Lecture 11. PROJECTIVE GEOMETRY II

§1. Desargues' Theorem

Given $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$ s.t. $(A_1B_1 \cap A_2B_2 \cap A_3B_3) = S$; $P_1 := A_2A_3 \cap B_2B_3$, $P_2 := A_1A_3 \cap B_1B_3$, $P_3 := A_1A_2 \cap B_1B_2$, then P_1, P_2, P_3 are collinear. Fig. 1

Proof. • We first prove the Desargues theorem in $\mathbb{R}^3$.

Denote $\Pi_1 := (A_1\hat{A}_2A_3)$, $\Pi_2 := (B_1\hat{B}_2B_3)$ and $\Lambda := \Pi_1 \cap \Pi_2$. Fig. 2

Let $Q_1 = (\hat{A}_2A_3) \cap (\hat{B}_2B_3)$, $Q_2 = (A_1A_3) \cap (B_1B_3)$, $Q_3 = (A_1\hat{A}_2) \cap (B_1\hat{B}_2)$.

Obviously $Q_1, Q_2, Q_3 \in \Lambda$

• Now we prove the Desargues theorem in the plane. Fig 3

Through SA_2 construct a plane Π_0 perpendicular to SB_1B_3, choose a point $\hat{A}_2 \in \Pi_0$ above A_2 and a point $\hat{B}_2$ above B_2 so that $S, \hat{A}_2, \hat{B}_2$ are collinear.

By the Desargues theorem in $\mathbb{R}^3$, the points Q_1, Q_2, Q_3 are collinear.

Now lower $(S\hat{B}_2)$ to (SB_2), then $\hat{A}_2$ lowers to A_2 and $Q \to P_1$, $Q_2 \to P_2$, $Q_3 \to P_3$.

Therefore, in the limit $Q_i = P_i$ and P_1, P_2, P_3 are collinear.

Remark If, say, $(A_2A_3) \parallel (B_2B_3)$, then P_1 lies on the line Λ_∞ at infinity. If also $(A_1A_3) \parallel (B_1B_3)$, then all three points P_1, P_2, P_3 lie on Λ_∞. Fig. 4 Fig. 5

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§2. Pappus' Theorem

Fig. 6

Given collinear points $A_1A_2A_3$ and $B_1B_2B_3$, $P_1 := (A_2A_3) \cap (B_2B_3)$, $P_2 = (A_1A_3) \cap (B_1B_3)$, $P_3 := (A_1A_2) \cap (B_1B_2)$, then (P_1, P_2, P_3) are collinar.

Proof By the Theorem in §5, Lecture 9, we can assume that the points $A_1A_2B_1B_2$ form a square, introduce coordinates $A_1(0,1)$, $A_2(1,1)$, $B_1(0,0)$, $B_2(1,0)$, calculate the coordinates $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$ and check that $(x_2-x_1)/(y_1-y_2) = (x_3-x_2)/(y_2-y_3)$.

Remark Pappus' theorem is correct when P_1 is at infinity and if P_1, P_2 are at infinity; in the latter case, $P_1, P_2, P_3 \in \Lambda_\infty$.

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§3. Pascal's Theorem

Fig 7

Given 6 points A,B,C,D,E,F on a conic, let $P_1 := (AB) \cap (ED)$, $P_2 := (FA) \cap (DC)$, $P_3 := (FE) \cap (BC)$; then P_1, P_2, P_3 are collinear

For the proof, see the next page

§3 cont'd

Definition. A 2-d degree curve (conic) in $\mathbb{R}P^2$ is non-degenerate if it part in $\mathbb{R}P^2 \setminus \Lambda_\infty$ is an ellipse, a hyperbola, or a parabola.

Lemma. All nondegenerate conics are projectively equivalent.

Proof. To transform a parabola into an ellipse, send its point at infinity to the origin; to transform a hyperbola into an ellipse, send its two points at infinity to two points in $\mathbb{R}P^2 \setminus \Lambda_\infty$; to transform an ellipse into another ellipse, use the Theorem in §5 of Lec. 9

The Lemma implies it is sufficient to prove the theorem for the circle. There are many proofs of this fact: using the cross-ratio of 4 points on the circle, using 3-dimensional space (as in §1 above), using the Menelaus theorem, etc.

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§4. Projective space $\mathbb{R}P^3$

Definition. The points of $\mathbb{R}P^3$ are lines in $\mathbb{R}^4$ containing the origin O, lines are planes in $\mathbb{R}^4$ containing O, planes are 3-dimensional hyper planes in $\mathbb{R}^4$ containing O, the group determining the geometry of $\mathbb{R}P^3$ is $PGL_4(\mathbb{R})$ and is defined similarly to $PGL_3(\mathbb{R})$.

There is a beautiful duality in $(\mathbb{R}P^3, PGL_4(\mathbb{R}))$ given by

point $\longleftrightarrow$ plane; line $\longleftrightarrow$ line.

This duality transforms theorems into theorems, for example

"Any two planes intersect in a unique line" is dual to

"Any two lines are contained in a unique plane".
