2019-8

Lecture 8. POINCARÉ HALF-PLANE MODEL

§1. Linear-fractional transformations

The transformation of the Riemann sphere $\overline{\mathbb{C}}$ given by the rules $z \mapsto \dfrac{az+b}{cz+d}$, where $cb-ad \neq 0$ and $-d/c \mapsto \infty$, $\infty \mapsto a/c$ is called *linear-fractional*.

Theorem 1. Linear-fractional transformations are bijections of $\overline{\mathbb{C}}$, they form a group (denoted $PGL_2(\mathbb{C})$), they preserve the cross ratio $\langle z_1, z_2, z_3, z_4\rangle = \dfrac{z_3-z_1}{z_3-z_2} : \dfrac{z_4-z_1}{z_4-z_2}$, they take lines or circles to lines or circles, they preserve angles. For any pair of triplets (z_1,z_2,z_3), (w_1,w_2,w_3) there exists a unique linear-fractional transformation taking z_i to w_i, $i=1,2,3$.

All these facts were proved in the exercise class (листок 6)

Hint: $z \mapsto cz+d =: z_1 \mapsto cz_1 =: z_2 \mapsto \dfrac{1}{z_2} =: z_3 \mapsto (bc-ad)z_3 =: z_4 \ldots$

$$\ldots z_4 \mapsto \frac{a}{c} + z_4 = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = \frac{az+b}{cz+d}$$

Examples (1) $\Omega : z \mapsto \dfrac{1+z}{1-z}$ takes $\{z : |z|<1\}$ to $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$

(2) the map $z \mapsto (az+b)/(cz+d)$ (or $z \mapsto (a(-\bar{z})+b)/(c(-\bar{z})+d)$, $a,b,c,d \in \mathbb{R}$ is a bijection of $\mathbb{C}_+$ to itself. Such maps form a group denoted $\mathcal{M}^*$

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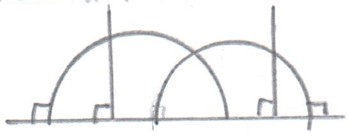
§2. The half-plane model

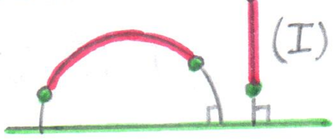
The geometry $(\mathbb{C}_+, \mathcal{M}^*)$ is called the *Poincaré half-plane model*, the line $\operatorname{Im} \mathbb{C} = 0$ is its absolute $\mathbb{A}$. Straight lines $\perp$ $\mathbb{A}$ and half circles with centers on $\mathbb{A}$ are called *lines*:

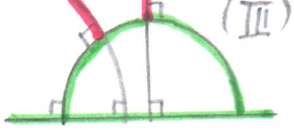


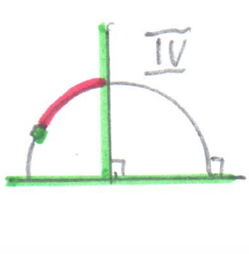
Theorem 2. The map Ω defines an isomorphism of geometries $(\mathbb{H}^2 : \mathcal{M}) \cong (\mathbb{C}_+ : \mathcal{M}^+)$ where the bijection between the open disk and $\mathbb{C}_+$ is given by Ω and the isomorphism between $\mathcal{M}$ and $\mathcal{M}^+$ is given by $\mathcal{M} \ni g \mapsto \Omega^{-1} \circ g \circ \Omega \in \mathcal{M}^+$

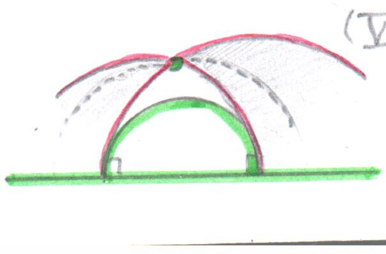
Corollary. The main properties I, III, IV, V (Lecture 7) also hold for the half-plane model.

See the figures:

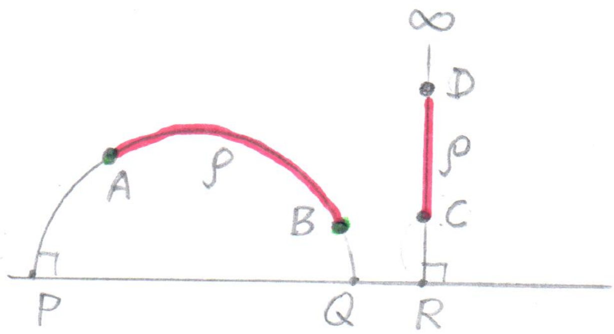








§3. Distances



Define the distance between points

$$\rho(A,B) := \left| \ln\left(\frac{z_A - z_P}{z_B - z_P} : \frac{z_A - z_Q}{z_B - z_Q} \right) \right|$$

if $\operatorname{Re} z_A \neq \operatorname{Re} z_B$;

$$\rho(C,D) = \left| \ln\left(\frac{z_C - z_R}{z_D - z_R} : \frac{z_C - z_\infty}{z_D - z_\infty} \right) \right| = \left| \ln \frac{CR}{DR} \right| \text{ if } \operatorname{Re} z_C = \operatorname{Re} z_D$$

Theorem 3. The function ρ is a distance on $\mathbb{C}_+$, and the isometry group of ρ is isomorphic to $\mathcal{M}^*$: $\operatorname{Isom}_\rho \mathbb{C}_+ \cong \mathcal{M}^* (\cong \mathcal{M})$.

We have proved that any element of $\mathcal{M}^*$ preserves distances (because it preserves cross-ratio). We omit the proof of the fact that any element of $\operatorname{Isom}_\rho(\mathbb{C}_+)$ is actually an element of $\mathcal{M}^*$.

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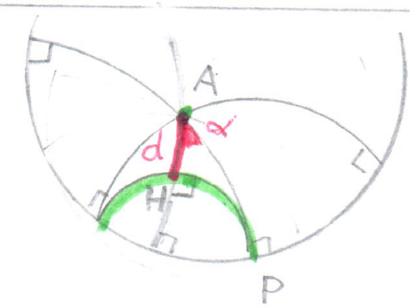
§4. Angle of parallelism



Theorem 4 The angle HAP, called the angle of parallelism depends only on $d = AH$ and is given by the formulas

$$\boxed{\tan \alpha/2 = e^{-d}} \quad \text{and} \quad \boxed{\tanh = \cos \alpha}.$$

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§5. Absolute constants

The area of the "infinite triangle" and the Schweikart constant $\sigma = OH$ are both absolute constants of hyperbolic geometry.