PROBLEMS for Lecture 10

10.1. Prove that stereographic projection is conformal.

10.2. Prove that the map β constructed in 10.1.2 is bijective and show that any chord of $\mathbb{H}^2$ (i.e., any line in the Cayley–Klein model) is taken by β to the arc of the circle passing through X and Y and orthogonal to the absolute (i.e., to a line in the Poincaré disk model).

10.3. Prove the main relations between the hyperbolic functions indicated in Section 10.3.

10.4. Prove the hyperbolic sine theorem.

10.5. Prove the hyperbolic cosine theorem.

10.6. Prove that two triangles with equal sides are congruent in hyperbolic geometry.

10.7. Prove that in hyperbolic geometry two triangles having an equal angle and equal sides forming this angle are congruent.

10.8. Show that homothety is not conformal in hyperbolic geometry.

10.9. (a) Prove the formula for the angle of parallelism α for a point A and a line l:

$$\tanh(d) = \cos(\alpha)$$

where d is the distance from A to l (thereby showing that the angle of parallelism depends only on the distance from the point to the line).

(b) Prove that the previous formula is equivalent to the following one (obtained independently by Bolyai and Lobachevsky):

$$\tan\frac{\alpha}{2} = e^{-d}$$

10.10. Prove that in a triangle with right angle γ the sides a, b, c and their opposite angles $\alpha, \beta, \gamma = \pi/2$ satisfy the following relations:

$$\sinh a = \sinh c \sin\alpha;\ \tanh b = \tanh c \cos\alpha;\ \cot\alpha\cot\beta = \cosh c;\ \cos\alpha = \cosh a \sin\beta.$$

What do these relations tend to as a, b, c become very small?

10.11. Prove that the sides a, b, c and opposite angles α, β, γ of any triangle on the hyperbolic plane satisfy the following relations:

$$\text{(a)}\quad \cosh a \sin\beta = \cosh b \sin\alpha\cos\beta + \cos\alpha\sin\gamma;$$

$$\text{(b)}\quad \cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

10.12. Prove that if the corresponding angles of two triangles are equal, then the triangles are congruent.

10.13. Prove that all the points of the (Euclidean) straight line $y = kx$ that lie in the upper half plane $y > 0$ are equidistant from the (hyperbolic) straight line Oy.

10.14. (a) Prove that any hyperbolic circle contained in any one of the Poincaré models of hyperbolic geometry is actually a Euclidean circle.

(b) For the Poincaré upper half plane model, find the Euclidean center and radius of the hyperbolic circle of radius r centered at the point (a, b).

(c) For the Poincaré model in the unit disk D, find the relationship between the radii of the Euclidean and the hyperbolic circles centered at the center of D.

10.15. Prove the triangle inequality for the distance in the Poincaré half plane model.

10.16. Prove that the three (a) bissectors (b) medians (c) altitudes of any hyperbolic triangle intersect at one point.