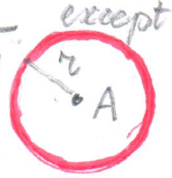
Lecture 5. SPHERICAL & ELLIPTIC GEOMETRIES

§1. Spherical geometry (basics)

Spherical geometry is the geometry $(\mathbb{S}^2, G_S)$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and G_S is the isometry group of the sphere (also denoted $O(3)$). Lines in spherical geometry are great circles, the distance between points A and B is the measure (in radians) of the angle AOB (O is the centre of $\mathbb{S}^2$). The polar of a point A is the great circle whose points are at the distance $\pi/2$ from A (e.g. the North pole and the equator). The angle between two lines l_1, l_2 is the dihedral angle between the corresponding planes.

I. $\forall A\ \exists!$  except $A=-B$!

II. $\forall A\ \forall l\ \exists!$  except polars!

III. $\forall A\ \forall 0<r\leq \pi/2\ \exists!$  $r \neq$ Euclidean radius!

There are no parallels.

There are two types of isometries: reflections in planes $\Pi \ni 0$ and rotations about straight lines $L \ni 0$.

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§2 Area measure

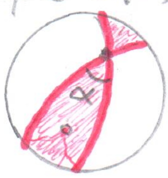
Jordan measure on $\mathbb{S}^2$; $: \{\text{polygons on } \mathbb{S}^2\} \to \mathbb{R}_+$ s.t.

(I) Additivity: $\mathcal{S}(\bigcup_i P_i) = \sum_i \mathcal{S}(P_i)$ if $i \neq j \Rightarrow \operatorname{Int} P_i \cap \operatorname{Int} P_j = \emptyset$

(II) Invariance: $\mathcal{S}(\varphi(P)) = \mathcal{A}(P)$ if $\varphi \in O(3)$

(III) Normalization: $\mathcal{S}(\mathbb{S}^2) = \pi$

(IV) Biangle formula: $\mathcal{S}($$) = 2\alpha$



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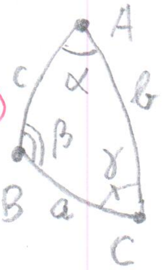
§3 Triangles

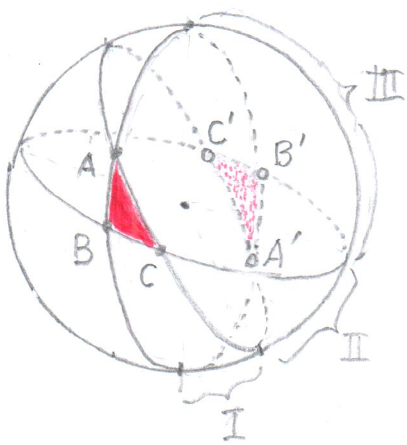
Area: $\boxed{\mathcal{S}(\Delta ABC) = \alpha+\beta+\gamma-\pi}$ (*)



Proof $4\pi = \mathcal{S}(\mathrm{I}) + \mathcal{S}(\mathrm{II}) + \mathcal{S}(\mathrm{III}) -$

$- 2\mathcal{S}(ABC) - 2\mathcal{S}(A'B'C') =$

$= 4\alpha + 4\gamma + 4\beta - 4\mathcal{S}(ABC) \Rightarrow$ (*)



Corollary The sum of angles of a spherical triangle is greater than π.

§3 Triangles (cont'd)

Sine theorem: $\boxed{\dfrac{\sin a}{\sin\alpha} = \dfrac{\sin b}{\sin\beta} = \dfrac{\sin c}{\sin\gamma}}$ (*)

Proof:

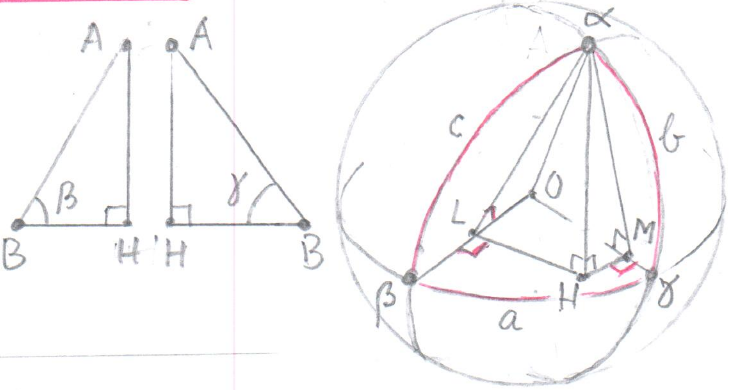
$\Delta ALH \Rightarrow AH = AL\sin\beta =$

$\quad (1) \quad = \sin c \sin\beta$

$\Delta AMH \Rightarrow AH = AM \sin\gamma$

$\quad (2) \quad = \sin b \sin\gamma$

$(1)\&(2) \Rightarrow (*)$



Cosine theorems: $\cos a = \cos b \cos c + \sin b \sin c \cos\alpha$

$\cos\alpha + \cos\beta\cos\gamma = \sin\beta\sin\gamma\cos a$

Pythagoras theorem: $\cos a = \cos b \cos c$

§4. Coxeter triangles on $\mathbb{S}^2$.

Def. A triangle with angles $\pi/k, \pi/l, \pi/m$ is Coxeter if it is the fundamental domain of a Klein geometry on $\mathbb{S}^2$.

Theorem There are three Coxeter triangles: $(\pi/2, \pi/3, \pi/3)$, $\boxed{\text{Рис. 6.3}}$ $(\pi/2, \pi/3, \pi/4)$, $(\pi/2, \pi/3, \pi/5)$ and a series $(\pi/2, \pi/2, \pi/n)$, $n \geq 2$)

Fact. The group $O(3) \equiv \mathrm{Isom}\,\mathbb{S}^2$ has five finite subgroups (the isometry groups of the five Platonic solids) and one infinite series (the dihedral groups: isometry groups of regular n-gons, $n \geq 2$)

Fact $\Rightarrow$ Theorem. Let the triangle $(\pi/k, \pi/l, \pi/m)$ be Coxeter and let N triangles cover $\mathbb{S}^2$. Then $N(\pi/k + \pi/l + \pi/m - \pi) = 4\pi \Rightarrow$ $N/k + N/l + N/m = N + 4$; Fact $\Rightarrow N \in \{24, 48, 120\} \Rightarrow$ three triangles from $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ and series from $(2,2,n)$, $n \geq 2$

§5 Riemann's elliptic geometry...

... is the geometry $(\mathbb{S}^2/Ant : O(3))$. It is locally isometric to spherical and axioms I, II, III hold without exceptions.

Theorem There is a surjective morphism of geometries $Ant : (\mathbb{S}^2 : O(3)) \to (\mathbb{S}^2/Ant : O(3))$ given by $s \mapsto (s, -s)$.