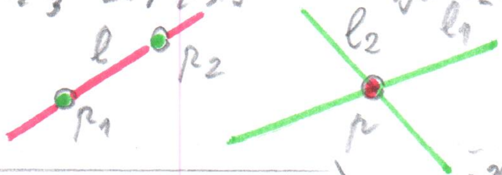
Lecture 10 PROJECTIVE GEOMETRY-1

§1. Definition

Points p of $\mathbb{R}P^2$ are lines passing through $O \in \mathbb{R}^3$, lines l are planes passing through O. Each point is determined by 3 coordinates $(x_1 x_2 x_3)$ (not all equal to O); the coordinates of a point are determined up to nonzero factor, so that $(x_1 x_2 x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$ for any $\lambda \neq 0$. The transformation group is $PGL_3(\mathbb{R}) := GL(3)/H$, where H is the subgroup of homotetheties. If $p \in \mathbb{R}P^2$ is $[(x_1, x_2, x_3)]$ and $\pi \in PGL_3(\mathbb{R})$ is $[((a_{ij}))]$, then $p\pi = [(x_1, x_2, x_3)]\pi := [((a_{ij}))(x_1, x_2, x_3)]$ which is well defined, i.e., does not depend on the choices $(x_1 x_2 x_3) \in [(x_1 x_2 x_3)]$ and $((a_{ij})) \in [((a_{ij}))]$.

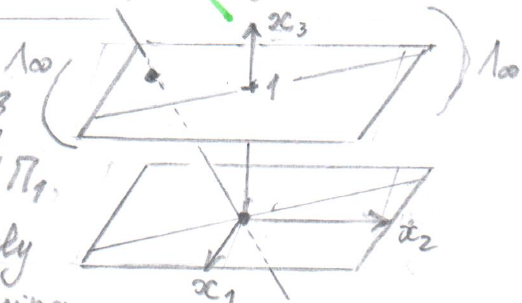
I. $\forall p_1 \neq p_2 \ \exists! \ l \ni p_1, p_2$

II. $\forall l_1 \neq l_2 \ \exists! \ p \in l_1, l_2$



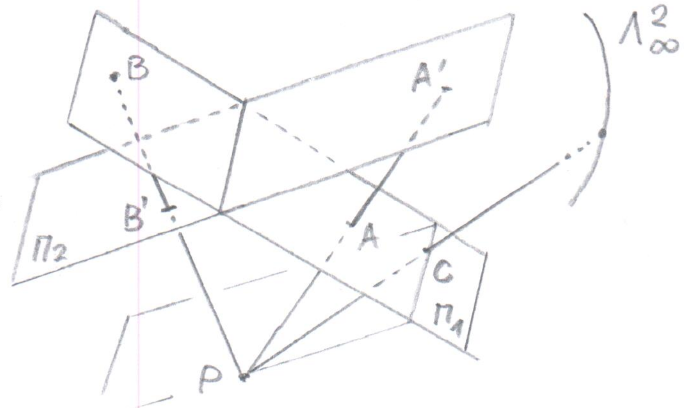
§2. The classical picture

Classically, $\mathbb{R}P^2$ is the plane $\Pi_1 = \{x_3 = 1\} \subset \mathbb{R}^3$, its points are points of Π_1, its lines are lines of Π_1 with the "line at infinity" Λ_∞ (which bijectively corresponds to lines in the plane $\Pi_0 = \{x_3 = 0\}$ passing through the origin. Note that $\mathbb{R}P^2$ has no metric.



§3. Why is it "projective"?

See the picture



§4. Homogeneous coordinates

In the model of §1, we denote the equivalence class $[(x_1, x_2, x_3)]$ by $(x_1 : x_2 : x_3)$ and call it homogeneous coordinates. In the model of §2, ordinary points have homogeneous coordinates $(x_1 : x_2 : 1)$ and points on the infinite line Λ_∞, $(x_1 : x_2 : 0)$.

Three points $(x_1^i : x_2^i : x_3^i)$, $i = 1, 2, 3$, are in general position if there is no (projective) line containing them $\Leftrightarrow$ the vectors (x_1^i, x_2^i, x_3^i), $i = 1, 2, 3$, form a basis of $\mathbb{R}^3 \Leftrightarrow$ the matrix $((x_j^i))$ is nondegenerate. Four points are in general position if any three are in general position

2019-10

§5 Four points in general position

Theorem (A,B,C,D) and (A',B',C',D') in general position $\Rightarrow \exists!\, \pi \in PGL_3(\mathbb{R})$
$(A,B,C,D) \xmapsto{\pi} (A',B',C',D')$. $(*)$

Proof. Let $\vec{OA'}=(a_1 a_2 a_3)$, $\vec{OB'}=(b_1,b_2,b_3)$, $\vec{OC'}=(c_1 c_2 c_3)$ in the basis $\vec{OA},\vec{OB},\vec{OC}$.

$$M:=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\quad A_g:=\begin{bmatrix} \lambda a_1 & \mu b_1 & \nu c_1 \\ \lambda a_2 & \mu b_2 & \nu c_2 \\ \lambda a_3 & \mu b_3 & \nu c_3 \end{bmatrix} \in PGL_3(\mathbb{R}).$$

Let us find λ,μ,ν such that $(*)$.

Let $\vec{OD}=(d_1,d_2,d_3)$, $\vec{OD'}=(d_1',d_2',d_3')$. Then we must have $D \to D'$:

$$\begin{cases} a_1 d_1 \lambda + b_1 d_2 \mu + c_1 d_3 \nu = d_1' \\ a_2 d_1 \lambda + b_2 d_2 \mu + c_2 d_3 \nu = d_2' \\ a_3 d_1 \lambda + b_3 d_2 \mu + c_3 d_3 \nu = d_3' \end{cases}$$

This system has a unique solution in (λ,μ,ν) because the deteminant of the equation is non zero, and for these values of λ,μ,ν we obtain $(*)$.

The proof of uniqueness is an exercise.

---

§6 Projective duality

The dual geometry of $(\mathbb{R}P^2 : PGL_3(\mathbb{R}))$ is $(D(\mathbb{R}P^2) : PGL_3(\mathbb{R}))$, where points of $D\mathbb{R}P^2$ are planes passing through $O \in \mathbb{R}^2$ and lines of $D\mathbb{R}P^2$ are lines passing through $O \in \mathbb{R}^2$. There is an obvious bijection:

$$D\mathbb{R}P^2 \ni q = a_1x_1 + a_2x_2 + a_3x_3 \longrightarrow (a_1 : a_2 : a_3) \in \mathbb{R}P^2 \quad (*)$$

Theorem The bijection $(*)$ defines an isomorphism between the geometries $(\mathbb{R}P^2 : PGL_3(\mathbb{R})) \cong (D\mathbb{R}P^2 : PGL_3(\mathbb{R}))$.

Corollary (Duality principle). The bijection $(*)$ between {lines} and {points} of $\mathbb{R}P^2$ takes any theorem of $\mathbb{R}P^2$ to a theorem of $\mathbb{R}P^2$
