2019-4 (1)

Lecture 4. TILINGS & FIODOROV GEOMETRIES

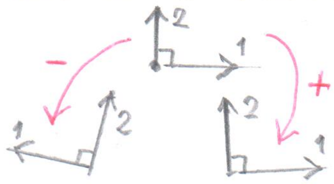
§1. Examples

Alhambra XIVc. Рис. 4.1 ; Escher Рис. 4.2 ; Penrose Рис. 4.4 ; Vanderberg Рис. 4.3 are examples of tilings of $\mathbb{R}^2$, which is filled with copies of a tile (which can be one-sided or two-sided: Some of them are "regular" (in a sense).

and they define Klein geometries whose transformation groups are subgroups of $Isom(\mathbb{R}^2)$. Here are six examples of such geometries Рис. 4.5

§2. Isometries of $\mathbb{R}^2$

Definitions An isometry of $\mathbb{R}^2$, $g \in Isom(\mathbb{R}^2)$, is orientation reversing if $\exists$ two orthonormal frames F_1, F_2 such that $\nexists$ a rotation or a shift that takes F_1 to F_2; otherwise g is called orientation preserving



Remark Orientation preserving isometries of $\mathbb{R}^2$ form a subgroup of $Isom(\mathbb{R}^2)$ denoted $Isom^+(\mathbb{R}^2)$.

Theorem Any isometry $g \in Isom(\mathbb{R}^2)$ is either a rotation, or a parallel translation (= shift), or a glide symmetry.

§3 Two-dimensional tilings

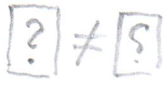
Definitions. A tile is a figure in $\mathbb{R}^2$ bounded by a closed curve or a closed broken line. A regular tiling with a one-sided (two-sided) tile T is a Klein geometry $(\mathbb{R}^2 : G_T)$, where G_T is a subgroup of $Isom^+(\mathbb{R}^2)$ (of $Isom(\mathbb{R}^2)$, resp.) and T is the fundamental domain of G_T, i.e., $\mathbb{R}^2 = \bigcup_{g \in G_T} Tg$ and $Int\, Tg \cap Int\, T_{g'} \neq 0 \Leftrightarrow g = g'$.

Theorem There are 17 regular tilings of $\mathbb{R}^2$: Рис. 4.5 + Рис. 4.6

Five of them are one-sided, twelve are two sided.

The corresponding Klein geometries are called Fiodorov, and the groups acting in them, Fiodorov (= Russian crystallographer, 1891) groups.

We will prove the theorem only in the one-sided case

Remark: Tiles (including one-sided tiles) are oriented 

§4. Classification of one-sided tilings of $\mathbb{R}^2$

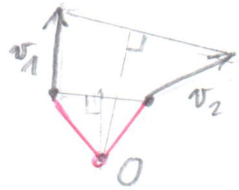
Proof. Let $(\mathbb{R}^2, G_T)$ be regular tiling with tile T and $G_T \in Isom^+\mathbb{R}^2$.

(i) G_T contains no rotations $\Rightarrow T =$; (a); $\uparrow \rightarrow$

(ii) G_T contains rotations of orde 2 only $\Rightarrow T \in$; (δ); π — Рис. 4.5

(iii) G_T contains a rotation of order $k \geq 3$, $(R_A, 2\pi/k)$

Main Lemma (iii) $\Rightarrow \begin{cases} G_T \text{ contains two more rotations of orders } l, m \\ \& \ 1/k + 1/l + 1/m = 1 \quad (*) \end{cases}$

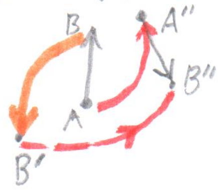


Example: $(k, l, m) = (3,3,3)$ Рис. 333 Рис. 4.5

Auxilliary statements:

(a) $\forall v_1, v_2 \subset \mathbb{R}^2$, $|v_1| = |v_2|$ $\exists!$ rotation R_0 s.t. $R_0(v_1) = R_0(v_1)$

(b) $\forall (R_A, \alpha), (R_B, \beta)$ there is a simple construction that takes AB to A''B'' and determines $R_C = R_A * R_B$ by using (a).



End of the proof. Eq. (*) has three solutions (3,3,3), (4,4,2), (3,6,2)

See Рис. 442 Рис 362 Рис. 4.5

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§5. Two-sided tilings of $\mathbb{R}^2$

There are twelve: Рис. 4.6

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§6. Regular tilings of $\mathbb{R}^3$

There are 230 [Fiodorov 1891]
