2019-1

Lecture 1: KLEIN GEOMETRIES

§1. Main definition

$(X:G)$ is a set with a transformation group acting on it if X is a set, $\emptyset \neq G \subset \mathrm{Bij}\,X$ ($g: X \to X$, $g: x \mapsto xg$) such that

(i) $g \in G \Rightarrow g^{-1} \in G$ (ii) $g_1, g_2 \in G \Rightarrow g_1 * g_2 \in G$

(iii) $x(g_1 * g_2) = (xg_1)g_2$

Remarks • $id \in G$ (Proof: $\emptyset \neq G \Rightarrow \exists g \in G \overset{(i)}{\Rightarrow} g^{-1} \in G \overset{ii}{\Rightarrow} G \ni g g^{-1} = id.$)

• G satisfies the axioms for groups (для тех, кто знает)

Short terminology: $(X:G)$ is a Klein geometry

§2 Examples

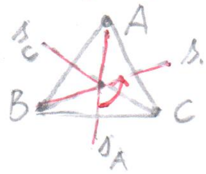
(0) Euclidean plane geometry: $X = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$, $G = \mathrm{Ismtr}\,\mathbb{R}^2$

Вы должны знать о строении Ismtr: п. перенос, поворот, скользящая сим.

(1) Symmetries of Δ: $X = \Delta$, $G = \mathrm{Ismtr}(\Delta) \equiv \mathrm{Sym}\,\Delta$

$\mathrm{Sym}(\Delta) = \{r_0, r_1, r_2, s_A, s_B, s_C\}$ $s_A * r_1 \neq r_1 * s_A$

$A \xrightarrow[s_A]{} A \xrightarrow[r_1]{} B$ $A \xrightarrow[r_1]{} B \xrightarrow[s_A]{} C$ Таблицу умножения!

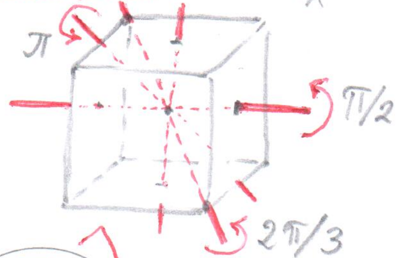


(2) Rotations of Π^3: $(\Pi^3 : \mathrm{Rot}\,\Pi^3)$

id, $3 \times \{\pi/2, \pi, 3\pi/2\} + 6 \times \{\pi\} + 4 \times \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$

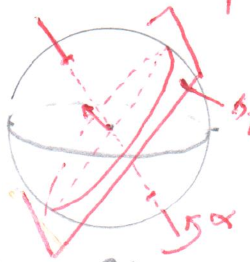
$\mathrm{ord}(\mathrm{Rot}\,\Pi^3) = 1 + 9 + 6 + 8 = 24$

Найдите $\mathrm{ord}(\mathrm{Sym}\,\Pi^3)$



(3) Spherical geometry: $(\mathbb{S}^3 : \mathrm{Sym}\,\mathbb{S}^3)$

$\mathrm{Sym}\,\mathbb{S}^3 = \{\{\ell_\alpha\}, \{\pi_\ell\}\}$ $\mathrm{ord}\,\mathrm{Sym}\,\mathbb{S}^3 = \infty$



$\mathrm{Rot}\,\mathbb{S}^3 = SO^3$

$\mathrm{Sym}\,\mathbb{S}^3 = O^3$

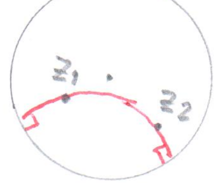
(4) Riemann's elliptic geometry: $X = \{(x, -x) \mid x \in \mathbb{S}^2\}$, $G = O^3$

Точка ≡ пара точек(!) прямые!

(5) Lobachevsky's hyperbolic geometry $X = \{z \in \mathbb{C} : |z| < 1\}$,

$G = \mathrm{Ismtr}_d$, where $d(z_1, z_2) = \ln \dfrac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{|1 - z_1\bar{z}_2| - |z_1 - z_2|}$

Три лекции! прямые!



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(6) Same as (3), (4), (5) but in dimension three

Remark • any "geometry" is a Klein geometry, but

• a Klein geometry is not necessarily a "geometry"

§3 Abstract groups

A set G with a binary operation $*$ is a group, if

(i) $\exists e \in G$ s.t. $e * g = g * e = g \ \forall g$ (neutral element)

(ii) $\forall g \in G \ \exists! g' \in G$ s.t. $g * g' = g' * g = e$ (inverse element)

(iii) $*$ is associative, i.e. $(g*h)*k = g*(h*k) \ \forall g,h,k \in G$

Обычно значок $*$ опускается

A group is Abelian (=commutative) if $gh = hg \ \forall g,h \in G$

Proposition If $(X:G)$ is a Klein geometry, then G is a group.

Proof Let $*$ be composition (which is associative) and $e := id$.

Expls: see 2.0–2.5 and exercise class.

Defs Let G, H be groups, let $h: G \to H$ be a map s.t $h(gg') = h(g)h(g')$

homomorphism, mono, epi, iso

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§4 Morphisms of geometries

Геометрии по Клейну образуют категорию ...

Definitions A morphism $\mu: (X:G) \to (Y:H)$ is a pair of maps $\varphi: X \to Y$, $\gamma: G \to H$ s.t. γ is a homomorphism and the diagram is commutative for all $g \in G$, i.e.

$$\varphi(xg) = (\varphi(x))(\gamma(g)) \quad \forall g \in G \ \forall x \in X$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "g"'] & Y \arrow[d, "\gamma(g)"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Remark "morphism" $\sim$ "equivariant map"

- If φ is a bijection and γ is an isomorphism, then μ is called an isomorphism of geometries
- If φ is injective and γ is a monomorphism, then μ is called an injective morphism and we say that (X,G) is a subgeometry of (X', G'). • If φ is surjective ...

Examples. (i) $(\square : Sym\,\square) \hookrightarrow (⬚ : Sym(⬚))$

(ii) $(\mathbb{R}^2 : Isom_d\,\mathbb{R}^2) \xrightarrow{\cong} (\mathbb{R}^2 : Isom_{2d}\,\mathbb{R}^2)$

(iii) $(\mathbb{S}^2 : O^3) \to (\mathbb{S}^2/Ant, O^3)$ surjective morphism

(iv) $(\triangle ; Rot\,\triangle) \hookrightarrow \mathbb{I}^3, (Rot\,\mathbb{I}^3)$