PROBLEMS for Lecture 3

3.1. A regular pyramid of six lateral sides is inscribed in the sphere $\mathbb{S}^2$. Find its symmetry (i.e., isometry) group and its group of motions. How does your answer relate to the theorem on finite subgroups of $SO(3)$?

3.2. Answer the same questions as in Problem 3.1 for

(a) the regular prism of six lateral sides;

(b) the regular truncated pyramid of five lateral sides;

(c) the double regular pyramid of six lateral sides (i.e., the union of two regular pyramids of six lateral sides with common base and vertices at the poles of the sphere);

3.3. Let G^+ be a finite subgroup of $SO(3)$ acting on the sphere $\mathbb{S}^2$ and F the set of all the points fixed by nontrivial elements of G^+; prove that F is invariant with respect to the action of G^+ and

$$|F| = |G^+| \cdot |A| - 2(|G^+| - 1),$$

where $A \subset F$ is a set containing exactly one point from each orbit of the action of G^+ on the set F.

3.4. Does the motion group of the cube have a subgroup isomorphic to the motion group of the regular tetrahedron?

3.5. Does the motion group of the dodecahedron have a subgroup isomorphic to the motion group of the cube?

3.6. In the motion group of the cube, find all groups isomorphic to $\mathbb{Z}_n$ and $\mathbb{D}_n$ for various values of n. Does it have any other subgroups?

3.7. Prove the existence of the dodecahedron in detail.

3.8. Given a cube inscribed in the sphere, let the set F consist of all the vertices of the cube, all the intersection points of the lines joining the centers of its opposite faces, and of the lines joining the midpoints of opposite edges, and let G^+ be the motion group of the cube. Prove that G^+ acts on F, find all the orbits of this action and the stabilizers of all the points of F. Compare your findings with the proof of Theorem 3.1 in Case 4.

3.9. Given a regular tetrahedron inscribed in the sphere, let the set F consist of all its vertices and of the lines joining the midpoints of the edges, and let G^+ be the motion group of the tetrahedron. Prove that G^+ acts on F, find all the orbits of this action and the stabilizers of all the points of F. Compare your findings with the proof of Theorem 3.1 in Case 3.

3.10. Given a dodecahedron inscribed in the sphere, let the set F consist of all the vertices of the dodecahedron, all the intersection points of the lines joining the centers of its opposite faces and of the lines joining the midpoints of the edges, and let G^+ be the motion group of the dodecahedron. Prove that G^+ acts on F and complete the proof of Theorem 3.1 in Case 5.

3.11. Prove Theorem 3.1 in Case 4 by constructing an octahedron (instead of a cube) from the points of F.

3.12*. Prove the classification theorem for regular polyhedra in dimension four.