PROBLEMS for Lecture 2

2.1. Describe all the finite groups of order 6 or less and supply each with a geometric interpretation.

2.2. Describe all the (nontrivial) normal subgroups and the corresponding quotient groups of
(a) the isometry group of the equilateral triangle;
(b) the isometry group of the regular tetrahedron.

2.3. Let G be the motion group of the plane, P its subgroup of parallel translations, and R its subgroup of rotations with fixed center O. Prove that the subgroup P is normal and the quotient group G/P is isomorphic to R.

2.4. Prove that if the order of a subgroup is equal to half the order of the group (i.e., the subgroup is of *index* 2), then the subgroup is normal.

2.5. Find all the orbits and stabilizers of all the points of the group $G \subset S_{10}$ generated by the permutation $[5, 8, 3, 9, 4, 10, 6, 2, 1, 7] \in S_{10}$ acting on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

2.6. Find the maximal order of elements in the group (a) S_5; (b) S_{13}.

2.7. Find the least natural number n such that the group S_{13} has no elements of order n.

2.8. Prove that the permutation group S_n is generated by the transposition $(1\,2) := [2, 1, 3, 4, \ldots, n]$ and the cycle $(1\,2 \ldots n) := [2, 3, \ldots, n, 1]$.

2.9. Present the symmetry group of the equilateral triangle by generators and relations in two different ways.

2.10. How many homomorphisms of the free group in two generators into the permutation group S_3 are there? How many of them are epimorphisms?

2.11. Prove that the group presented as $\langle a, b \mid a^2 = b^n = a^{-1}bab = 1\rangle$ is isomorphic to the dihedral group $\mathcal{D}_n$ (defined in Chapter 3).

2.12. Show that if the elements a and b of a group satisfy the relations $a^5 = b^3 = 1$ and $b^{-1}ab = a^2$, then $a = 1$.