Problems to the course "Lecture course Modern Monte-Carlo and optimization methods for optimal stopping problems in financial mathematics"

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1. The Snell-Envelope Process

$$Y_j^*(X_j) = \sup_{\tau \in \{j,,\ldots,\mathcal{J}\}} \mathrm{E}\left[G_\tau(X_\tau)|X_j\right], \quad j = 0, \ldots, \mathcal{J},$$

   fulfills the dynamic programming principle (DPP):

$$\begin{aligned} Y_{\mathcal{J}}^* &= G_{\mathcal{J}}(X_{\mathcal{J}}), \\ Y_j^* &= \max\left\{G_j(X_j), \mathrm{E}[Y_{j+1}^*(X_{j+1}))|X_j = x]\right\}. \end{aligned}$$

   Derive from the DPP that the process

$$C_j^* := \mathrm{E}[Y_{j+1}^*(X_{j+1})|X_j], \quad j = 0, \ldots, \mathcal{J} - 1$$

   solves

$$C_j^* = \mathrm{E}[\max(G_{j+1}(X_{j+1}), C_{j+1}^*)|X_j]$$

   for $j = 0, \ldots, \mathcal{J} - 1$.

2. Let $Y_j \leq Y_j^*$, $j = 0, \ldots, \mathcal{J}$. Show that

$$\begin{aligned} Y_0^{up} &:= \mathrm{E}[G_{\mathcal{J}}(X_{\mathcal{J}})] + \mathrm{E}\left[\sum_{i=0}^{\mathcal{J}-1} [G_i(X_i) - \mathrm{E}[Y_{i+1}|X_i]]^+\right] \\ &= Y_0 + \mathrm{E}\left[\sum_{i=0}^{\mathcal{J}-1} [\max\{G_i(X_i), \mathrm{E}[Y_{i+1}|X_i]\} - Y_i]\right]. \end{aligned}$$

3. Derive from the DPP the following representation for the Snell-Envelope prozess:

$$Y_j^* = \mathrm{E}[G_{\mathcal{J}}(X_{\mathcal{J}})|X_j] + \mathrm{E}\left[\left.\sum_{i=j}^{\mathcal{J}-1} [G_i(X_i) - \mathrm{E}[Y_{i+1}^*|X_i]]^+ \right| X_j\right].$$

4. Let M^* be the (unique) Doob-Meyer martingale part of $(Y_j^*)_{0\leq j\leq \mathcal{J}}$, i.e. M_j^* is an $(\mathcal{F}_j)$-martingale which satisfies

$$Y_j^* = Y_0^* + M_j^* - A_j^*, \quad j = 0, ..., \mathcal{J}$$

with $M_0^* := A_0^* := 0$, where A_j^* is increasing process which $\mathcal{F}_{j-1}$- measurable. Prove that

$$Y_0^* = \max_{0\leq j\leq \mathcal{J}} \left[Z_j - M_j^* \right], \quad a.s.$$

5. Let $(\widehat{\alpha}_1, \ldots, \widehat{\alpha}_K)$ be a solution of the least squares optimization problem

$$\operatorname*{arginf}_{\alpha\in\mathbb{R}^K} \sum_{m=1}^{M} \left[\widehat{V}_{j+1,M}(X_{j+1}^{(m)}) - \alpha_1\psi_1(X_j^{(m)}) - \ldots - \alpha_K\psi_K(X_j^{(m)}) \right]^2$$

with $\widehat{V}_{j+1,M}(x) = \max\left\{ G_{j+1}(x), \widehat{C}_{j+1,M}(x) \right\}$. Show that

$$(\widehat{\alpha}_1, \ldots, \widehat{\alpha}_K)^\top = (B^{-1}b)^\top$$

with

$$B_{p,q} = \frac{1}{M} \sum_{m=1}^{M} \psi_p(X_j^{(m)})\psi_q(X_j^{(m)})$$

and

$$b_p = \frac{1}{M} \sum_{m=1}^{M} \psi_p(X_j^{(m)})\widehat{V}_{j+1,M}(X_{j+1}^{(m)}),$$

where $p, q \in \{1, \ldots, K\}$.

6. Let $(Z_t)_{t\in[0,T]}$ be an uniformly integrable submartingale. Then Z_t admits the so-called Doob-Meyer decomposition:

$$Z_t = Z_0 + M_t + A_t,$$

where M_t with $M_0 = 0$ is a uniformly integrable martingale and A_t is an increasing predictable process.

- Show that

$$Y^* := \sup_{\tau\in\mathcal{T}[0,T]} \mathrm{E}[Z_\tau] = \mathrm{E}[Z_T] = Z_0 + \mathrm{E}[A_T].$$

- Prove that

$$Y^* = \mathrm{E}\left[\sup_{t\in[0,T]} (Z_t - M_t) \right]$$

  but $Y^* \neq \sup_{t\in[0,T]}(Z_t - M_t)$ with positive probability, if A_T is not deterministic.

- Define

$$M_t^* = M_t + \mathrm{E}[A_T|\mathcal{F}_t] - \mathrm{E}[A_T].$$

  Prove that $Y^* = \sup_{t\in[0,T]}(Z_t - M_t^*)$ with probability 1.