Advanced Monte Carlo and Optimization Methods for Optimal Stopping Problems: Part 0

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18.09.2012

Stochastic Processes

Definition (Brownian Motion)

W_t has the properties:

- *W_t is a continuous stochastic process*
- $W_0 = 0$
- $W_t \sim \mathcal{N}(0, t)$
- *For any sequence of time points $t_0 < \ldots < t_n$, the increments $W_{t_k} - W_{t_{k-1}}$, $k = 1, \ldots, n$, are independent*

Remark

Brownian motion is the main building block for constructing other stochastic processes

Stochastic Prozesses

Algorithm (Simulation of Brownian Motion)

- *Initialization:* $t_0 = 0, W_0 = 0$
- $W_0 = 0$
- *For* $j = 1, 2, \dots, n$ *:*
  - $t_j = t_{j-1} + \Delta t$
  - *simulate* $Z \sim \mathcal{N}(0, 1)$
  - $W_j = W_{j-1} + Z\sqrt{\Delta t}$
- $W_0, W_{t_1}, \dots, W_{t_n}$

Stochastic Differential Equations

Definition

Itô-SDE is the integral equation of the form:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)\, ds + \int_{t_0}^{t} b(s, X_s)\, dW_s.$$

Differential symbolic form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

Solutions of SDE are called stochastic diffusions or Itô processes. The term $a(X_t, t)$ is called a drift, and $b(X_t, t)$ is called a diffusion term.

Stochastic Differential Equations

Algorithm (Simulation of a diffusion process via Euler scheme)

Let y_j be an approximation for X_{t_j}

- *Start:* $t_0 = 0,\ y_0 = X_0,\ W_0 = 0$
- $W_0 = 0$
- *For* $j = 1, 2, \ldots, N$ *:*
  - $t_j = t_{j-1} + \Delta t$
  - *simulate* $Z \sim \mathcal{N}(0, 1)$
  - $y_{j+1} = y_j + a(t_j, y_j)\Delta t + b(t_j, y_j)Z\sqrt{\Delta t}$
- $y_0, y_1, \ldots, y_N$

Monte Carlo Methods for Pricing European Options

Let S_t be stock price and let r be an interest rate

- A fair price of European Option with Payoff Ψ is given by

$$V = \mathrm{E}_Q[\Psi(S_T)|S_0],$$

  where Q is a risk-neutral martingale measure, i.e., $\exp(-rt)S_t$ is a Q-Martingale ($\mathrm{E}_Q[\exp(-rT)S_T|\mathcal{F}_t] = \exp(-rt)S_t$).

Aim

We would like to compute the value of the European option

$$V(S_0) = \mathrm{E}_Q[\Psi(S_T)|S_0]$$

Monte Carlo Methods for Pricing European Options

Algorithm

- *Simulate N values of stock at time T under the risk neutral measure Q, all starting from* S_0*, so we obtain* $S_T^{(1)}, \ldots, S_T^{(N)}$
- *Compute the values of payoff* $\Psi(S_T)$ *to get* $\Psi(S_T^{(n)})$, $n = 1, \ldots, N$
- *Build the mean value:*

$$\widehat{V} = \frac{1}{N} \sum_{n=1}^{N} \Psi(S_T^{(n)})$$

Precision

Let

$$\widehat{V} = \frac{1}{N}\sum_{n=1}^{N}\Psi\left(S_T^{(n)}\right),$$
$$\widehat{S}^2 = \frac{1}{N-1}\sum_{n=1}^{N}\left(\Psi\left(S_T^{(n)}\right) - \widehat{V}\right)^2.$$

Due to the central limit theorem:

$$\widehat{V} - \mathrm{E}[\widehat{V}] \sim \mathcal{N}(0, \widehat{S}^2/N), \quad N >> 1.$$

The standard deviation of $\widehat{V}$ approximately behaves like $\widehat{S}/\sqrt{N}$. In order to minimize the s.d. one can

- either reduce $\widehat{S}$ (variance reduction)
- or increase N (more simulations)

Precision

Bias

$$\mathrm{E}[\widehat{V}] - V$$

Example

The Euler scheme leads to a bias. While for GBM the use of the scheme:

$$S_{t_{j+1}} = S_{t_j} \exp((r - \sigma^2/2)\Delta t + \sigma \Delta W)$$

leads to exact results, the Euler scheme

$$S_{t_{j+1}} = S_{t_j}(1 + r\Delta t + \sigma \Delta W)$$

is biased.

Precision

Question

Where should one invest more computational efforts ?

- *variance reduction*
- *simulation of more paths N*
- *bias reduction, i.e., increase the number of steps*

Remark

The advantage of Monte Carlo methods is that the MC error is basically independent of the dimension