D}(n) \subset \mathrm{SO}(3)$ and the symmetries of $D(n)$ with respect to the horizontal lines are actually *rotations* in space (about these lines by angles of 180°).

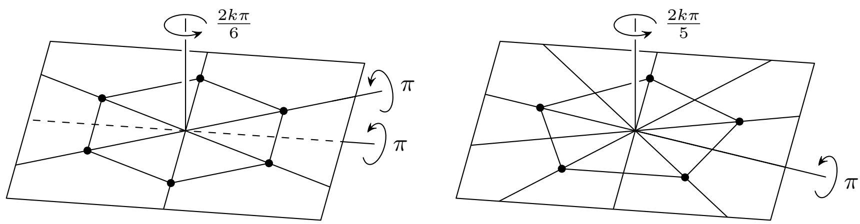

Fig. 3.4. The dihedral group $\mathbb{D}_n$ for $n = 5$ and $n = 6$

(iii) The *isometry group of the regular tetrahedron* inscribed in the sphere $\mathbb{S}^2$ (24 elements), denoted by $\mathrm{Sym}(\Delta^3)$ (we will see later that $\mathrm{Sym}(\Delta^3)$ is isomorphic to the permutation group S_4) and its (12 element) rotation subgroup:

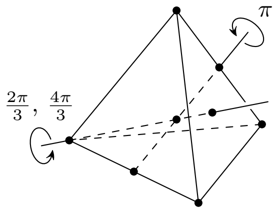

Fig. 3.5. $\mathrm{Sym}(\Delta^3)$

$$\mathrm{Rot}(\Delta^3) = \mathrm{Sym}^+(\Delta^3) \subset \mathrm{Sym}(\Delta^3),$$

consisting of 8 rotations about 4 axes (containing one vertex) by angles of $2\pi/3$ and $4\pi/3$, three rotations by π (describe them!) and the identity.

(iv) The *isometry group* $\mathrm{Sym}(I^3)$ *of the cube* (48 elements) and its rotation subgroup (consisting of (24 elements)):

$$\mathrm{Rot}(I^3) = \mathrm{Sym}^+(I^3) \subset \mathrm{Sym}(I^3)$$

(see the previous lecture). If we join the center of each of the 6 faces of the cube by segments to the four neighboring centers, we obtain the carcass of the *octahedron* dual to the cube (see Fig. 3.6 (a). The octahedron has 6 vertices and 8 triangular faces; its isometry group is obviously the same as that of the cube.

(v) The *isometry group* $\mathrm{Sym}(\mathrm{Dod})$ *of the dodecahedron* (120 elements) and its (60 element) rotation subgroup

$$\mathrm{Rot}(\mathrm{Dod}) = \mathrm{Sym}^+(\mathrm{Dod}) \subset \mathrm{Sym}(\mathrm{Dod}).$$

The dodecahedron is the (regular) polyhedron (inscribed in the sphere $\mathbb{S}^2$) with 12 faces (congruent regular pentagons), 30 edges, and 20 vertices (see Fig. 3.6 (b)). The existence of such a polyhedron will be proved at the end of this lecture. Joining the centers of the faces of the dodecahedron

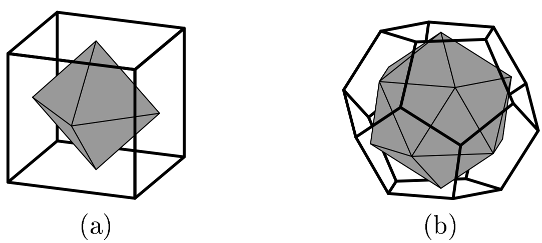

Fig. 3.6. Dual pairs cube-octahedron and dodecahedron-icosahedron

having a common edge (see Fig. 3.6 (b) again, as well as Fig. 3.2), we get the *icosahedron* dual to the dodecahedron, which has the same transformation group.

The following theorem states that there are no other finite subgroups.

Theorem 3.1. *Any finite subgroup of* $G^+ = \mathrm{Sym}^+(\mathbb{S}^2) = \mathrm{SO}(3)$ *is isomorphic to one of the following groups*:

(i) $\mathbb{Z}_n$, (ii) $\mathbb{D}_n$, (iii) $\mathrm{Rot}(\Delta^3)$, (iv) $\mathrm{Sym}^+(I^3)$, (v) $\mathrm{Sym}^+(\mathrm{Dod})$.

Sketch of the proof. It is known (see the linear algebra course and the Appendix) that any element of SO(3) (and hence of G^+) is a rotation about a diameter of the sphere $\mathbb{S}^2$ and has two fixed points (the ends of the diameter). Let F be the set of fixed points of the group G^+:

$$F = \{x \in \mathbb{S}^2 \mid \exists\, g \in G^+ - \mathrm{id},\ g(x) = x\}.$$

Consider the (finite) geometry $(F : G^+)$ and let A be a set containing one point in each orbit of G^+ in F. First we claim that the number of points in F is $|F| = |A||G^+| - 2(|G^+| - 1)$. This will be proved in the exercise class. Using the class formula from § 3.1, we can write $|F| = \sum_{x\in A} |G^+|/v(x)$, where $v(x) = |\mathrm{St}(x)|$. Note that $v(x)$ is the order of the rotation subgroup of G^+ determined by the diameter containing x. Replacing $|F|$ by its value found above and dividing by $|G^+|$, we obtain

$$\boxed{2 - \frac{2}{|G^+|} = \sum_{x\in A}\left(1 - \frac{1}{v(x)}\right)}.$$

The left-hand side of this formula is less than 2; hence the sum in the right-hand side can contain only 2 or 3 summands; therefore there can be only 2 or 3 orbits. Denote by x_1, x_2, x_3 points of these three orbits; denote by v_1, v_2, v_3 the values of $v(x)$ (in nondecreasing order). It is not difficult to see that only the following cases are allowed by the formula above:

	v_1	v_2	v_3	$\lvert G^+\rvert$
case 1	n	n	—	n
case 2	2	2	n	$2n$
case 3	2	3	3	12
case 4	2	3	4	24
case 5	2	3	5	60

The five cases correspond to (i)—(v), respectively. □

(For the details of the proof, see, for example, the book *Géométrie* by Marcel Berger, pp. 102—108).

Corollary. *Any finite subgroup of* $G = \mathrm{Sym}(S^2)$ *is isomorphic to one of the following groups*:

(i) $\mathbb{Z}_n$, (ii) $\mathbb{D}_n$, (iii) S_4, (iv) $\mathrm{Sym}(I^3)$, (v) $\mathrm{Sym}(\mathrm{Dod})$.

§ 3.3. The five regular polyhedra. A *regular polyhedron* is defined as a convex polyhedron (inscribed in the sphere $\mathbb{S}^2$) such that

(i) all its faces are congruent regular polygons of k sides for some $k > 2$;

(ii) the endpoints of all the edges issuing from each vertex lie in one plane and form a regular l-gon for some $l > 2$.

Theorem 3.2. *There are exactly five different regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.*

Proof. This theorem follows from the Corollary to Theorem 3.1. Indeed, the definition implies that the isometry group of a regular polyhedron is finite and therefore must be one of the groups listed in Theorem 3.1. The two "series" (i) and (ii) do not give any (nondegenerate) polyhedra (why?). In case (iii), we get the tetrahedron (because its symmetry group is isomorphic to the permutation group S_4). In case (iv), we get the cube and its dual, the octahedron, in case (v), the dodecahedron and its dual, the icosahedron. □

Thus we obtain five geometries with three different group actions (tetrahedron, cube ~ octahedron, dodecahedron ~ icosahedron). To understand the group actions in these geometries, it is useful to construct their fundamental regions.

§ 3.4. The five Kepler cubes. Kepler observed that the cube can be inscribed in five different ways into the dodecahedron. Here we will perform the opposite construction: starting from the cube, we will construct a dodecahedron circumscribed to the cube. This will prove the existence of the dodecahedron.

Consider two copies $ABCDE$ and $A'B'C'D'E'$ of the regular pentagon with diagonals of length 1. Place these pentagons in the plane of the unit square $PQRS$ so that the diagonals BE and $B'E'$ are identified with PS and QR, respectively, and CD is parallel to $C'D'$. By rotating the pentagons in space about PS and QR, identify the sides CD and $C'D'$ above the square $PQRS$.

Now suppose $PQRS$ is the top face of the unit cube $PQRSP'Q'R'S'$. Place two more pentagons on the face $SRR'S'$ of the cube the same way as before, so that their parallel sides are parallel to SR. Now rotate these two pentagons until these parallel sides are identified. Then it is not hard to prove that the upper endpoint of the identified segment will coincide with one of the endpoints of the common (identified) segment of the first two pentagons. Perform similar constructions on the other faces of the cube. The polyhedron thus obtained will be the dodecahedron.

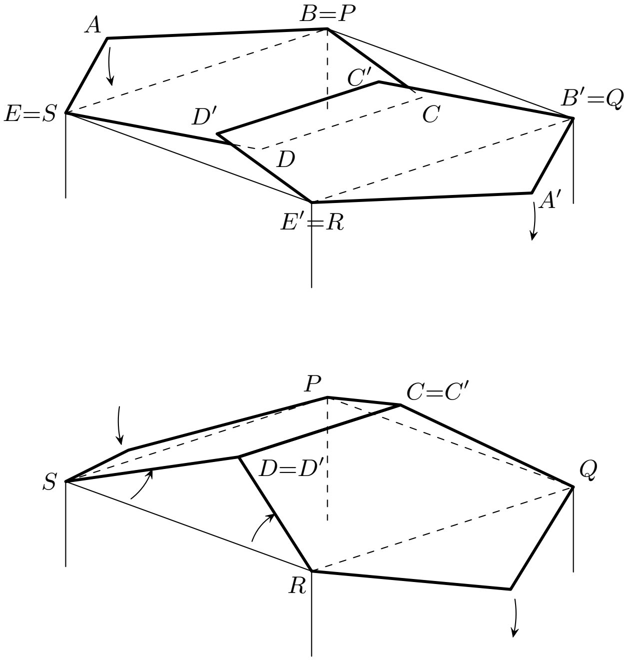

Fig. 3.7. Constructing the dodecahedron

§ 3.5. Problems.

3.1. A regular pyramid of six lateral sides is inscribed in the sphere $\mathbb{S}^2$. Find its symmetry (i.e., isometry) group and its group of motions. How does your answer relate to the theorem on finite subgroups of SO(3)?

3.2. Answer the same questions as in Problem 3.1 for

(a) the regular prism of six lateral sides;

(b) the regular truncated pyramid of five lateral sides;

(c) the double regular pyramid of six lateral sides (i.e., the union of two regular pyramids of six lateral sides with common base and vertices at the poles of the sphere);

3.3. Let G^+ be a finite subgroup of SO(3) acting on the sphere $\mathbb{S}^2$ and F the set of all the points fixed by nontrivial elements of G^+; prove that F is invariant with respect to the action of G^+ and

$$|F| = |G^+| \cdot |A| - 2(|G^+| - 1),$$

where $A \subset F$ is a set containing exactly one point from each orbit of the action of G^+ on the set F.

3.4. Does the motion group of the cube have a subgroup isomorphic to the motion group of the regular tetrahedron?

3.5. Does the motion group of the dodecahedron have a subgroup isomorphic to the motion group of the cube?

3.6. In the motion group of the cube, find all groups isomorphic to $\mathbb{Z}_n$ and $\mathbb{D}_n$. Does it have any other subgroups?

3.7. Prove the existence of the dodecahedron in detail.

3.8. The set F consists of all the vertices, all the midpoints of the edges, and the centers of the faces of the cube, and let G^+ be the motion group of the cube. Prove that G^+ acts on F and find all the orbits of this action and the stabilisators of all the points.

3.9. Same question for

(a) the regular tetrahedron; (b)* the dodecahedron.