PROBLEMS for Lecture 8

8.1. Prove that

(a) linear-fractional transformations preserve the cross-ratio of four points on the Riemann sphere $\overline{\mathbb{C}}$;

(b) a linear-fractional transformation is uniquely determined by three points and their images.

8.2. Let l be a straight line in the Euclidean plane, γ a circle with center O on l, P a point not on l and not on the perpendicular to l from O. Prove that there exists a unique circle passing through P, orthogonal to γ, and centered on l.

8.3. Let l be a straight line in the Euclidean plane, γ a circle with diameter AB on l, P a point not on l and not in γ. Prove that there exists a unique circle passing through P and A with center on l, and a unique circle passing through P and B with center on l.

8.4. Prove that all motions (i.e., orientation-preserving isometries) of the Poincaré disk model are of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\overline{a}},$$

where a and b are complex numbers such that $|a|^2 = |b|^2 = 1$.

8.5. Show that there exists an isometry of the half-plane model that takes any flag to any other flag (a flag is a triple consisting of a line in the hyperbolic plane, one of the two half-planes that the line bounds, and a point on that line).

8.6*. Find a formula for the area of a triangle in hyperbolic geometry.