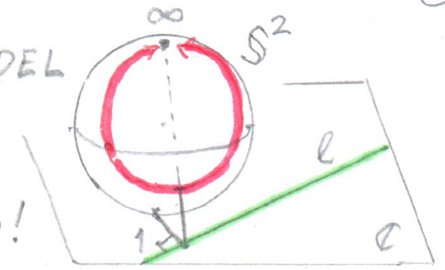
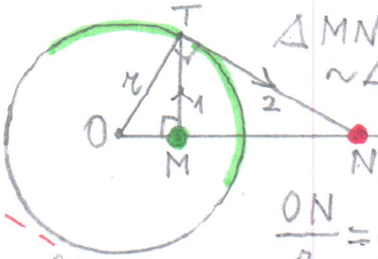
Lecture 6. INVERSION → POINCARÉ DISK MODEL



§1. Inversion : main properties

The Riemann sphere $\bar{\mathbb{C}} := \mathbb{C} \cup \infty$. All lines $\ni \infty$!

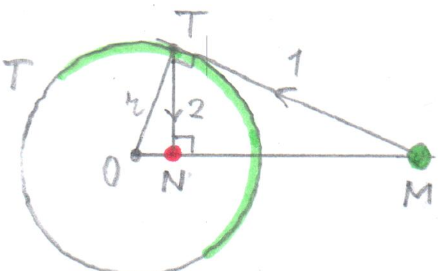
Inversion $I(O, r^2) : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ s.t. $M \mapsto N$, $N \in [OM\rangle$, $ON \cdot OM = r^2$,
$\infty \mapsto O$, $O \mapsto \infty$.



$\Delta MNT \sim \Delta ONT$

$\Downarrow$

$\frac{ON}{r} = \frac{r}{OM}$



Properties : $I(O, r^2)$ is

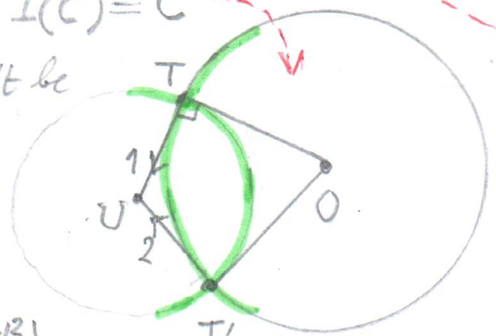
(1) bijective involution

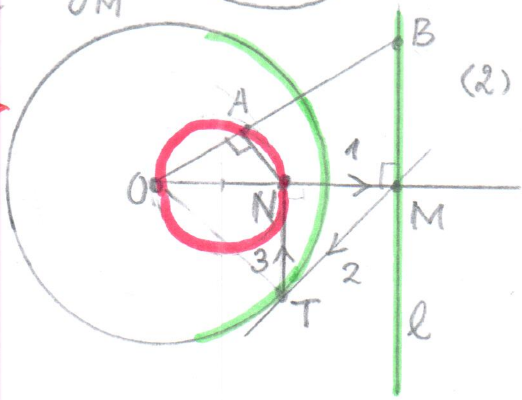
(2) $I : \{\text{circle} \vee \text{line}\} \circlearrowleft$

(3) preserves angles (conformal)

(4) $C \perp C_I \Rightarrow I(C) = C$

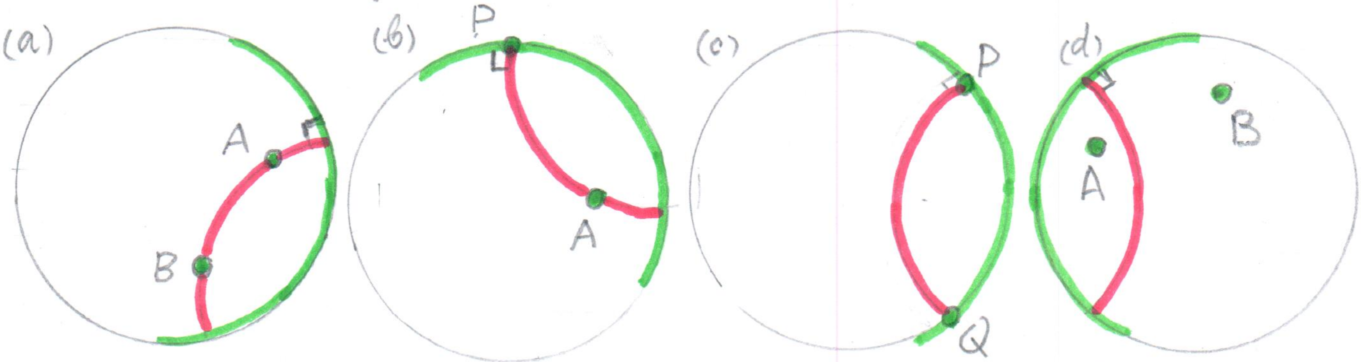
Proof: $I(C)$ can't be a line (why?) and so it has to be a circle, namely $(U, |UT|^2)$

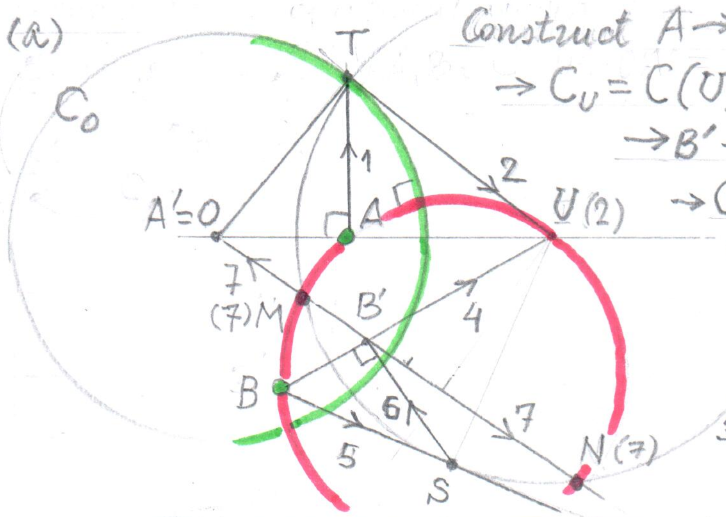




(5) $C \perp C(0,1) \Rightarrow I_C$ is a bijection of $\{z \mid |z| < 1\}$ to itself.

§2. Inversion : special properties $\exists!$ circle $\perp$ to $C_0 = C(0,1)$:

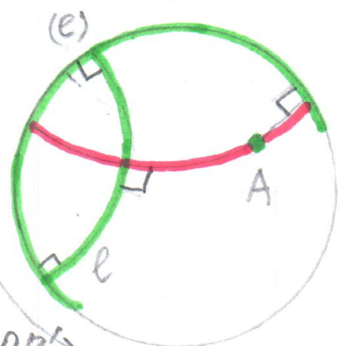




Construct $A \to T \to U \to$
$\to C_U = C(U, UT^2) \to S \to$
$\to B' \to M \,\&\, N \to$
$\to C_+ \ni M, N, U$.

Then $C_+ \ni A, B$ because $C_+ \leftrightarrow \langle OB' \rangle$

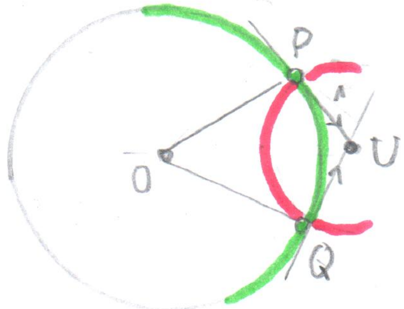
3 and $C_+ \perp C_0$ because so is the line $\langle OB' \rangle$



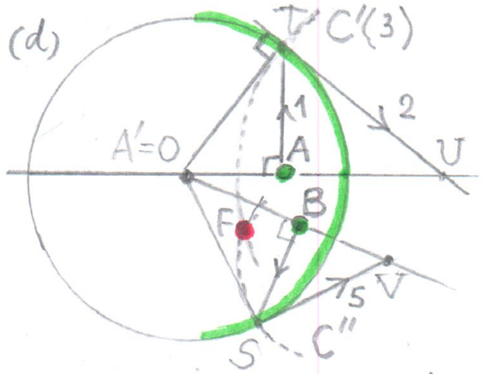
§2 (cont'd)

(b) Proof similar to (a): $A \to T \to U \to I(U, UT^2): A \to 0$ and $P \to P'$ etc.

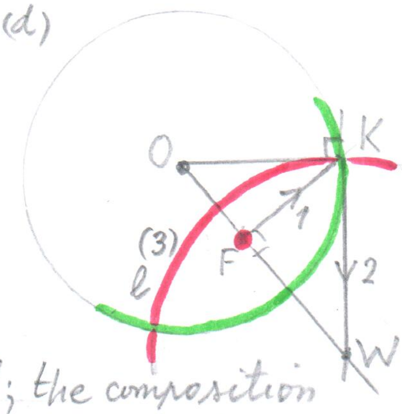
(c)



(d)

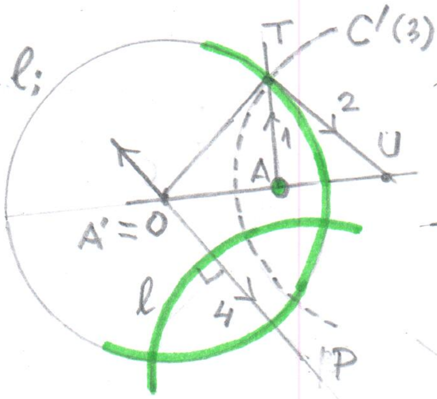


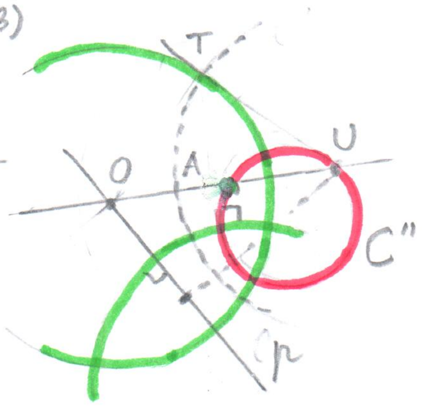
(d)



(d) $A \to T \to U \to C'$; $B \to S \to V \to C''$; $F := C' \cap C''$; the composition of the inversions in C' and C'' takes A to B leaves F in place; $\exists!$ line l which leaves F in place.

(e) $A \to T \to U \to C'$; $\langle OP \rangle \perp l$; the inversion in C' takes $\langle OP \rangle$ to a circle C'' that passes through U (why?) and A (why), and is $\perp$ to the given line l.





§3. Definition of the Poincaré disk model

Denote by $\mathbb{H}^2$ the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and by $\mathcal{M}$ the Möbius group (acting on $\mathbb{H}^2$) and generated by reflections in circles orthogonal to $\mathbb{A} = \partial \mathbb{H}^2$ and in diameters of $\mathbb{H}^2$. It follows from (5) that each $\mu \in \mathcal{M}$ is a bijection of $\mathbb{H}^2$ and from (1) that $\mathcal{M}$ is a group acting on $\mathbb{H}^2$.

The geometry $(\mathbb{H}^2 : \mathcal{M})$ is called the (Poincaré disk model of) the Lobachevsky plane; its lines are the diameters of $\mathbb{H}$ and the arcs (contained in $\mathbb{H}^2$) of circles orthogonal to $\mathbb{A} = \partial \mathbb{H}^2$, which is called the absolute.