7. SINGULAR HOMOLOGY

Problem 1. (a) (Alexander) If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ are closed and homeomorphic then $H_k(\mathbb{R}^n \setminus A) \cong H_k(\mathbb{R}^n \setminus B)$. (b) Can one replace $\mathbb{R}^n$ with S^n in the previous statement? (c) (Jordan's lemma in $\mathbb{R}^n$) If $A \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}, then $\mathbb{R}^n \setminus A$ consists of two connected components.

Problem 2. (a) Let $\Delta_n \stackrel{\text{def}}{=} \{(x_0, \dots, x_n) \mid x_0, \dots, x_n \geq 0, x_0 + \dots + x_n = 1\}$ be a standard simplex with the vertices $a_i = (0, \dots, 1, \dots, 0)$ (1 at the i-th position). Let $K_s \subset \Delta_n \times [0,1]$ be a simplex with the vertices $(a_i, 0)$, $s \leq i \leq n$ и $(a_i, 1)$, $0 \leq i \leq s$. Prove that the simplices $K_0, \dots, K_n$ have no common internal points and that their union is the whole $\Delta_n \times [0,1]$. (b) Generalize the construction above and describe a splitting of the product $\Delta_n \times \Delta_m$ into simplices ($[0,1]$ is equivalent to the simplex Δ_1).

Problem 3 (Künneth formula)**.** Let $X = \dots \to X_n \xrightarrow{\partial n,X} X_{n-1} \xrightarrow{\partial n-1,X} \dots$ and $Y = \dots \to Y_n \xrightarrow{\partial n,Y} Y_{n-1} \xrightarrow{\partial n-1,Y} \dots$ be complexes of Abelian groups, and $X \otimes Y = \dots \to X_n \otimes Y_n \xrightarrow{\partial n,X \otimes \partial n,Y} X_{n-1} \otimes Y_{n-1} \xrightarrow{\partial n-1,X \otimes \partial n-1,Y} \dots$ be their tensor product. (a) Prove that if X and Y are complexes of vector spaces then $H_*(X \otimes Y) = H_*(X) \otimes H_*(Y)$ (i.e. $H_n(X \otimes Y) = \sum_{k=0}^n H_k(X) \otimes H_{n-k}(Y)$). (b) Prove that in general $H_*(X \otimes Y) = \big(H_*(X) \otimes H_*(Y)\big) \oplus G$ for some Abelian group G that depends on $H_*(X)$ and $H_*(Y)$ only. Give an example where G is nontrivial.

Remark. Problem 3 is motivated by the fact that the CW-complex of the product $P \times Q$ of two CW-spaces P and Q is the tensor product of the CW-complexes of P and Q.

Problem 4. Let X be a topological space, and $U_1, \dots, U_N \subset X$ be its open subsets such that $X = \bigcup_{i=1}^N U_i$ and any finite intersection $U_{i_1} \cap \dots \cap U_{i_k}$ is either empty or homeomorphic to $\mathbb{R}^n$. Let $N(U_1, \dots, U_N)$ be a simplicial complex contatining vertices $a_1, \dots, a_N$ and a simplex $a_{i_1} \dots a_{i_k}$ if and only if $U_{i_1} \cap \dots \cap U_{i_k} \neq \varnothing$. (a) Prove that $N(U_1, \dots, U_N)$ is homotopy equivalent to X. (b) Is the statement above still true if $U_{i_1} \cap \dots \cap U_{i_k}$ is allowed to be homeomorphic to a disjoint union of several copies of $\mathbb{R}^n$? Consider, in particular, the case when X is a finite discrete space.