**IUM, Spectral Geometry**
**Exam. 07.12.2013.**

*This is a take-home exam. The deadline for solutions is the lecture on December, 21.*

*The point-to-grade transfer for IUM students is the following: $\geqslant$50 points = 5, $\geqslant$40 points = 4, $\geqslant$30 points = 3.*

*The point-to-grade transfer for HSE students is the following: $\geqslant$50 points = 10, $\geqslant$47 points = 9, $\geqslant$44 points = 8, $\geqslant$40 points = 7, $\geqslant$37 points = 6, $\geqslant$33 points = 5, $\geqslant$30 points = 4.*

**Problem 1.** Let $e_1, \ldots, e_n$ be a local orthonormal basis in vector fields in a neighbourhood of a point $p$ on a manifold $M$, and $c_1(t), \ldots, c_n(t)$ be geodesics such that $c_i(0) = p$ and $c_i'(0) = e_i$. Prove that $\Delta f(p) = -\sum_{i=1}^{n} \frac{d^2}{dt^2} f(c_i(t))|_{t=0}$. *(5 points).*

**Problem 2.** We know that the Dirichlet spectrum of a domain has monotonicity property, i.e. if $\Omega_1 \subset \Omega \subset \mathbb{R}^n$, then $\lambda_i(\Omega_1) \geqslant \lambda_i(\Omega)$.

Prove that the Neumann spectrum of a domain does not have monotonicity property, i.e. provide examples where the spectrum increases and decreases when one goes from a domain to a subdomain *(10 points).*

**Problem 3.** Find upper and lower bounds for Dirichlet eigenvalues of the domain $ABCDEF$, where $A = (0,0)$, $B = (0,2)$, $C = (1,2)$, $D = (1,1)$, $E = (2,1)$, $F = (2,0)$. *(up to 10 points depending on found bounds).*

**Problem 4.** We proved Faber-Krahn theorem saying that the disc minimizes the first Dirichlet eigenvalue $\lambda_1(\Omega)$ among all planar domains of the same area. Prove that the disjoint union of two discs of the same radius minimizes $\lambda_2(\Omega)$ among all planar domains of the same area. *(10 points).*

**Problem 5.** Prove that the disjoint union of $n$ discs of the same radius cannot minimize Dirichlet eigenvalues $\lambda_n(\Omega)$ for all $n$ *(5 points).*

**Problem 6.** Prove that the spectrum of the square of side length 1 with the Dirichlet condition on three sides and the Neumann condition on the remaining side coincides with the spectrum of the rectangular triangle of catheti length $\sqrt{2}$ with the Dirichlet conditions on the hypotenuse and one cathetus and the Neumann condition on another cathetus *(10 points).*

**Problem 7.** Construct an isometric immersion of the clifford torus in spheres using eigenfunctions of $\Delta$ corresponding to $\lambda_1$. Prove that the metric $g_{Cl}$ on the Clifford torus is extremal for the functional $\Lambda_1(\mathbb{T}^2, g)$. Find the value $\Lambda_1(\mathbb{T}^2, g_{Cl})$ *(5 points).*

**Problem 8.** Construct suitable isometric immersions of the clifford torus in a sphere using eigenfunctions of $\Delta$ and prove that the metric $g_{Cl}$ on the Clifford torus is extremal for infinite set of functionals $\Lambda_j(\mathbb{T}^2, g)$. Find at least three such values of $j$ *(10 points).*

**Problem 9$^*$.** Using the same approach as in problem 7 prove that the metric $g_{eq}$ on the equilateral torus is extremal for the functional $\Lambda_1(\mathbb{T}^2, g)$. Find $\Lambda_1(\mathbb{T}^2, g_{eq})$. Prove that the metric $g_{Cl}$ on the Clifford torus is not maximal for the functional $\Lambda_1(\mathbb{T}^2, g)$ *(20 points).*

**Problem 10$^*$.** Find the minimal isometric immersion of the real projective plane $\mathbb{R}P^2$ equipped with the canonical metric into a sphere using eigenfunctions correponding to the first eigenvalue. Prove that this is one of Veronese maps. *(25 points).*

**Problem 11.** Using your solution of problem 10 find an upper bound for $\Lambda_1(\mathbb{R}P^2, g)$ for any metric $g$. *(10 points).*