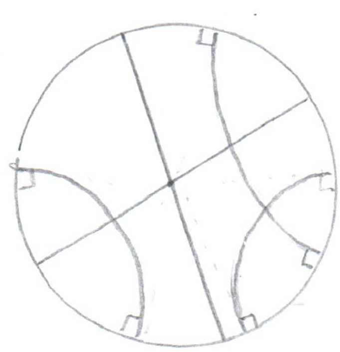
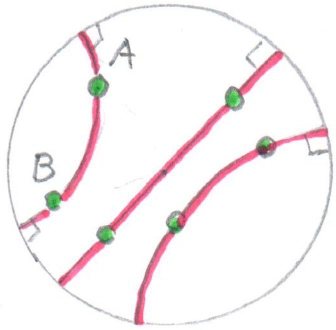
Lecture 7 POINCARÉ DISK MODEL

§1 Points and lines



The Poincaré disk model of the hyperbolic plane is the geometry $(\mathbb{H}^2, \mathcal{M})$, where $\mathcal{M}$ is the Möbius group and $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$.

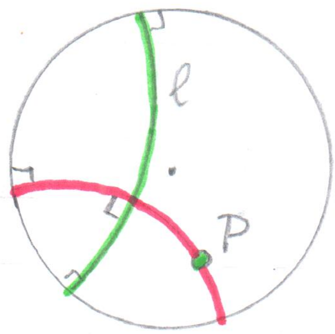


The points of the model are ordinary points of the open disk $\mathbb{H}^2$, the lines are either arcs of circles orthogonal to the boundary A of $\mathbb{H}^2$ (called the absolute) or diameters of A.

Fact I $\forall A, B \in \mathbb{H}^2, A \neq B, \exists! \, l : l \ni A, B$

This follows from (a) in Lecture 6.

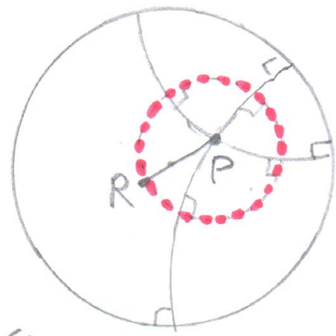
§2. Perpendiculars



Fact III $\forall P \forall l \; \exists! \, l' : P \in l' \perp l$

This follows from (e) in Lect. 6 if $P \notin l$. If $P \in l$, send P to O, etc. If l is a diameter, then send P to O, in that case the construction is obvious

§3. Rotations and circles



A rotation about a point $P \in \mathbb{H}^2$ is the composition of two reflections in lines passing through P. A circle of center P and radius PR is the set of all images of R under all rotations about P; notation $C(P; PR)$.

Fact IV $\forall P \forall R \; \exists! \, C(P; PR)$.

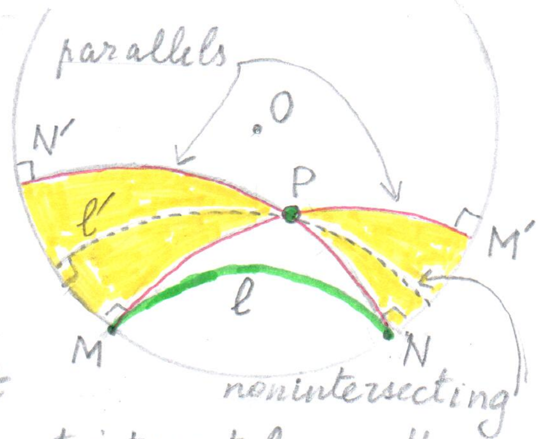
The proof is the definition of circles

Theorem Any hyperbolic circle of center P is a Euclidean circle whose center I differs from P.

Exercise Prove this theorem and, using complex numbers, calculate the Euclidean distance IP.

§4. Parallels and nonintersecting lines

Given $P \notin \ell$, let $\ell \cap A = M \cup N$. Using (b) from Lect. 6, construct lines PM and PN; they are called parallels to ℓ through P. Any line ℓ' through P between $\langle MPM' \rangle$ and $\langle NPN' \rangle$ does not intersect ℓ. We have



Fact V If $P \notin \ell = \langle MN \rangle$, then there exist infinitely many lines through P that do not intersect ℓ and they lie between $\langle MPM' \rangle$ and $\langle NPN' \rangle$.

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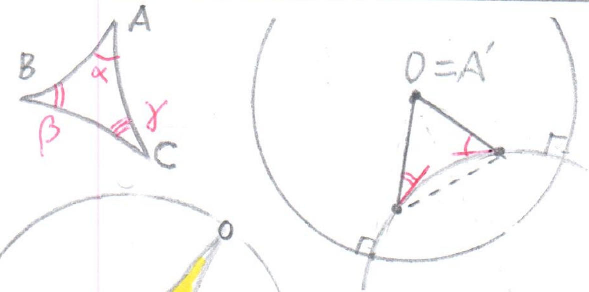
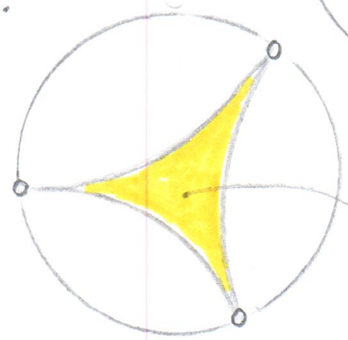
§5. Angle sum of a triangle.



Theorem. $\alpha + \beta + \gamma < \pi$.

Proof. If $A = O$, then it is obvious. In the general case, send A to O by a reflection and back again, and use that all reflections preserve angles.



It is interesting that small triangles have large angle sums, while very large triangles have small angle sums.

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§6. Hyperbolic geometry and the physical world

See the book pp 127–128.
