Advanced Monte Carlo and Optimization Methods for Optimal Stopping Problems: Part V

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Optimal stopping problems

Consider an **optimal stopping problem**:

$$Y^* = \sup_{\tau \in \mathscr{T}([0,T])} \mathrm{E}[Z_\tau],$$

where

- $(Z_t)_{t \geq 0}$ is a process on the probability space $(\Omega, (\mathscr{F}_t)_{t \geq 0}, \mathrm{P})$, s. t. $\mathrm{E}[\sup_{t \in [0,T]} |Z_t|] < \infty$,
- $\mathscr{T}([0,T])$ is the set of stopping times with values in $[0, T]$.

Question

How to approximate Y^* in the case when the expectation $\mathrm{E}[Z_\tau]$ cannot be computed in a closed form ?

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Dual approach

Consider a martingale $(M_t)_{t\geq 0}$ with $M_0 = 0$ adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. We have

$$Y^* = \sup_{\tau\in\mathscr{T}[0,T]} \mathrm{E}\,[Z_\tau - M_\tau] \leq \mathrm{E} \sup_{t\in[0,T]} [Z_t - M_t].$$

Observation

The r.h.s. with an arbitrary martingale gives an upper bound for Y^*.

It can be shown

$$Y^* = \inf_{M\in\mathscr{A}} \mathrm{E} \sup_{t\in[0,T]} [Z_t - M_t], \tag{1}$$

where $\mathscr{A}$ is the set of all adapted martingales starting at 0.

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Dual approach

Question

What martingales do solve (1) *and is the solution unique ?*

- ✓ One solution of (1) is M^* - the Doob martingale of the Snell process

$$Y_t^* = \sup_{\tau \in \mathcal{T}[t,T]} \mathrm{E}[Z_\tau | \mathcal{F}_t],$$

i.e., M_t^* is an $(\mathcal{F}_t)$-martingale which satisfies

$$Y_t^* = Y_0^* + M_t^* - A_t^*, \quad t \in [0,T]$$

with $M_0^* := A_0^* := 0$.

- ✓ There are many martingales solving (1) and some of solutions are "better" than others.

Dual approach

Note that

$$Y^* = \sup_{t\in[0,T]} [Z_t - M_t^*], \quad a.s.$$

Hence M^* also solves the penalized optimization problem

$$\inf_{M\in\mathscr{A}} \left\{ \mathrm{E}\left[\sup_{t\in[0,T]} (Z_t - M_t) \right] + \lambda \sqrt{\mathrm{Var}\left[\sup_{t\in[0,T]} (Z_t - M_t) \right]} \right\} \tag{2}$$

for any $\lambda > 0$ and M^* is a “good” solution of (1).

Observation

In fact, even the problem (2) *has infinitely many solutions.*

Dual approach

- Let M be an adapted martingale with $M_0 = 0$.
- Simulate a set of trajectories

$$(Z_t^{(1)}, M_t^{(1)}), \ldots, (Z_t^{(n)}, M_t^{(n)}), \quad t \in [0, T].$$

- Define $Z^{(j)}(M) = \sup_{s\in[0,T]}(Z_s^{(j)} - M_s^{(j)})$, $j = 1, \ldots, n$.

Monte Carlo estimate

$$Y_n(M) = \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M),$$

has the variance $\mathrm{Var}(Z(M))/n$.

Observation

To speed up the convergence of $Y_n(M)$ we would like to have martingales M with a smaller variance of $Z(M) = \sup_{s\in[0,T]}(Z_s - M_s)$.

Penalized empirical dual approach

Let $\mathscr{M} \subset \mathscr{A}$ be a family of adapted martingales with $M_0 = 0$. Consider

$$M_n = \operatorname*{arginf}_{M \in \mathscr{M}} \left(\frac{1}{n} \sum_{j=1}^{n} Z^{(j)}(M) + \lambda \sqrt{V_n(M)} \right), \quad \lambda > 0,$$

where $Z^{(j)}(M) = \sup_{s \in [0,T]} (Z_s^{(j)} - M_s^{(j)})$ and

$$V_n(M) = \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} (Z^{(i)}(M) - Z^{(j)}(M))^2.$$

Question

How large are the variance of $Z(M_n) = \mathrm{E}[\sup_{t \in [0,T]} (Z_t - M_{n,t})]$ *and the bias* $\mathrm{E}[Z(M_n)] - Y^*$ *?*

Penalized empirical dual approach

- (Ψ, ϱ) is a metric space.
- $\mathscr{M} = \{M(\psi) : \psi \in \Psi\}$ is a family of adapted continuous martingales such that

$$\sup_{\psi,\phi\in\Psi} \frac{\sqrt{\langle M(\psi) - M(\phi)\rangle_T}}{\varrho(\psi,\phi)} < C, \quad \text{a.s.}$$

- A set $\Psi^* \subset \Psi$ such that

$$Y^* = \sup_{t\in[0,T]} (Z_t - M_t(\psi)), \quad \text{a.s., for any} \quad \psi \in \Psi^*$$

is not empty.

Penalized empirical dual approach

- (Ψ_n, ϱ) is a sequence of finite-dimensional approximating spaces (**sieves**) such that for any $n \in \mathbb{N}$ and some $\psi^* \in \Psi^*$ there exists an element $\pi_n \psi^*$ in Ψ_n satisfying $\varrho(\psi^*, \pi_n \psi^*) \to 0$ as $n \to \infty$.

Theorem

Denote

$$\mathfrak{C}_n = \int_0^1 \varepsilon^{-1} J_n(\varepsilon)\, dJ_n(\varepsilon)$$

with

$$J_n(\delta) = \int_0^\delta \sqrt{\log[1 + N(\varepsilon, \Psi_n, \varrho)]}\, d\varepsilon.$$

Then

$$\mathrm{E}[Z(M_n)] - Y^* \xrightarrow{\mathrm{P}} 0, \quad \mathrm{Var}(Z(M_n)) \xrightarrow{\mathrm{P}} 0, \quad n \to \infty,$$

provided $\mathfrak{C}_n / \sqrt{n} = O(1)$.

Martingales via martingale representations

$$Z_t = G(t, X_t), \quad t \in [0, T]$$

- $G : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is a Hölder function
- X_t is a d-dimensional Markov process solving the system of SDE's:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x$$

Theorem

If M_t is square integrable and is adapted to the filtration generated by W_t, then there is a square integrable (row vector valued) process $H_t = (H_t^1, \ldots, H_t^m)$ satisfying

$$M_t = \int_0^t H_s dW_s.$$

Martingales via martingale representations

Under some conditions we have $H_s = \psi(s, X_s)$ and

$$M_t = M_t(\psi) = \int_0^t \psi(s, X_s) dW_s$$

for some ψ satisfying $\int_0^T \mathrm{E}[|\psi(s, X_s)|^2]\, ds < \infty$.
Define

$$\mathscr{M} = \left\{ M(\psi), \quad \psi \in L_{2,P}([0,T] \times \mathbb{R}^d) \right\}$$

Lemma

$$\sqrt{\langle M - M' \rangle_T} \leq \sqrt{T} \sup_{(s,x) \in [0,T] \times \mathbb{R}^d} |\psi(s,x) - \psi'(s,x)| := \sqrt{T} \cdot \varrho(\psi, \psi')$$

with $M_t = M_t(\psi)$, $M_t' = M_t(\psi')$, $\psi, \psi' \in L_{2,P}([0,T] \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$.

Martingales via martingale representations

- Introduce a linear sieve

$$\widetilde{\Psi}_K = \{\beta_1\phi_1 + \ldots + \beta_K\phi_K : \beta_1, \ldots, \beta_K \in \mathbb{R}\},$$

  where $\phi_1, \ldots, \phi_K \in L_{2,P}([0,T] \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$.

- Define

$$\mathfrak{M}_K = \left\{M_t(\psi) : \psi \in \widetilde{\Psi}_K\right\}.$$

- Set

$$M_n = \arg \inf_{M \in \mathfrak{M}_{K_n}} \left(\frac{1}{n} \sum_{j=1}^{n} Z^{(j)}(M) + (\kappa + \lambda_n) \sqrt{V_n(M)} \right),$$

  where $K_n \to \infty$ as $n \to \infty$.

Martingales via martingale representations

Observation

For many linear sieves it holds

$$\log[1+N(\varepsilon,\widetilde{\Psi}_K,\varrho)] \lesssim K^{d+1}\log(1/\varepsilon), \quad \varepsilon\to 0$$

Theorem

With probability at least $1-\delta$

$$\mathrm{E}[Z(M_n)]-Y^* \lesssim a_n, \quad \sqrt{V(M_n)} \lesssim a_n,$$

where $a_n=\inf_{\psi\in\widetilde{\Psi}_{K_n},\psi^*\in\widetilde{\Psi}^*}\|\psi-\psi^*\|_\infty$, *provided* $\lambda_n=\sqrt{2\log(2/\delta)}/\sqrt{n}$ *and* $K_n^{d+1}/\sqrt{n}=O(1)$ *for* $n\to\infty$.

Andersen-Broadie dual approach

- Discrete time optimal stopping problems: $0 = t_0 < t_1 < \ldots < t_L = T$

$$Y^* = \sup_{\tau \in \{t_0, \ldots, t_L\}} \mathrm{E}\left[Z_\tau\right],$$

Lemma

It holds

$$M^*_{t_{j+1}} - M^*_{t_j} = Y^*_{t_{j+1}} - \mathrm{E}[Y^*_{t_{j+1}} | \mathcal{F}_{t_j}], \quad j = 0, \ldots, L-1.$$

- Replace Y^* by its approximation Y obtained, for example, using a regression approach.
- Find an approximation M of M^* using sub-simulation and the formula

$$M_{t_{j+1}} - M_{t_j} = Y_{t_{j+1}} - \mathrm{E}[Y_{t_{j+1}} | \mathcal{F}_{t_j}].$$

Comparison with the standard approach

Our new approach

- is directly applicable in the case of continuous time optimal stopping problems,
- delivers "true" upper bound without use of sub-simulation, thus resulting in a non-nested Monte Carlo,
- does not exclusively concentrate on finding Doob martingale and takes advantage of the richness of the class $\mathscr{A}^*$ of adapted martingales satisfying

$$Y_t^* = \sup_{t \in [0,T]} (Z_t - M_t), \quad \text{a.s.},$$

- the variance of the r.v $Z(M_n) = \sup_{s \in [0,T]}(Z_s - M_{n,s})$ is, with high probability, bounded by a multiple of

$$\inf_{M \in \mathscr{M}, M' \in \mathscr{A}^*} d(M, M'),$$

where d is a deterministic metric on $\mathscr{A}$.

Penalization vs. No peanlization

Question

What happens if we do not penalize by empirical variance?

- Consider a class of processes Z_t defined as

$$Z_t = \int_0^t f(s, W_s)\, dW_s + \int_0^t g(s, W_s)\, ds, \quad t \geq 0,$$

where $(W_t)_{t \geq 0}$ is the standard Brownian motion and f, g are two functions satisfying

$$\int_0^T \mathrm{E}\,|f(s, W_s)|^2\, ds < \infty, \quad \int_0^T \mathrm{E}\,|g(s, W_s)|\, ds < \infty.$$

Penalization vs. No peanlization

Observation

If $g \geq 0$, then the process Z_t is uniformly integrable submartingale and

$$Y^* = \sup_{\tau \in \mathcal{T}([0,T])} \mathrm{E}[Z_\tau] = \mathrm{E}\left[\int_0^T g(s, W_s)\, ds\right].$$

- Take $T = 1$, $f(s,x) = \sin^4(x)$ and $g(s,x) = x^2$, then $Y^* = 1/2$.
- Consider a set of functions on $[0, T] \times \mathbb{R}$:

$$(\phi_1(t,x), \ldots, \phi_7(t,x)) = \{1, x, tx, \sin(x), \cos(x), \sin(2x), \cos(2x)\}.$$

- Define a sieve $\widetilde{\Psi}$ via

$$\widetilde{\Psi} = \{\beta_1\phi_1 + \ldots + \beta_7\phi_7 : \beta_1, \ldots, \beta_7 \in \mathbb{R}\}.$$

Penalization vs. No peanlization

- Simulate n paths of the Brownian motion W_t on $[0,T]$
- Consider two optimization problems

$$\psi_n = \operatorname*{arginf}_{\psi \in \widetilde{\Psi}} \left\{ \frac{1}{n} \sum_{j=1}^{n} Z^{(j)}(\psi) \right\}$$

and

$$\psi_{n,\lambda} = \operatorname*{arginf}_{\psi \in \widetilde{\Psi}} \left\{ \frac{1}{n} \sum_{j=1}^{n} Z^{(j)}(\psi) + \frac{\lambda}{n(n-1)} \sum_{1 \le i < j \le n} (Z^{(i)}(\psi) - Z^{(j)}(\psi))^2 \right\}$$

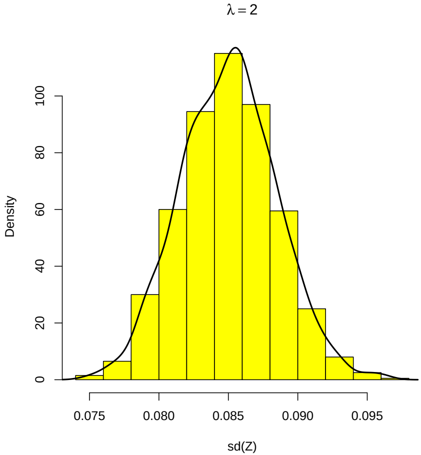
with

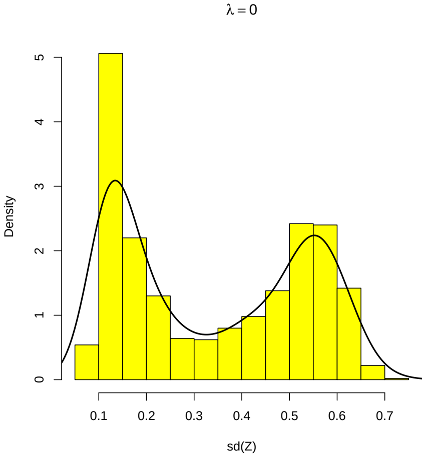
$$Z^{(j)}(\psi) = \sup_{t \in [0,T]} \left[Z_t^{(j)} - \int_0^t \psi(s, W_s^{(j)})\, dW_s^{(j)} \right],$$

$$Z_t^{(j)} = \int_0^t f(s, W_s^{(j)})\, dW_s^{(j)} + \int_0^t g(s, W_s^{(j)})\, ds.$$

Penalization vs. No peanlization

The histograms of the standard deviations of the r.v. $Z(\psi_{n,2})$ (left) and $Z(\psi_n)$ (right) based on 1000 realizations of the solutions $\psi_{n,2}$ and ψ_n





Belomestny, D. (2012).
Solving optimal stopping problems by empirical dual optimization and penalization, to appear in *Annals of Applied Probability*.