Advanced Monte Carlo and Optimization Methods for Optimal Stopping Problems: Part IV

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Optimal Stopping Problems

Consider the following discrete time optimal stopping problem:

$$Y_0^* = \sup_{\tau \in \{0,\ldots,\mathcal{J}\}} \mathrm{E}\left[Z_\tau\right],$$

where

- $(Z_j)_{j\geq 0}$ is an adapted process on a probability space $(\Omega, (\mathcal{F}_j)_{j\geq 0}, \mathrm{P})$
- τ is a stopping time with values in $\{1, \ldots, \mathcal{J}\}$, i.e. $\{\tau = j\} \in \mathcal{F}_j$

Question

How to approximate Y_0^* in the case when the expectation $\mathrm{E}[Z_\tau]$ cannot be computed in a closed form ?

Dual upper bounds

Consider a discrete martingale $(M_j)_{j=0,\ldots,\mathcal{J}}$ with $M_0 = 0$ w. r. t. the filtration $(\mathcal{F}_j)_{j=0,\ldots,\mathcal{J}}$. Observe that

$$Y_0^* = \sup_{\tau \in \{0,\ldots,\mathcal{J}\}} \mathrm{E}^{\mathcal{F}_0} [Z_\tau - M_\tau] \le \mathrm{E}^{\mathcal{F}_0} \max_{0 \le j \le \mathcal{J}} \left[Z_j - M_j \right].$$

Hence the r.h.s. with an arbitrary martingale gives an upper bound for the Bermudan price Y_0.

Question

What martingale does lead to equality ?

Dual upper bounds

Theorem (Rogers (2001), Haugh & Kogan (2001))

Let M^* be the (unique) Doob-Meyer martingale part of $(Y_j^*)_{0 \leq j \leq \mathcal{J}}$, i.e. M_j^* is an $(\mathcal{F}_j)$-martingale which satisfies

$$Y_j^* = Y_0^* + M_j^* - A_j^*, \quad j = 0, ..., \mathcal{J}$$

with $M_0^* := A_0^* := 0$, where A_j^* is increasing process which $\mathcal{F}_{j-1}$-measurable. Then

$$Y_0^* = \max_{0 \leq j \leq \mathcal{J}} \left[Z_j - M_j^* \right], \quad a.s.$$

Riesz upper bounds

Doob-Meyer decomposition:

$$Y_j^* = Y_0^* + M_j^* - A_j^*, \quad j = 0, ..., \mathcal{J},$$

and $Y_{\mathcal{J}}^* = Z_{\mathcal{J}}$ imply Riesz decomposition:

$$Y_j^* = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j}(A_{\mathcal{J}}^* - A_j^*)$$

Since $A_{i+1}^* - A_i^* = Y_i^* - E^{\mathcal{F}_i} Y_{i+1}^* = [Z_i - E^{\mathcal{F}_i} Y_{i+1}^*]^+$, we get

$$Y_j^* = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - E^{\mathcal{F}_i} Y_{i+1}^*]^+.$$

Riesz upper bounds

Theorem

If Y_i is a lower approximation for Y_i^*, i.e., $Y_i \le Y_i^*$ a.s., then

$$Y_j^{up} = \mathrm{E}^{\mathcal{F}_j} Z_{\mathcal{J}} + \mathrm{E}^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - \mathrm{E}^{\mathcal{F}_i} Y_{i+1}]^+$$

is an upper approximation for Y_j^*, that is

$$Y_j \le Y_j^* \le Y_j^{up}, \quad j = 0, \ldots, \mathcal{J}.$$

Riesz upper bounds

Proposition

- *Monotonicity*

$$\widetilde{Y}_i \geq Y_i \longrightarrow \widetilde{Y}_i^{up} \leq Y_i^{up}$$

- *Locality*
  Let $\{Y_i^\alpha, \alpha \in I_i\}$ be a family of local lower bounds at i, then

$$Y_j^{\alpha,up} = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - \max_{\alpha \in I_{i+1}} E^{\mathcal{F}_i} Y_{i+1}^\alpha]^+$$

  is an upper bound.

Doob-Meyer Martingale

For any martingale M_{T_j}, starting at $M_0 = 0$,

$$Y_0^{up}(M) := \mathrm{E}^{\mathcal{F}_0}\left[\max_{0 \le j \le \mathcal{J}} (Z_j - M_j)\right]$$

is an upper bound for Y_0^*.

Exact value Y_0^* is attained at the martingale part M^* of the Snell envelope:

$$Y_j^* = Y_0^* + M_j^* - A_j^*,$$

where $M_0^* = A_0^* = 0$ and A_j^* is $\mathcal{F}_{j-1}$ measurable.

Doob-Meyer Martingale

Let Y_j be an approximation for the Snell envelope Y_j^* with the Doob decomposition

$$Y_j = Y_0 + M_j - A_j,$$

where $M_0 = A_0 = 0$ and A_j is $\mathcal{F}_{j-1}$ measurable.
It then holds:

$$M_{j+1} - M_j = Y_{j+1} - \mathrm{E}^{\mathcal{F}_j}[Y_{j+1}]$$

Observation

The martingale M can be used to obtain an upper bound Y_0^{up} via

$$Y_0^{up} = \mathrm{E}\left[\max_{j=0,\ldots,\mathcal{J}}(Z_j - M_j)\right].$$

Andersen-Broadie Approach

Doob martingale corresponding to Y:

$$M_j = \sum_{i=1}^{j} \left(Y_i - \mathrm{E}_{\mathcal{F}_{i-1}} [Y_i] \right), \quad j = 0, \ldots, \mathcal{J}.$$

Estimate the conditional expectations by Monte Carlo to get

$$M_j^k = \sum_{i=1}^{j} \left(Y_i - \frac{1}{k} \sum_{l=1}^{k} \xi_i^{(l)} \right), \quad k \in \mathbb{N},$$

where, conditionally on $\mathcal{F}$, all random variables $\xi_i^{(l)}$, $l = 1, ..., k$, $i = 1, ..., \mathcal{J}$, are independent and

$$\mathrm{E}_{\mathcal{F}} \left[\xi_i^{(l)} \right] = \mathrm{E}_{\mathcal{F}_{i-1}} \left[\xi_i^{(l)} \right] = \mathrm{E}_{\mathcal{F}_{i-1}} \left[\xi_i^{(1)} \right] = \mathrm{E}_{\mathcal{F}_{i-1}} [Y_i] .$$

Andersen-Broadie Approach

Fix some natural numbers N and K, and consider the estimator:

$$
\begin{aligned}
Y^{N,K} &= \frac{1}{N}\sum_{n=1}^{N} \max_{j=0,\ldots,\mathcal{J}} (Z_j^{(n)} - M_j^{K,(n)}) \\
&=: \frac{1}{N}\sum_{n=1}^{N} \mathcal{Z}^{(n)}(M^K)
\end{aligned}
$$

based on a set of trajectories

$$
\left\{ (Z_j^{(n)}, M_j^{K,(n)}),\ n = 1, \ldots, N,\ j = 0, \ldots, \mathcal{J} \right\}
$$

of the vector process (Z, M^K).

Andersen-Broadie Approach

Denote

$$Y_0^{up} := \mathrm{E}\left[\max_{j=0,\ldots,\mathcal{J}}(Z_j - M_j)\right] = \mathrm{E}[\mathcal{Z}(M)]$$

Observation

$$\mathrm{E}\left[(M_j^k - M_j)^2\right] = \mathrm{E}\left[\sum_{i=1}^{j}\left(\mathrm{E}_{\mathcal{F}_{i-1}}[Y_i] - \frac{1}{k}\sum_{l=1}^{k}\xi_i^{(l)}\right)\right]^2$$

$$= \frac{1}{k}\sum_{i=1}^{\mathcal{J}}\mathrm{E}\left[\mathrm{E}_{\mathcal{F}_{i-1}}[Y_i] - \xi_i^{(1)}\right]^2 \leq \frac{1}{k}\sum_{i=1}^{\mathcal{J}}\mathrm{E}[Y_j^2] = O(1/k),$$

provided $\mathrm{E}[Y_j^2] < \infty$ *for* $j = 0, \ldots, \mathcal{J}$.

Andersen-Broadie Approach

Then it holds

$$\mathrm{E}\left[Y^{N,K} - Y_0^{up}\right]^2 \le N^{-1}\,\mathrm{Var}(\mathcal{Z}(M^K)) + CK^{-\beta}$$
$$=: N^{-1} v_K + CK^{-1}, \quad K \to \infty.$$

In order to get $\sqrt{\mathrm{E}\left[Y^{N,K} - Y(M)\right]^2} \le \varepsilon$, we may take

$$K = \left\lceil \frac{2C}{\varepsilon^2} \right\rceil,$$

and then

$$N = \left\lceil \frac{2v_K}{\varepsilon^2} \right\rceil$$

with $\lceil x \rceil$ denoting the first integer which is larger than or equal to x.

Andersen-Broadie Approach

If v_K is non-increasing, then, given an accuracy ε, the complexity is, up to a constant,

$$\mathcal{C}^{N,K}(\varepsilon) := NK \lesssim \frac{v_{\left\lceil \frac{2C}{\varepsilon^{2/\beta}} \right\rceil}}{\varepsilon^4}.$$

Observation

If $\mathrm{Var}(\mathcal{Z}(M)) = 0$ *(e.g.,* $M = M^*$*) we have*

$$v_K = \mathrm{Var}(\mathcal{Z}(M^K)) \leq \mathrm{E}(\mathcal{Z}(M^K) - \mathcal{Z}(M))^2 \leq BK^{-1},$$

and as a result

$$\mathcal{C}^{N,K}(\varepsilon) \lesssim \frac{B}{2C}\frac{1}{\varepsilon^2}.$$

Multilevel Approach

Let $L \in \mathbb{N}$ and $\mathbf{k} = (k_0, \ldots, k_L)$ with $1 \le k_0 < k_1 < \ldots < k_L$

$$
\begin{aligned}
Y(M^{k_L}) &= Y(M^{k_0}) + \sum_{l=1}^{L} [Y(M^{k_l}) - Y(M^{k_{l-1}})] \\
&= \mathrm{E}[\mathcal{Z}(M^{k_0})] + \sum_{l=1}^{L} \mathrm{E}[\mathcal{Z}(M^{k_l}) - \mathcal{Z}(M^{k_{l-1}})]
\end{aligned}
$$

with

$$
Y(M^k) := \mathrm{E}\left[\max_{j=0,\ldots,\mathcal{J}} (Z_j - M_j^k)\right] = \mathrm{E}[\mathcal{Z}(M^k)]
$$

Multilevel Algorithm

- Fix a sequence $\mathbf{n} = (n_0, \ldots, n_L) \in \mathbb{N}^L$ with $1 \leq n_0 < \ldots < n_L$
- Simulate the initial set of trajectories:

$$\left\{ \left(Z_j^{(i)}, M_j^{k_0,(i)} \right), \quad i = 1, ..., n_0, \quad j = 0, \ldots, \mathcal{J} \right\}$$

  of the vector process (Z, M^{k_0})
- For each level $l = 1, \ldots, L$, generate independently a set of trajectories:

$$\left\{ \left(Z_j^{(i)}, M_j^{k_{l-1},(i)}, M_j^{k_l,(i)} \right), \quad i = 1, ..., n_l, \quad j = 0, \ldots, \mathcal{J} \right\}$$

  of the vector process $(Z, M^{k_{l-1}}, M^{k_l})$.

Multilevel Algorithm

Consider the approximation

$$Y^{\mathbf{n},\mathbf{k}} := \frac{1}{n_0}\sum_{i=1}^{n_0} \mathcal{Z}^{(i)}(M^{k_0}) + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}\left[\mathcal{Z}^{(i)}(M^{k_l}) - \mathcal{Z}^{(i)}(M^{k_{l-1}})\right]$$

with

$$\mathcal{Z}^{(i)}(M^k) := \max_{j=0,\ldots,\mathcal{J}}\left(Z_j^{(i)} - M_j^{k,(i)}\right), \quad i = 1, \ldots, n_l \quad k \in \mathbb{N}.$$

Multilevel Algorithm

Theorem

Let $k_l = k_0 \kappa^l$, $l = 0, \ldots, L$, *for some* $k_0 \in \mathbb{N}$ *and* $\kappa > 1$. *Set*

$$L = \left\lceil -(\gamma \ln \kappa)^{-1} \ln \frac{\sqrt{k_0}\varepsilon}{C\sqrt{2}} \right\rceil$$

Let

$$n_l = \left\lceil 2\varepsilon^{-2}(L+1)k_0^{-1}\kappa^{-l} \right\rceil,$$

Then

$$\sqrt{\mathrm{E}[Y^{\mathbf{n},\mathbf{k}} - Y(M)]^2} \leq \varepsilon,$$

while the computational complexity of the estimator $Y^{\mathbf{n},\mathbf{k}}$ *is of order*

$$\varepsilon^{-2} \ln^2 \varepsilon.$$

Belomestny, D. and Milstein, G.
Monte Carlo evaluation of American options using consumption processes. *Int. J. of Theoretical and Applied Finance*, 02(1):65–69, 2000.

Belomestny, D. et al. (2009).
True upper bounds for Bermudan products via non-nested Monte Carlo. *Mathematical Finance*, **19**(1), 53–71.

elomestny, D. and Schoenmakers, J. (2012).
Multilevel dual approach for pricing American type derivatives, forthcoming in *Finance and Stochastics*.