2019-12

Lecture 12. "PROJECTIVE GEOMERY IS ALL GEOMETRY"

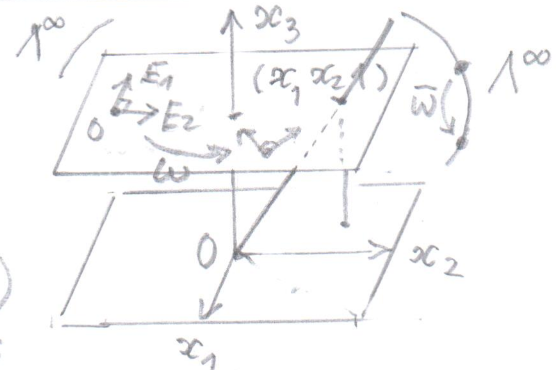
§1. Main Theorem

Def. $(A:H)$ is a subgeometry of $(X:G)$ if there exists an injective map $i: A \hookrightarrow X$ and a monomorphism $\mu: H \to G$ which are compatible, i.e.:
$i(ah) = (i(a))(\mu(h))$ $\forall a \in A, h \in H$.

$$\begin{array}{ccc} A & \xrightarrow{h} & A \\ i\downarrow & & \downarrow i \\ X & \xrightarrow{\mu(h)} & X \end{array}$$

Theorem All the geometries in this course are subgeometries of $(\mathbb{R}P^3 : PGL_4(\mathbb{R}))$.

§2. $\mathbb{R}^2$ and $\mathbb{R}^3$



$(\mathbb{R}^2 : O(2))$ is a subgeometry of $(\mathbb{R}P^2 : PGL_3(\mathbb{R}))$.

Proof $i: \mathbb{R}^2 \to \mathbb{R}P^2 : (x_1, x_2) \mapsto (x_1 : x_2 : 1)$

$\mu: O(2) \to PGL_3(\mathbb{R})$; $O(2) \ni \omega$ takes a frame (OE_1E_2) to $(O'E_1'E_2')$; extend ω to $\mathbb{R}P^2$ by defining it on Λ_∞ in the natural way. The map i and μ are compatible.

Similarily $(\mathbb{R}^3, O(3)) \hookrightarrow (\mathbb{R}P^3, PGL_4(\mathbb{R}))$

§3. $\mathbb{H}^2$ (Cayley-Klein model)

Fig. 1

$(\mathbb{H}^2, Isom_\rho(\mathbb{H}^2))$ is a subgeometry of $(\mathbb{R}P^2 : PGL_3(\mathbb{R}))$.

Proof $i: (x_1, x_2) \mapsto (x_1 : x_2 : 1)$; given $\omega \in Isom_\rho(\mathbb{H}^2)$, take (A, B, C, D) in general position and let $(A', B', C', D') = (\omega(A), \omega(B), \omega(C), \omega(D))$. Then $\exists!$ projective transformation $p: \mathbb{R}P^2 \to \mathbb{R}P^2$ s.t. $(A, B, C, D) \to (A', B', C', D')$ by the Theorem in §5 of Lecture 10.

The correspondence $\omega \mapsto p$ defines a monomorphism $Isom_\rho(\mathbb{H}^2) \xrightarrow[\mu]{} PGL_3(\mathbb{R})$ because $(A, B, C) \to (A', B', C')$ uniquely defines ω.

The map i and the monomorphism μ are compatible by construction. Note that lines in $\mathbb{R}P^2$ are extensions of lines in $\mathbb{H}^2$

§4. The elliptic plane

$(\mathbb{S}^2/Ant : Isom\,\mathbb{S}^2)$ is a subgeometry of $(\mathbb{RP}^2 : PGL_3(\mathbb{R}))$.

- $i : \mathbb{S}^2/Ant \to \mathbb{RP}^2 = \Pi_1 \cup \Lambda_\infty$ is the projection of the lower half sphere onto $\mathbb{RP}^2$ from $(0,0,2)$ (the center of $\mathbb{S}^2$). Fig. 2
- $\mu : Isom\,\mathbb{S}^2 \to PGL_3(\mathbb{R})$; let $g \in Isom\,\mathbb{S}^2$; take $A,B,C,D \in \mathbb{S}^2$ in general position, let $g(A,B,C,D) =: A_1, B_1, C_1, D_1$, let $i(A,B,C,D) =: (A',B',C',D')$, $i(A_1,B_1,C_1,D_1) =: (A_1',B_1',C_1',D_1')$; $\exists!\ \omega \in PGL_3(\mathbb{R})$ such that $\omega(A',B',C',D') = (A_1',B_1',C_1',D_1')$. Put $\mu : g \mapsto \omega$.

Then i is injective and μ is a monomorphism, and they are compatible by construction.

§5 Spherical geometry

$(\mathbb{S}^2 : Isom\,\mathbb{S}^2 = O(2)) \hookrightarrow (\mathbb{R}^3 : O(2)) \hookrightarrow (\mathbb{R}^3 : O(3)) \hookrightarrow (\mathbb{RP}^3 : PGL_4(\mathbb{R}))$

§6. Hierachy of geometries

See Fig 3 below: